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On the Use of Roche Equipotentials in Analysing the Problems of Binary and Rotating Stars

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Abstract. Kopal (*Adv. Astron. Astrophys.*, **9**, 1, 1972) introduced the concept of Roche equipotentials to analyse the effects of rotational and tidal distortions in case of stars in binary systems. In this approach a mathematical expression for the potential of a star in a binary system is obtained by approximating its inner structure with Roche model. This expression for the potential has been used in subsequent analysis by various authors to analyse the problems of structures and oscillations of synchronous and nonsynchronus binary stars as well as single rotating stars. Occasionally, doubts have been expressed regarding the validity of the use of this approach for analysing nonsynchronous binaries and rotationally and tidally distorted single stars. In this paper we have tried to clarify these doubts.

Key words. Methods: analytic—binary: close—stars: rotation.

1. Introduction

The mathematical problem of determining the effects of rotation and tidal forces on the equilibrium structures of rotating stars and stars in binary systems is quite complex. Therefore, attempts have been made to carry out such studies in some approximate ways. Kopal (1972) introduced the concept of Roche equipotentials to analyse the problems of binary stars. In this approach an expression is obtained for the potential of the primary component of a binary system to account for the effects of rotational and tidal forces by assuming Roche model (model in which the total mass of the star is assumed to be concentrated at its centre, and this point mass is surrounded by an evanescent envelope in which density varies as the square of the distance from its centre) for its inner structure. Subsequently, Mohan & Saxena (1983, 1985) used Kopal's concept of Roche equipotentials in conjunction with the averaging technique of Kippenhahn & Thomas (1970) to incorporate the rotational and tidal effects in the equations of equilibrium structure and equations governing the small adiabatic pseudo-radial and nonradial oscillations of rotating stars and stars in the binary systems. The method has been subsequently used by Mohan, Saxena,

& Agarwal (1990, 1991), Mohan, Lal & Singh (1992, 1998), Lal, Mohan & Singh (2006), Lal *et al*. (2006) and Lal, Pathania & Mohan (2009) to compute the equilibrium structures and periods of oscillations of certain rotationally and/or tidally distorted stellar models. Occasionally comments have been made casting doubts on the validity of the use of this approach in analysing the problems of nonsynchronous binaries and single stars distorted by the tidal effects alone. The objective of this paper is to clarify these points.

2. Roche equipotentials of a binary system

Following Kopal (1972), let M_0 and M_1 be the masses of the two components of a close binary system separated by a distance D. The primary component of mass M_0 of this binary system is supposed to be much more massive than its companion star of mass M_1 which is regarded as a point mass ($M_0 \gg M_1$). Suppose that the position of the two components of such a binary system is referred to a rectangular system of cartesian coordinates with origin at the center of gravity of mass M_0, then x-axis will be along the line joining the mass centers of the two components and z-axis will be perpendicular to the plane of the orbit of the two components (Fig. 1).

In this figure $r_0 = \sqrt{x^2 + y^2 + z^2}$, $r_1 = \sqrt{(D-x)^2 + y^2 + z^2}$ and $r = \sqrt{(x-d_1)^2 + y^2 + z^2}$ represent the distances of a point $P(x, y, z)$ from the centers of gravity of the primary star with center at O, secondary star with center at O_1 and center of gravity C ((d_1, 0, 0) where $d_1 = M_1 D/(M_0 + M_1)$) of the system, respectively. Also Ω_1 represents the angular velocity of rotation of the primary component about the z-axis passing through center O of the primary component whereas Ω denotes the angular velocity of revolution of the system about an axis parallel to the z-axis and perpendicular to the xy-plane that passes through the center of gravity C of the system.

