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Black Hole Dynamic Potentials

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Abstract. In the following paper, certain black hole dynamic potentials have been developed definitively on the lines of classical thermodynamics. These potentials have been refined in view of the small differences in the equations of the laws of *black hole* dynamics as given by Bekenstein and those of thermodynamics. Nine fundamental black hole dynamical relations have been developed akin to the four fundamental thermodynamic relations of Maxwell. The specific heats $C_{\Omega,\Phi}$ and $C_{J,Q}$ have been defined. For a black hole, these quantities are *negative*. The $\kappa \mathrm{d}A$ equation has been obtained as an application of these fundamental relations. Time reversible processes observing constancy of surface gravity are considered and an equation connecting the internal energy of the black hole *E*, the additional available energy defined as the first free energy function *K*, and the surface gravity κ, has been obtained. Finally as a further application of the fundamental relations, it has been proved for a *homogeneous gravitational field in black hole space times or a de Sitter black hole* that $C_{\Omega,\Phi} - C_{J,Q} = \kappa \left[\left(\frac{\partial J}{\partial \kappa}\right)_{\Omega,\Phi} \left(\frac{\partial \Omega}{\partial \kappa}\right)_{J,Q} + \left(\frac{\partial Q}{\partial \kappa}\right)_{\Omega,\Phi} \left(\frac{\partial \Phi}{\partial \kappa}\right)_{J,Q} \right]$. This is dubbed as the *homogeneous fluid approximation* in context of the black holes.

Key words. Black hole dynamics—black hole thermodynamics—thermodynamic potentials.

The classical theory of relativistic gravity – the general theory of relativity asserts that a gravitationally collapsing star of mass *M* will shrink, in short time as measured by an observer on the surface, to a radius of the order of $2GM/c^2$, known as its *gravitational radius* or *Schwarzschild radius*, at which the gravitational field becomes so strong that no further radiation or anything else can escape to infinity, the region of space time from which it is not possible to escape to infinity is a black hole. The boundary of the black hole, called as the *event horizon*, is an outgoing null *hypersurface* that just fails to reach infinity. The existence of finite entropy (Bekenstein 1973, 1974) as predicted for the black holes implied the possibility of radiating black holes (Hawking 1972, 1974, 1975), in turn proven conclusively by the quantum considerations of a black hole neighborhood. We now have a standard result verified many times (see for example DeWitt (1975) and references therein) that black holes emit radiation, now known as the Hawking radiation. The noteworthy feature of the Hawking radiation tunneling through the event horizon of the black hole is that it possesses an exact thermal spectrum. Consequently, it is observed that the dynamics of black holes runs competitively on the same lines as that of classical thermodynamics. The thermal radiation emitted by a black hole corresponds to a temperature of

$$T_{\mathrm{H}} = \frac{\kappa h}{2\pi k_{\mathrm{B}} c}, \tag{1}$$

where κ is the surface gravity of the black hole given by

$$\kappa = \frac{4\pi \left(r_{+}c^{2} - GM\right)}{A}. \tag{2}$$

This once again implies the premise of finite entropy which is expressed by the Bekenstein–Hawking formula as

$$S_{\mathrm{BH}} = \frac{k_{\mathrm{B}}}{4\, l_{\mathrm{Pl}}^{2}} A, \tag{3}$$

where l_{Pl} is the Planck's length expressed by $l_{\mathrm{Pl}} = \sqrt{\frac{G h}{c^{3}}} \approx 10^{-33}$cm.

In the above and henceforth, A is the area of the event horizon of the black hole which is mathematically defined as

$$A = \frac{4\pi G}{c^{4}} \left(2GM^{2} - Q^{2} + 2\sqrt{G^{2}M^{4} - J^{2}c^{2} - GM^{2}Q^{2}}\right). \tag{4}$$

The quantity r_{+} in eq. (2) is defined by the relation:

$$r_{+} = \frac{1}{c^{2}} \left(GM + \sqrt{G^{2}M^{2} - \frac{J^{2}c^{2}}{M^{2}} - GQ^{2}}\right). \tag{5}$$

