J. Astrophys. Astr. (2014) **35,** 107–119

Stability of Triangular Equilibrium Points in the Photogravitational Restricted Three-Body Problem with Oblateness and Potential from a Belt

Jagadish Singh[1,*] & Joel John Taura[2,†]

[1]*Department of Mathematics, Faculty of Science, Ahmadu Bello University, Zaria, Nigeria.*
[2]*Department of Mathematics and Computer Science, Faculty of Science, Federal University, Kashere, Gombe State, Nigeria.*
e-mail: *[*]jgds2004@yahoo.com*
[†]taurajj@yahoo.com

Received 23 November 2013; accepted 24 February 2014

Abstract. We have examined the effects of oblateness up to J_4 of the less massive primary and gravitational potential from a circum-binary belt on the linear stability of triangular equilibrium points in the circular restricted three-body problem, when the more massive primary emits electromagnetic radiation impinging on the other bodies of the system. Using analytical and numerical methods, we have found the triangular equilibrium points and examined their linear stability. The triangular equilibrium points move towards the line joining the primaries in the presence of any of these perturbations, except in the presence of oblateness up to J_4 where the points move away from the line joining the primaries. It is observed that the triangular points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \leq \mu \leq \frac{1}{2}$, where μ_c is the critical mass ratio affected by the oblateness up to J_4 of the less massive primary, electromagnetic radiation of the more massive primary and potential from the belt, all of which have destabilizing tendencies, except the coefficient J_4 and the potential from the belt. A practical application of this model could be the study of motion of a dust particle near a radiating star and an oblate body surrounded by a belt.

Key words. Restricted three-body problem—photogravitational—zonal harmonic effect—potential from the belt.

1. Introduction

The general three-body problem requires the motion of three celestial bodies under their mutual gravitational interaction. However, the complete solution of the general problem remains a formidable challenge. The restricted three-body problem is a simplified form of the general three-body problem, in which one of the bodies is of infinitesimal mass, and therefore does not influence the motion of the remaining two massive bodies called the primaries (Chenciner 2007; Bruno 1994; Gutzwiller 1998; Valtonen & Karttunen 2006). Three-body problem also exists in general relativity: the geodetic or de Sitter effect (Renzetti 2012a).

At the Newtonian level, the circular restricted three-body problem (in brief CR3BP) deals with the motion of the infinitesimal mass in the gravitational field of the primaries, which revolve in circular orbits around their centre of mass. It becomes photogravitational when one or both of the masses of the primaries are intense emitters of electromagnetic radiation. The CRTBP admits five equilibrium points in the plane of motion of the primaries; three, the collinear points L_1, L_2, L_3 lying on the line connecting the primaries, while the other two are the triangular points L_4, L_5 forming equilateral triangles with the primaries. The latter are linearly stable for the mass ratio μ of the primaries less than $\mu_0 = 0.03852$ (Szebehely 1967).

Studies on the locations and stability of the equilibrium points with perturbations have received a lot of attention in recent times. Radzievskii (1950, 1953) found that an allowance for direct solar radiation pressure results in a change in the positions of the equilibrium points. Jiang and Yeh (2003, 2004a, b,) studied the Chermnykh's problem by considering the influence from a belt for planetary systems and found that the probability to have equilibrium points around the inner part of the belt is larger than the one near the outer part. The influence from the belt makes the structure of the dynamical system quite different so that new equilibrium points exist under certain condition (Jiang & Yeh 2003; Yeh & Jiang 2006; Kushvah 2008). The orbital motion of a test particle around a primary is greatly affected in the presence of a massive belt (Iorio 2007, 2012).

The CR3BP generally assumes that the shapes of the participating bodies are spherical, but it is found that celestial bodies, such as the Sun (Rozelot *et al*. 2004; Rozelot & Damiani 2011; Rozelot & Fazel 2013), the solar system's planetary satellites (Anderson *et al*. 2001; Iess *et al*. 2010; Yan *et al*. 2013) and some astrophysical compact objects (Geroch 1970; Hansen 1974) are oblate. The lack of sphericity or the oblateness of the planets causes large perturbations from a two-body orbit. The most striking example of perturbations due to oblateness in the solar system is the orbit of the fifth satellite of Jupiter, Amalthea. This planet is very oblate and the satellite's orbit is very small that its line of apsides advances about 900° in one year (Moulton 1914). Such oblateness-driven effects are competing disturbing effects for qualitatively similar general relativistic effects (Iorio 2005, 2009; Renzetti 2012b, 2013a; Iorio *et al*. 2011, 2013). At the Newtonian level, the orbital effects of the quadrupole, i.e. J_2, and the octupole, i.e. J_4, on the orbital motion of a particle in the field of an ashperical primary has been recently worked out in the general case of an arbitrarily oriented spin axis (Iorio 2011; Renzetti 2013b). At the post-Newtonian level, the quadrupole J_2 has orbital consequences as well, which could be measured in a foreseeable future in space-based mission (Soffel *et al*. 1988; Heimberger *et al*. 1990; Brumberg 1991; Iorio 2013).

Abouelmagd (2012) proved that the locations of the triangular points and their linear stability are affected by the oblateness up to J_4 of the more massive primary in the planar CR3BP. Abouelmagd (2013) examined the effects of photogravitational force and oblateness in the perturbed restricted three-body problem. Kushvah (2008) studied analytically and numerically the effects of electromagnetic radiation pressure of more massive primary, oblateness of less massive primary and gravitational potential from a circum-binary belt on the linear stability of equilibrium points in

R3BP. Singh and Taura (2013) studied the combined effect of electromagnetic radiation and oblateness up to J_2 of both primaries, together with additional gravitational potential from the circum-binary belt on the motion of an infinitesimal body in the binary stellar systems within the framework of CR3BP.