Following Kopal (1972), for a system as described above the total potential at point $P(x, y, z)$ due to gravitational and rotational forces is given as

$$\psi = \frac{GM_0}{r_0} + \frac{GM_1}{r_1} + \frac{1}{2}\Omega^2\left[\left(x - \frac{M_1 D}{M_0 + M_1}\right)^2 + y^2\right]. \quad (1)$$

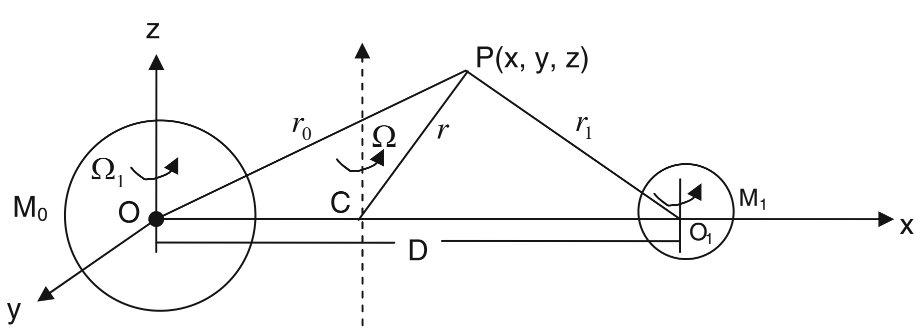

Figure 1. Axis of reference for a binary system.

On assuming Roche model for the primary component, equation (1) holds everywhere including the points in the interior of the primary component. The approximation of the inner structure of the star by the Roche model considerably simplifies subsequent analysis and this approximation is reasonably valid for a majority of real stars (cf. Chandrasekhar (1939)). In principle, Ω_1 should be equal to Ω otherwise the bulge would keep on shifting with time. However, since we assume that Ω_1 and Ω are both small, this shifting problem when $\Omega_1 \neq \Omega$ is ignored. Moreover since in our analysis we use Roche equipotentials, these average out this effect. In non-dimensional form the above equation can be written as

$$\psi_1 = \frac{1}{r_0^*} + \frac{q}{r_1^*} + \frac{1}{2}\omega^2 \left[x_1^2 + y_1^2 - 2\,x_1 \frac{q}{1+q} + \left(\frac{q}{1+q}\right)^2 \right], \tag{2}$$

where

$$\psi_1 = \frac{D\psi}{GM_0}, \quad r_0^* = \frac{r_0}{D}, \quad r_1^* = \frac{r_1}{D}, \quad \omega^2 = \frac{\Omega^2 D^3}{G\,M_0}, \quad x_1 = \frac{x}{D},$$
$$y_1 = \frac{y}{D} \quad \text{and} \quad q = \frac{M_1}{M_0}. \tag{3}$$

Here G is the gravitational constant and q is the tidal distortion parameter that accounts for the effect due to tidal distortions caused by the companion star. Equation (1) (or (2)) represents the general expression for the total potential of a rotationally and tidally distorted star in a binary system under the assumptions made earlier.

In the case of synchronous binaries it is assumed that the angular velocity of rotation of the primary component of the binary system is the same as the angular velocity of revolution of the system, that is,

$$\Omega_1 = \Omega = \Omega_{\mathrm{k}}, \tag{4}$$

where $\Omega_{\mathrm{k}}^2 = G(M_0 + M_1)/D^3$ is the Keplerian angular velocity. Using this value of Keplerian angular velocity, we get a relation between the angular velocity and tidal distortion parameter in nondimensional form as

$$\omega^2 = 1 + q = 2\,n. \tag{5}$$

Here ω^2 is the square of the angular velocity of rotation (as well as revolution) in nondimensional form. Also n is a rotational parameter that accounts for the effect due to rotational distortions in a star. Using (5) in (2) and changing to spherical system of coordinates ($x_1 = r_0^*\lambda,\ y_1 = r_0^*\mu,\ z_1 = r_0^*\nu$, where $\lambda = \sin\theta\,\cos\phi,\ \mu = \sin\theta\sin\varphi$ and $\nu = \cos\theta$), Kopal (1972) finally obtained an expression for the potential at point P in nondimensional form as

$$\psi^* = \frac{1}{r_0^*} + q\left[\frac{1}{\sqrt{1 - 2\lambda r_0^* + r_0^{*2}}} - \lambda r_0^*\right] + n r_0^{*2}(1 - \nu^2), \tag{6}$$

where

$$\psi^* = \psi_1 - \frac{q^2}{2\,(1+q)} = \frac{D\psi}{GM_0} - \frac{M_1^2}{2M_0\,(M_0 + M_1)}.$$

This expression for the potential has been subsequently used by Kopal as well as various authors such as Mohan & Saxena (1983, 1985), Mohan, Saxena & Agarwal (1990, 1991), Mohan, Lal & Singh (1992, 1998), Lal, Mohan & Singh (2006), Lal *et al*. (2006) and Lal, Pathania & Mohan (2009) to determine the effects of rotational and tidal distortions on the equilibrium structures and periods of oscillations of rotationally and/or tidally distorted stellar models.