The present work considers the effects of oblateness of less massive primary up to the coefficient J_4 and electromagnetic radiation of the more massive primary, together with additional gravitational potential from a circum-binary belt on the stability of triangular equilibrium points in CR3BP.

The triangular equilibrium solutions of the CR3BP are widely used in many branches of astronomy, both for constant and variable masses (e.g., in the Roche model for binary stars systems) (Luk'yanov 1989). These equilibrium points are important in astronomy as they mark places where particles can be trapped (L_4 and L_5). Thus, the results obtained in this study will have practical applications in astrophysics.

This paper is organized as follows: Section 2 deals with the derivation of equations of motion and zero velocity surfaces. Section 3 is devoted to the location of triangular equilibrium points. The linear stability of these points is examined in section 4, while the discussion and conclusion are presented in sections 5 and 6 respectively.

2. Equations of motion and zero velocity surfaces

We consider an infinitesimal body of mass m moving under the gravitational influence of the more massive and less massive primaries of masses m_1 and m_2, respectively. We take a co-ordinate system $oxyz$ with origin at the centre of mass of the primaries and the x-axis is the line joining the primaries, the y-axis is perpendicular to it, while the z-axis is perpendicular to the orbital plane of the primaries. We assume that the spin axis of the less massive primary is directed along the z-axis. The distances between m and the primaries m_1 and m_2 are r_1 and r_2 respectively, and the distance between the primaries is R. The co-ordinates of m_1, m_2 and m are $(x_1, 0)$, $(x_2, 0)$ and (x, y) respectively; and the circum-binary belt has its centre at the origin of the co-ordinate system $oxyz$ (Fig. 1). In the above barycentric co-ordinate system rotating about the z-axis with constant angular velocity n, the kinetic energy of the infinitesimal body *is* given by

$$\begin{aligned} \text{K.E.} &= \tfrac{1}{2}m[n^2(x^2+y^2) + 2n(x\dot{y} - \dot{x}y) + (\dot{x}^2 + \dot{y}^2)] \\ &= T_0 + T_1 + T_2, \end{aligned}$$

where

$$\begin{aligned} T_0 &= \tfrac{1}{2}mn^2(x^2+y^2), \\ T_1 &= mn(x\dot{y} - \dot{x}y), \\ T_2 &= \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2), \end{aligned}$$

and the dots denote differentiations with respect to time t.

Now, since the electromagnetic radiation pressure force F_{p} changes with distance by the same law as the gravitational attraction force F_{g} and acts opposite to it, the resulting force on the particle is

$$F = F_{\text{g}} - F_{\text{p}} = F_{\text{g}}\left(1 - \frac{F_{\text{p}}}{F_{\text{g}}}\right) = qF_{\text{g}},$$

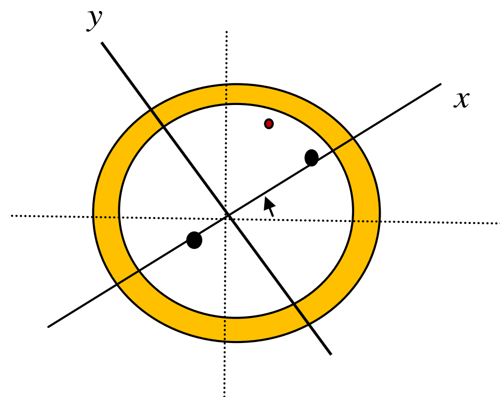

Figure 1. The planar view of the problem.

where $q = \left(1 - \frac{F_p}{F_g}\right)$, a constant for a given particle, which is the mass reduction factor. We denote the electromagnetic radiation factor for the more massive as $q_1 = 1 - p_1, 0 < p_1 = \frac{F_{p_1}}{F_{g_1}} \ll 1$.

In free space the gravitational potential exterior to an oblate body can be expanded in terms of Legendré polynomials (Peter & Lissauer 2001) as

$$V_o(r_o, \phi, \theta) = -\frac{Gm_o}{r_o}\left[1 - \sum_{n=2}^{\&} J_n P_n(\cos\theta)\left(\frac{R_o}{r_o}\right)^n\right]. \quad (1)$$

Equation (1) is expressed in standard spherical co-ordinates, with ϕ the longitude and θ representing the angle between the body's symmetry axis and the vector to a particle r_o (i.e., the co-latitudes). R_o is the mean radius of the body. The terms $P_n(\cos\theta)$ are the Legendré polynomials, given by

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n.$$

J_n is the gravitational moments and are determined by the body's mass distribution. If the body's mass is distributed symmetrically about its equator, then J_n are zero for odd n and are called zonal harmonic coefficients. We denote the oblateness coefficients for the less massive primary as $B_i, 0 < B_i = J_{2i} R_2^{2i} \ll 1, \quad i = 1, 2$, where J_{2i} are zonal harmonic coefficients and R_2 is the mean radius of m_2.

Then, the potential energy of the infinitesimal body, under the influence of the oblateness of less massive primary, electromagnetic radiation of the more massive primary and the circum-binary belt, can be expressed as

$$V = -Gm\left\{m_1\frac{q_1}{r_1} + m_2\left(\frac{1}{r_2} + \frac{B_1}{2r_2^3} - \frac{3B_2}{8r_2^5}\right) + \frac{M_b}{(r^2 + T^2)^{\frac{1}{2}}}\right\}$$

with

$$r_1^2 = (x - x_1)^2 + y^2, \quad r_2^2 = (x - x_2)^2 + y^2.$$

G is the gravitational constant, $\frac{M_b}{(r^2+T^2)^{\frac{1}{2}}}$ is the potential due to the belt (Miyamoto & Nagai 1975) where M_b is the total mass of the belt, r is the radial distance of the infinitesimal body and is given by $r^2 = x^2 + y^2$, $T = a + b$, a and b are parameters which determine the density profile of the belt. The parameter a controls the flatness of the profile and is known as the *flatness parameter*. The parameter b controls the size of the core of the density profile and is called the *core parameter*. When $a = b = 0$, the potential equals to one by a point mass.

The Lagrangian of the problem can be written as

$$L = T_1 + T_2 - U \quad \text{with} \quad U = V - T_0.$$

Thus, the equations of motion of the infinitesimal body are

$$\ddot{x} - 2n\dot{y} = -\frac{1}{m}\frac{\partial U}{\partial x}, \quad \ddot{y} + 2n\dot{x} = -\frac{1}{m}\frac{\partial U}{\partial y}. \tag{2}$$

Now, we choose units for the mass and length such that the sum of the masses of the primaries and their separation distance are unity. The unit of time is so chosen that the gravitational constant is unity. Hence, $m_1 = 1 - \mu$, $m_2 = \mu$ where $0 < \mu = \frac{m_2}{m_1+m_2} \leq \frac{1}{2}$ is the mass ratio. It implies that the co-ordinates of m_1 and m_2 are $(-\mu, 0)$ and $(1 - \mu, 0)$ respectively. Thus, in the dimensionless synodic co-ordinate system, the equations of motion (2) reduce to

$$\ddot{x} - 2n\dot{y} = \Omega_x, \quad \ddot{y} + 2n\dot{x} = \Omega_y \tag{3}$$

with

$$\Omega = \frac{n^2}{2}[(1-\mu)r_1^2 + \mu r_2^2] + \frac{(1-\mu)q_1}{r_1} + \mu\left(\frac{1}{r_2} + \frac{B_1}{2r_2^3} - \frac{3B_2}{8r_2^5}\right) + \frac{M_b}{(r^2+T^2)^{\frac{1}{2}}},$$

$$r_1^2 = (x+\mu)^2 + y^2, \quad r_2^2 = (x+\mu-1)^2 + y^2, \tag{4}$$

and n *is* the mean motion, given by Singh and Taura (2013) and Peter and Lissauer (2001) as

$$n^2 = 1 + \frac{3}{2}\left(B_1 - \frac{5}{4}B_2\right) + \frac{2M_b r_c}{(r_c^2+T^2)^{\frac{3}{2}}}, \tag{5}$$

r_c is the radial distance of the infinitesimal body in the classical restricted three-body problem.

These are equations of motion of the infinitesimal body for the problem under consideration. For simplicity, we set $T = 0.01$ for all numerical investigations.

Now, using equation (3) we obtain the energy integral of the problem as

$$\dot{x}^2 + \dot{y}^2 = 2\Omega(x, y) - C, \tag{6}$$

where constant C is known as Jacobi's constant. The equation of zero velocity surfaces is given by

$$2\Omega(x, y) = C,$$

that is,

$$n^2(x^2+y^2)+\frac{2(1-\mu)q_1}{r_1}+\frac{2\mu}{r_2}+\frac{\mu B_1}{r_2^3}-\frac{3\mu B_2}{4r_2^5}+\frac{2M_b}{(r^2+T^2)^{\frac{1}{2}}}=C. \tag{7}$$

We observe that with large values of x and y, the curves of (7) approximate very close to a circle $x^2+y^2=\left(\frac{\sqrt{C-\varepsilon}}{n}\right)^2$ of radius $\frac{\sqrt{C-\varepsilon}}{n}$, where $\varepsilon=\frac{2(1-\mu)}{r_1}+\frac{2\mu}{r_2}+\frac{\mu B_1}{r_2^3}-\frac{3\mu B_2}{4r_2^5}+\frac{2M_{\mathrm{b}}}{(r^2+T^2)^{\frac{1}{2}}}$ is very small.

If the values of x and y are very small, then the curves of (7) approximate closely to the following equipotential surface:

$$\frac{2(1-\mu)}{\sqrt{(x+\mu)^2+y^2}}+\frac{2\mu}{\sqrt{(x+\mu-1)^2+y^2}}+\frac{\mu B_1}{((x+\mu-1)^2+y^2)^{\frac{3}{2}}}$$
$$-\frac{3\mu B_2}{4((x+\mu-1)^2+y^2)^{\frac{3}{2}}}+\frac{2M_{\mathrm{b}}}{(x^2+y^2+T^2)^{\frac{1}{2}}}=C-(x^2+y^2)=C-\varepsilon.$$

The zero velocity surfaces are presented in Figures 2(a)–(e), in which the two singular regions are shown by conic shapes.