3. Validity of the approach

Although this approach seems to be reasonably correct in the case of synchronous binaries, occasionally doubts have been raised regarding the validity of this approach in analysing the problems of nonsynchronous binaries and single stars distorted by the effects of rotational and tidal forces. In this section we briefly address these doubts.

In Mohan & Saxena (1983, 1985) and in all the subsequent studies as mentioned earlier in this paper, authors have used equation (6) to study the effects of rotational and tidal forces on the equilibrium structures of rotating stars and stars in binary systems. The authors have also used the same equation to study the effect of rotational distortions by setting tidal distortion parameter $q = 0$ and effects of tidal distortions by setting rotational distortion parameter $n = 0$ in expression (6).

In case of pure rotation ($q = 0$, the effect due to tidal distortions), expression (1) reduces to

$$\psi = \frac{GM_0}{r_0} + \frac{1}{2}\Omega^2[x^2 + y^2] \tag{7}$$

which in nondimensional form becomes

$$\psi^* = \frac{1}{r_0^*} + nr_0^{*2}(1 - \nu^2), \tag{8}$$

where $\psi^* = \psi_1 = D\psi/GM_0$, $\omega^2 = \Omega^2 D^3/GM_0$ and D is the measure of distance. In the present case D can be taken to be the outermost equilibrium radius of the undistorted single star under consideration.

Now, if we put $q = 0$ (as done in Mohan & Saxena (1983, 1985) and subsequent studies) in (6) we get the same expression as (8). This shows that in case of pure rotation of a single star, the expression for Roche equiopotential obtained from (6) by setting $q = 0$ is justified.

For analysing the tidal effects of a small companion star on a given star we set $n = 0$ (effect due to rotational distortions zero, as done in Mohan & Saxena (1983, 1985) and subsequent studies) in (6) to obtain

$$\psi^* = \frac{1}{r_0^*} + \frac{q}{r_1^*} - \lambda r_0^* q. \tag{9}$$

Technically this is not the same as we would have obtained by setting $\Omega = 0$ (effect due to rotational distortions) in (1). This expression yields

$$\psi^* = \frac{1}{r_0^*} + \frac{q}{r_1^*} - \frac{q^2}{2\,(1+q)} \tag{10}$$

that differs from (9). Thus, the Roche equipotential obtained from (1) and (6) are not the same in case of pure tidally distorted stars. That is, for the stars that are distorted only by tidal effects of the companion star (no rotational effects), the expression for Roche equipotential obtained from (6) by setting $n = 0$ is not justified. However, it may be noted that when we talk of the tidal effects of a companion star, we are talking of the tidal effects of a companion star in a binary system in which the effects of rotation (though present) is being ignored. In such cases we are analysing the exclusive effect of tidal forces ignoring the effects of rotational terms. In fact physically we cannot have a stable binary system of stars in the absence of rotation. In the absence of rotation the two stars in a binary system will coalesce into each other sooner than later.

Furthermore, they have also used equation (6) for studying the nonsynchronous binaries. They have assumed that for nonsynchronous binaries $2n \neq q+1$. This condition does not seem to be technically correct as it is not justified in the framework of the assumptions considered in their study because the basic equipotential equation (6) that they have used has been derived for synchronous binaries (Kopal 1972) and not for nonsynchronous binaries. However, as discussed in Pathania & Medupe (2013), using the first approximation theory of Limber (1963) that considers the more general case of nonsynchronous binaries (single rotating stars and synchronous binaries are the particular cases of this theory), the use of the Roche equipotential for the nonsynchronous binaries as that done in Mohan & Saxena (1983, 1985) is justified.

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