3. Locations of equilibrium and triangular points

3.1 *Locations of equilibrium points*

The equilibrium points are the solutions of the equations

$$\Omega_x=0,\quad \Omega_y=0,$$

which yield

$$n^2x-\frac{(1-\mu)(x+\mu)q_1}{r_1^3}-\frac{\mu(x+\mu-1)}{r_2^3}-\frac{3\mu(x+\mu-1)B_1}{2r_2^5}$$
$$+\frac{15\mu(x+\mu-1)B_2}{8r_2^7}-\frac{M_{\mathrm{b}}x}{(r^2+T^2)^{\frac{3}{2}}}=0,$$
$$n^2y-\frac{(1-\mu)q_1y}{r_1^3}-\frac{\mu y}{r_2^3}-\frac{3\mu B_1 y}{2r_2^5}+\frac{15\mu B_2 y}{8r_2^7}-\frac{M_{\mathrm{b}}y}{(r^2+T^2)^{\frac{3}{2}}}=0.$$

Re-writing these equations, we have

$$x\left(n^2-\frac{(1-\mu)q_1}{r_1^3}-\frac{\mu}{r_2^3}-\frac{3}{2}\frac{\mu B_1}{r_2^5}+\frac{15}{8}\frac{\mu B_2}{r_2^7}-\frac{M_{\mathrm{b}}}{\left(r^2+T^2\right)^{\frac{3}{2}}}\right)$$
$$-\frac{(1-\mu)\mu q_1}{r_1^3}-\frac{\mu(\mu-1)}{r_2^3}-\frac{3\mu(\mu-1)B_1}{2r_2^5}+\frac{15\mu(\mu-1)B_2}{8r_2^7}=0, \tag{8}$$

$$y\left(n^2-\frac{(1-\mu)q_1}{r_1^3}-\frac{\mu}{r_2^3}-\frac{3}{2}\frac{\mu B_1}{r_2^5}+\frac{15}{8}\frac{\mu B_2}{r_2^7}-\frac{M_{\mathrm{b}}}{\left(r^2+T^2\right)^{\frac{3}{2}}}\right)=0. \tag{9}$$

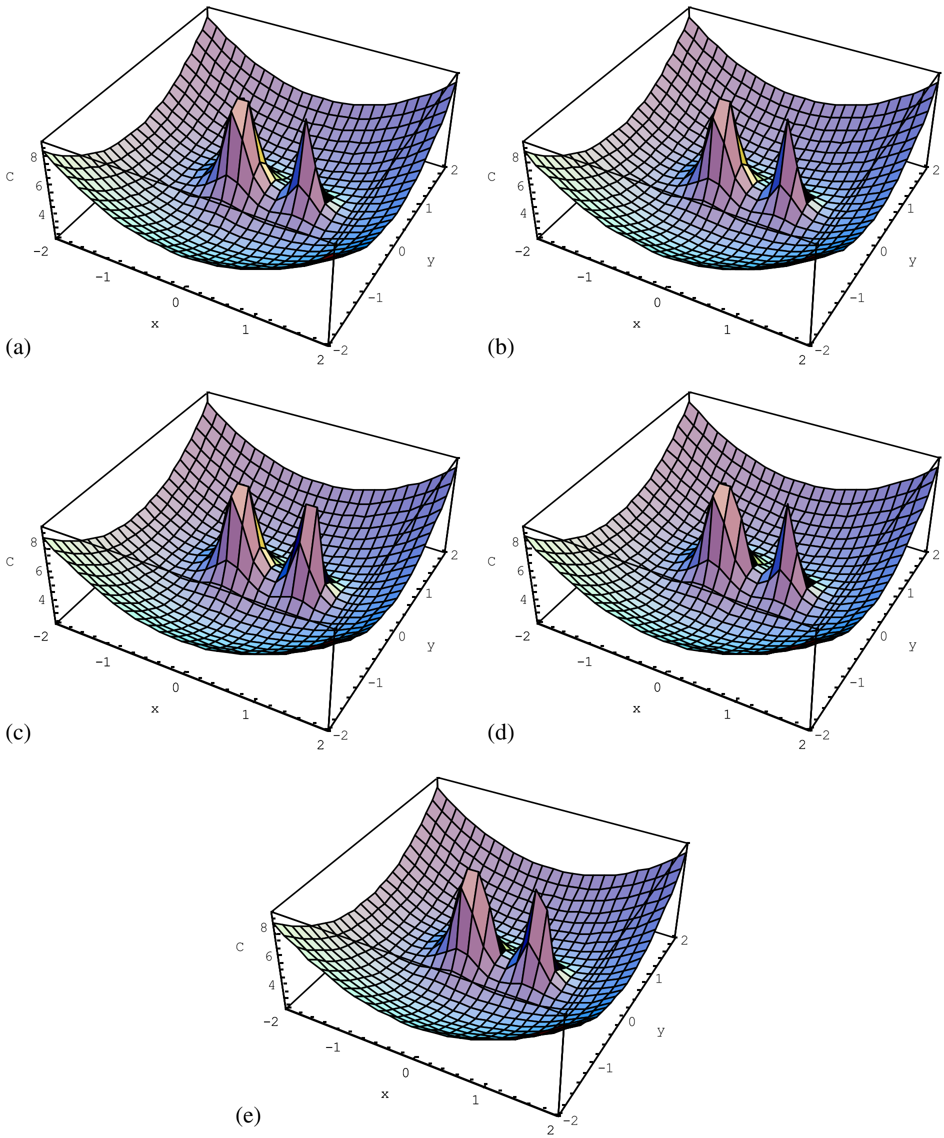

Figure 2. (**a**) $\mu = 0.3$, $q_1 = 1$, $B_1 = B_2 = M_b = 0$, (**b**) $\mu = 0.3$, $q_1 = 0.95$; $B_1 = B_2 = M_b = 0$, (**c**) $\mu = 0.3$, $q_1 = 1$, $B_1 = 0.03$, $B_2 = 0$, $M_b = 0$, (**d**) $\mu = 0.3$, $q_1 = 1$, $B_1 = B_2 = 0$, $M_b = 0.01$, (**e**) $\mu = 0.3$, $q_1 = 1$, $B_1 = 0.03$, $B_2 = 0.02$, $M_b = 0.01$.

3.2 *Locations of triangular points*

The triangular points are the solutions of equations (8) and (9) when $y \neq 0$. Hence, we obtain

$$n^2 - \frac{q_1}{r_1^3} - \frac{M_b}{(r^2 + T^2)^{\frac{3}{2}}} = 0, \tag{10}$$

$$n^2 - \frac{1}{r_2^3} - \frac{3B_1}{2r_2^5} + \frac{15B_2}{8r_2^7} - \frac{M_b}{(r^2+T^2)^{\frac{3}{2}}} = 0. \tag{11}$$

When the effects of electromagnetic radiation on the more massive primary, oblateness of the less massive primary and the potential from the belt are neglected, these equations are reduced to the classical case with solutions $r_1 = r_2 = 1$. Thus, we can assume the solutions of equations (10) and (11) to be

$$r_1 = 1 + \varepsilon_1, \qquad r_2 = 1 + \varepsilon_2, \tag{12}$$

where ε_1 , ε_2 are very small quantities.

Now, using equation (5) and substituting $q_1 = 1 - p_1$ in equations (10) and (11) and restricting ourselves to the linear terms in ε_i, B_i, p_1, M_b and neglecting their products, we obtain

$$\varepsilon_1 = -\frac{p_1}{3} - \frac{B_1}{2} + \frac{5B_2}{8} - \frac{M_b(2r_c-1)}{3(r_c^2+T^2)^{\frac{3}{2}}},$$
$$\varepsilon_2 = -\frac{M_b(2r_c-1)}{3(r_c^2+T^2)^{\frac{3}{2}}}.$$

Substituting these values in equation (12), we have

$$r_1 = 1 - \left(\frac{p_1}{3} + \frac{B_1}{2} - \frac{5B_2}{8} + \frac{M_b(2r_c-1)}{3(r_c^2+T^2)^{\frac{3}{2}}}\right),$$
$$r_2 = 1 - \frac{M_b\,(2r_c-1)}{3(r_c^2+T^2)^{\frac{3}{2}}}, \tag{13}$$

where $r_c^2 = 1 - \mu + \mu^2$.

Substituting (13) in (4) and then solving them, we obtain the triangular equilibrium points L_4 and L_5 as

$$x = \frac{1}{2} - \mu - \left(\frac{p_1}{3} + \frac{B_1}{2} - \frac{5B_2}{8}\right),$$
$$y = \pm\sqrt{3}\left(\frac{1}{2} - \frac{p_1}{9} - \frac{B_1}{6} + \frac{5B_2}{24} - \frac{2M_b(2r_c-1)}{9(r_c^2+T^2)^{\frac{3}{2}}}\right). \tag{14}$$

In order to examine the effects of various perturbations on the positions of triangular points $L_{4(5)}$ (Table 1), we have used equation (14) in the following cases:

(1) Absence of electromagnetic radiation, oblateness and potential from the belt (classical case).
(2) Oblateness of the less massive primary up to J_2 only.
(3) Oblateness of the less massive primary up to J_4 only.
(4) Electromagnetic radiation of the more massive primary only.
(5) Potential from the belt only.
(6) Electromagnetic radiation of the more massive primary, oblateness of the less massive primary up to J_4 and potential from the belt.

Table 1. Effects of various perturbations on the positions of the triangular points $L_{4,5}$ ($\mu = 0.04$ and $T = 0.01$).

Case	q_1	B_1	B_2	M_b	$L_{4,5}(x, y)$
1	1	0	0	0	0.46000, ± 0.86603
2	1	0.03	0	0	0.44500, ± 0.85737
	1	0.04	0	0	0.44000, ± 0.85448
	1	0.05	0	0	0.43500, ± 0.85159
3	1	0.03	0.01	0	0.45125, ± 0.86097
	1	0.04	0.02	0	0.45250, ± 0.86170
	1	0.05	0.03	0	0.45375, ± 0.86242
4	0.90	0	0	0	0.42667, ± 0.84678
	0.85	0	0	0	0.41000, ± 0.83716
	0.80	0	0	0	0.39333, ± 0.82754
5	1	0	0	0.01	0.46000, ± 0.86210
	1	0	0	0.02	0.46000, ± 0.85818
	1	0	0	0.03	0.46000, ± 0.85426
6	0.90	0.03	0.01	0.01	0.41792, ± 0.83781
	0.85	0.04	0.02	0.02	0.40250, ± 0.82498
	0.80	0.05	0.03	0.03	0.38708, ± 0.81216

4. Stability of equilibrium and triangular points

4.1 *Stability of equilibrium points*

In order to study the stability of an equilibrium point (x_0, y_0), we write $x = x_0 + \alpha$ and $y = y_0 + \beta$, where α and β are small displacements. Substituting these values in (3), we obtain the variational equations of motion as

$$\ddot{\alpha} - 2n\dot{\beta} = \Omega^0_{xx}\alpha + \Omega^0_{xy}\beta,$$
$$\ddot{\beta} + 2n\dot{\alpha} = \Omega^0_{yx}\alpha + \Omega^0_{yy}\beta. \tag{15}$$

Here, only linear terms in α and β have been taken. The second partial derivatives of Ω are denoted by subscripts. The superscript 0 indicates that the derivatives are to be evaluated at the equilibrium point (x_0, y_0).

The characteristic equation corresponding to (16) is

$$\lambda^4 + (4n^2 - \Omega^0_{xx} - \Omega^0_{yy})\lambda^2 + \Omega^0_{xx}\Omega^0_{yy} - \Omega^{0\,2}_{xy} = 0. \tag{16}$$

4.2 *Stability of triangular points*

At the triangular points (eq. (14)), we have

$$\Omega^0_{xx} = \frac{3}{4} + a_1 + \mu b_1 + \frac{5M_b\,(2r_c - 1)}{4(r_c^2 + T^2)^{\frac{3}{2}}} + \frac{3M_b(\frac{1}{4} - \mu + \mu^2)}{(r_c^2 + T^2)^{\frac{5}{2}}},$$

$$\Omega^0_{yy} = \frac{9}{4} + a_2 + \mu b_2 + \frac{7M_b\,(2r_c - 1)}{4(r_c^2 + T^2)^{\frac{3}{2}}} + \frac{3M_b\left(\frac{3}{4}\right)}{(r_c^2 + T^2)^{\frac{5}{2}}},$$

$$\Omega^0_{xy} = \sqrt{3}\left\{\frac{3}{4} + a_3 + \mu\left(b_3 - \tfrac{3}{2} - \frac{11M_\mathrm{b}\,(2r_\mathrm{c}-1)}{6(r_\mathrm{c}^2+T^2)^{\frac{3}{2}}}\right)\right.$$
$$\left. + \frac{11M_\mathrm{b}\,(2r_\mathrm{c}-1)}{12(r_\mathrm{c}^2+T^2)^{\frac{3}{2}}} + \frac{\frac{3}{2}M_\mathrm{b}\left(\frac{1}{2}-\mu\right)}{(r_\mathrm{c}^2+T^2)^{\frac{5}{2}}}\right\},$$

where

$$a_1 = -\frac{p_1}{2} + \frac{3B_1}{8} - \frac{15B_2}{32},\quad b_1 = \frac{3p_1}{2} + 3B_1 - \frac{75B_2}{16},\quad a_2 = \frac{p_1}{2} + \frac{33B_1}{8} - \frac{165B_2}{32},$$
$$b_2 = -\frac{3p_1}{2} - \frac{45B_2}{16},\quad a_3 = -\frac{p_1}{6} + \frac{7B_1}{8} - \frac{35B_2}{32},\quad b_3 = -\frac{p_1}{6} - \frac{13B_1}{4} + 5B_2.$$

Here each of $|a_i|$, $|b_i|$ $(i = 1, 2, 3)$ is very small as $|B_v| \ll 1$, $|p_1| \ll 1$ $(v = 1, 2)$.

Substituting these values in equation (16), the characteristic equation becomes

$$\lambda^4 + b\lambda^2 + c = 0 \tag{17}$$

with

$$b = 1 + 6\left(B_1 - \frac{5B_2}{4}\right) - (a_1 + a_2) - \mu\,(b_1 + b_2) + \frac{M_\mathrm{b}\,(2r_\mathrm{c}+3)}{(r_\mathrm{c}^2+T^2)^{\frac{3}{2}}} - \frac{3M_\mathrm{b}r_\mathrm{c}^2}{(r_\mathrm{c}^2+T^2)^{\frac{5}{2}}},$$

$$c = \left(-\frac{27}{4} + 9b_3 - \frac{33M_\mathrm{b}\,(2r_\mathrm{c}-1)}{2(r_\mathrm{c}^2+T^2)^{\frac{3}{2}}} - \frac{27M_\mathrm{b}}{4(r_\mathrm{c}^2+T^2)^{\frac{5}{2}}}\right)\mu^2$$
$$+ \left(\frac{27}{4} + \frac{9b_1}{4} + \frac{3b_2}{4} - \frac{9b_3}{2} + 9a_3 + \frac{33M_\mathrm{b}\,(2r_\mathrm{c}-1)}{2(r_\mathrm{c}^2+T^2)^{\frac{3}{2}}} + \frac{27M_\mathrm{b}}{4(r_\mathrm{c}^2+T^2)^{\frac{5}{2}}}\right)\mu$$
$$+ \frac{9a_1}{4} + \frac{3a_2}{4} - \frac{9a_3}{2}.$$

Its roots are

$$\lambda^2 = \frac{-b \pm \sqrt{\Delta}}{2}, \tag{18}$$

where $\Delta = b^2 - 4c$ is the discriminant,

$$\Delta = \left(27 - 36b_3 + \frac{66M_\mathrm{b}\,(2r_\mathrm{c}-1)}{(r_\mathrm{c}^2+T^2)^{\frac{3}{2}}} + \frac{27M_\mathrm{b}}{(r_\mathrm{c}^2+T^2)^{\frac{5}{2}}}\right)\mu^2$$
$$- \left(27 + 11b_1 + 5b_2 - 18b_3 + 36a_3 + \frac{66M_\mathrm{b}\,(2r_\mathrm{c}-1)}{(r_\mathrm{c}^2+T^2)^{\frac{3}{2}}} + \frac{27M_\mathrm{b}}{(r_\mathrm{c}^2+T^2)^{\frac{5}{2}}}\right)\mu$$
$$+ 1 + 12\left(B_1 - \frac{5B_2}{4}\right) - 11a_1 - 5a_2 + 18a_3 + \frac{2M_\mathrm{b}\,(2r_\mathrm{c}+3)}{(r_\mathrm{c}^2+T^2)^{\frac{3}{2}}} - \frac{6M_\mathrm{b}r_\mathrm{c}^2}{(r_\mathrm{c}^2+T^2)^{\frac{5}{2}}}. \tag{19}$$

Now, Δ is a strictly decreasing function of μ in the interval (0,1/2) and it has values of opposite signs at the end points, hence there isonly one value of μ, say μ_c in the interval (0, 1/2) for which the discriminant vanishes. μ_c is called the critical mass parameter and it is given by

$$\mu_c = \mu_0 - \frac{2p_1}{27\sqrt{69}} + \frac{1}{9}\left(1 - \frac{13}{\sqrt{69}}\right)B_1 - \frac{5}{18}\left(1 - \frac{25}{2\sqrt{69}}\right)B_2 + \left[\frac{(76-8r_c)\left(r_c^2+T^2\right)}{27\sqrt{69}} - \frac{1+6r_c^2}{3\sqrt{69}}\right]\frac{M_b}{(r_c^2+T^2)^{\frac{5}{2}}}, \tag{20}$$

where

$$\mu_0 = \frac{1}{2}\left(1 - \sqrt{\frac{23}{27}}\right).$$

The value of the critical mass parameter to fifteen decimal places (T = 0.01, r_c = 0.9999) is

$$\mu_c = 0.038520896504551 - 0.008917470598946P_1 - 0.062779565568333B_1 + 0.140228656547810B_2 + 0.022320657882394M_b \tag{21}$$

Now, since $b > 0$, $\Delta > 0$ in the interval $0 < \mu < \mu_c$, it follows that the roots (18) are distinct pure imaginary numbers. Hence, the triangular point is stable in this region. In $\mu_c < \mu < 1/2$, $\Delta < 0$ the real parts of two of the roots (equation (18)) are positive. Therefore, the triangular point is unstable. If $\mu = \mu_c$, $\Delta = 0$ the roots (equation (18)) are double roots, which gives instability of the point. Hence, the triangular points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \le \mu \le \frac{1}{2}$, where the critical mass parameter μ_c depends on the combined effect of the electromagnetic radiation of the more massive primary, oblateness of the less massive primary and gravitational potential from the belt.

5. Discussion

The system (3) represents the equations of motion of the infinitesimal body; and they are different from those obtained by Kushvah (2008) due to the presence of oblateness up to J_4 of the less massive primary. If the potential from the belt and the oblateness coefficient B_2 of the less massive primary are ignored (i.e. $M_b = 0$, $B_2 = 0$), they become analogous to those of Singh and Ishwar (1999) in the absence of the electromagnetic radiation of the less massive primary and oblateness of more massive primary. In the presence of the potential from the belt only (i.e. $q_1 = 1$, $B_1 = B_2 = 0$), the equations of (3) are in agreement with those of Jiang and Yeh (2006) and, Yeh and Jiang (2006).

Equation (14) gives the co-ordinates of the triangular equilibrium points. If the oblateness coefficient B_2 of the less massive primary is neglected (i.e. $B_2 = 0$), these co-ordinates fully agree with those of Kushvah (2008) when only linear terms in small quantities are considered. In the absence of the oblateness coefficient B_2

of the less massive primary (i.e. $B_2 = 0$), the equations of (14) agree with Singh and Taura (2013) when the more massive primary is spherical and the less massive primary is non-radiating. On ignoring the potential from the belt (i.e. $M_b = 0$), the co-ordinates (14) coincide with those of Singh and Ishwar (1999) in the absence of the electromagnetic radiation of the less massive primary and oblateness of the more massive primary.

Table 1 shows that the positions of the triangular equilibrium points are affected by the oblateness of the less massive primary, electromagnetic radiation of the more massive primary and the potential from the belt. If the oblateness of the less massive primary is considered up to J_4 (i.e. $q_1 = 1$, $M_b = 0$), the triangular points move away from the line joining the primaries. However, in the presence of the oblateness up to J_2 of the less massive primary (i.e. $q_1 = 1$, $B_4 = M_b = 0$) or if the electromagnetic radiation of the more massive primary is considered (i.e. $B_1 = B_2 = M_b = 0$), or considering the potential from the belt (i.e. $q_1 = 1$, $B_1 = B_2 = 0$); the triangular points tend to move towards the line joining the primaries. In the presence of all the perturbations, the triangular points come closer to the line joining the primaries.

Equation (20) shows the resultant effect of oblateness of the less massive primary, electromagnetic radiation of the more massive primary and potential from the belt on the critical mass value μ_c. However, in the absence of these perturbations (i.e. $q_1 = 1$, $B_1 = B_2 = M_b = 0$), μ_c becomes μ_0 which corresponds to the classical CR3BP (Szebehely 1967). If the electromagnetic radiation of the more massive primary and oblatenesss up to J_2 of the less massive primary are considered (i.e. $B_2 = M_b = 0$), μ_c confirms the result of Sharma (1987).

Equation (21) indicates clearly that the oblateness up to J_2 of the less massive primary and the electromagnetic radiation of the more massive primary, reduce the range of stability, while the coefficient B_2 and the potential from the belt increase the range of stability.

6. Conclusion

We have studied the motion of an infinitesimal body under the influence of the electromagnetic radiation of the more massive primary and the oblateness up to J_4 of the less massive primary; together with the gravitational potential from the belt. We have found the equations that govern the motion of the infinitesimal body and the positions of the triangular equilibrium points. The equations are affected by these perturbations. Numerical investigations reveal that the triangular equilibrium points come closer to the line joining the primaries in the presence of any of the perturbations, except when considering the oblateness up to J_4. The triangular points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \leq \mu \leq \frac{1}{2}$, where μ_c is the critical mass parameter influenced by the potential from the belt; and the oblateness and electromagnetic radiation of the less massive primary and more massive primary respectively. The oblateness up to J_2 of the less massive primary and the electromagnetic radiation of the more massive primary have destabilizing tendencies, while coefficient J_4 (i.e. B_2) and the potential from the belt have stabilizing tendencies. Further studies can consider how our results depend on the combined effects of these parameters and on the spatial orientation of the spin axes of the bodies.

References

Abouelmagd, E. I. 2012, *Astrophys. Space Sci.*, **342**, 45–53.

Abouelmagd, E. I. 2013, *Astrophys. Space Sci.*, **346**, 51–69.

Anderson, J. D., Jacobson, R. A., McElrath, T. P., Moore, W. B., Schubert, G., Thomas, P. C. 2001, *Icarus*, **153**, 157.

Brumberg, V. A. 1991, *Essential relativistic celestial mechanics*, Adam Hilger.

Bruno, A. D. 1994, *The restricted 3-body problem: Plane periodic orbits*, Walter de Gruyter.

Chenciner, A. 2007, *Scholarpedia*, **2(10)**, 2111.

Geroch, R. 1970, *J. Math. Phys.*, **11**, 2580.

Gutzwiller, M. 1998, *Rev. Modern Phys.*, **70(2)**.

Hansen, R. O. 1974, *J. Math. Phys.*, **15**, 46.

Heimberger, J., Soffel, M., Ruder, H. 1990, *Celest*, *Mech. Dyn. Astron.*, **47(2)**, 205–217.

Iess, L., Rappaport, N. J., Jacobson, R. A., Racioppa, P., Stevenson, D. J., Tortora, P., Armstrong, J. W., Asmar, S. W. 2010, *Science*, **327**, 1367.

Iorio, L. 2005, *A&A*, **433(1)**, 385–39.

Iorio, L. 2007, *MNRAS*, **375(4)**, 1311–1314.

Iorio, L. 2009, *Space Sci. Rev.*, **148(1–4)**, 363–381.

Iorio, L. 2011, *Phys. Rev. D*, **84(12)**. id. 124001.

Iorio, L. 2012, *Earth Moon Planet.*, **108(3–4)**, 189–217.

Iorio, L. 2013, *Class. Quantum Gravity* **30**(19), article id 195011.

Iorio, L., Lichtenegger, H. I. M., Ruggiero, M. L, Corda, C. 2011, *Astrophys. Space Sci.*, **331(2)**, 351–395.

Iorio, L., Ruggiero, M. L., Corda, C. 2013, *Acta Astronautica*, **91**, 141–148.

Jiang, I. G., Yeh, L. C. 2003, *Int. J. Bifurcation Chaos*, **13**, 534–539.

Jiang, I. G., Yeh, L. C. 2004a, *Rev. Mex de A A*, **21**, 152–155.

Jiang, I. G., Yeh, L. C 2004b, *Astron. J.*, **128**, 923–932.

Jiang, I. G., Yeh, L. C. 2006, *Astrophys. Space Sci.*, **305**, 341–348.

Kushvah, B. S. 2008, *Astrophys. Space Sci.*, **318**, 41–50.

Luk'yanov, L. G. 1989, *Soviet Astr. (Tr: A. Zhurn.)*, **33**, 92–95.

Moulton, F. R. 1914, *An Introduction to Celestial Mechanics*, 2nd ed., New York: Dover.

Miyamoto, M., Nagai, R. 1975, *Publ*, *Astron. Soc. Jpn.*, **27**, 533–543.

Peter, I. D., Lissauer, J. J. 2001, *Planetary Science*, Cambridge University Press, New York.

Radzievskii, V. V. 1950, *Astron. J. Zh*, **27**, 250–256.

Radzievskii, V. V. 1953, *Astron. Zh.*, **30**, 225.

Renzetti, G. 2012a, *Earth, Moon Planet*, **109(1–4)**, 55–59.

Renzetti, G. 2012b, *Can. J. Phys.*, **90(9)**, 883–888.

Renzetti, G. 2013a, *Centr. Eur. J. Phys.*, **11(5)**, 531–544.

Renzetti, G. 2013b, *J. Astrophys. Astr.*, **34**, 341–348.

Rozelot, J. P., Damiani, C. 2011, *Eur. Phys. J. H*, **36**, 407.

Rozelot, J. P., Fazel, Z. 2013, *Solar Phys*, **287(1–2)**, 161–170.

Rozelot, J., Pireaux, S., Lefebvre, S., Corbard, T. 2004, arXiv:astro-ph/0403382.

Sharma, R. K. 1987, *Astrophys. Space Sci.*, **135**, 271–281.

Singh, J., Ishwar, B. 1999, *Bull. Astron. Soc. India*, **27**, 415–424.

Singh, J., Taura, J. J. 2013, *Astrophys. Space Sci.*, **343**, 95–106.

Soffel, M., Wirrer, R., Schastok, J., Ruder, H., Schneider, M. 1988, *Celest. Mech.*, **42(1–4)**, 81–89.

Szebehely, V. 1967, *Theory of Orbits: The Restricted Problem of Three Bodies*, Academic Press, San Diego.

Valtonen, M., Karttunen, H. 2006, *The Three-Body Problem*, Cambridge University Press.

Yan, J., Zhong, Z., Li, F., Dohm, J. M., Ping, J., Cao, J., Li, X. 2013, *Adv. Space Res.*, **52**, 512.

Yeh, L. C., Jiang, I. G. 2006, *Astrophys. Space Sci.*, **306**, 189–200.