The solution of the coupled Einstein–Maxwell field equations

$$R_{ab} - \frac{1}{2} g_{ab} R = -\frac{8\pi G}{c^{4}} T_{ab}, \tag{6}$$

$$F^{ab}_{;b} = 4\pi J^{a} \tag{7}$$

and

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0, \tag{8}$$

subject to the constraints imprinted by mass M, angular momentum J, and charge Q (at radial infinity) and subject to the existence of a physically non-singular horizon describes the *Kerr–Newman geometry* of a black hole with mass M, angular momentum J and charge Q. For a general Kerr–Newman black hole, Ω, the angular frequency of rotation of the hole is given by

$$\Omega = \frac{4\pi J}{MA}, \tag{9}$$

and Φ, the potential of the event horizon is given by

$$\Phi = \frac{4\pi Q r_{+}}{A}. \tag{10}$$

The theoretical foundations of the subject of black hole dynamics were laid by Bekenstein (1973, 1974), Hawking (1972, 1974, 1975), Bardeen *et al.* (1973), and the systematic and complete treatment is given by Straumann (2004). The statistical theory of internal (unobservable) configurations and micro-canonical ensemble in the black hole framework has been developed and discussed in considerable detail by Hawking (1976). As a consequence to the theory of black hole dynamics, the possibility of developing *black hole dynamic potentials*, mutually associating them and developing fundamental relations between the black hole parameters M, J, Q, κ, Ω and Φ, with the use of these potentials becomes inevitable and feasible.

Any Kerr–Newman black hole i.e., a black hole of given mass, angular momentum and charge can have a large number σ of different unobservable internal configurations which reflect the possible different initial configurations of the matter which collapsed to produce the hole. The logarithm of this number σ can be regarded as the entropy of the black hole and is a measure of the amount of information about the initial state, which was lost in the formation of the black hole. This is written mathematically as

$$S_{\mathrm{H}} = \ln\sigma. \tag{11}$$

Bekenstein suggested that the area of the event horizon of the black hole is a measure of its entropy which is what is embodied in eq. (3). Further, Bekenstein was the first to suggest that some multiple of κ should be regarded as representing in some sense, the temperature of the black hole. He noted that energy is conserved for black holes as well as it is for other phenomena that occur in the Universe. This is personified in the first law of black hole dynamics,

$$\mathrm{d}E = \mathrm{d}\left(Mc^2\right) = \frac{\kappa c^2}{8\pi G}\mathrm{d}A + \Omega\,\mathrm{d}J + \Phi\,\mathrm{d}Q \tag{12}$$

which connects the difference in the energy of two nearby black hole equilibrium states to the differences in the area A of the event horizons, in the angular momentum J, and in the charge Q. This is very similar to the first law of thermodynamics

$$\mathrm{d}E = T\,\mathrm{d}s - p\,\mathrm{d}v \tag{13}$$

suggesting that one should regard some multiple of A as the entropy of a black hole.

Given the background, we now proceed to the theoretical considerations of the paper. Considering eq. (12), we have

$$\left(\frac{\partial E}{\partial A}\right)_{J,Q} = \frac{\kappa c^2}{8\pi G}, \tag{14}$$

$$\left(\frac{\partial E}{\partial J}\right)_{A,Q} = Q \tag{15}$$

and

$$\left(\frac{\partial E}{\partial J}\right)_{A,J} = \Phi \tag{16}$$

Since dE is a perfect differential,

$$\left(\frac{\partial}{\partial J}\left(\frac{\partial E}{\partial A}\right)_{J,Q}\right)_{A,Q} = \left(\frac{\partial}{\partial A}\left(\frac{\partial E}{\partial J}\right)_{A,Q}\right)_{J,Q}. \tag{17}$$

Using eqs (14) and (15) we have

$$\left(\frac{\partial \kappa}{\partial J}\right)_{A,Q} = \frac{8\pi G}{c^2}\left(\frac{\partial \Omega}{\partial A}\right)_{J,Q}. \tag{18}$$

Next, we have

$$\left(\frac{\partial}{\partial Q}\left(\frac{\partial E}{\partial A}\right)_{J,Q}\right)_{A,J} = \left(\frac{\partial}{\partial A}\left(\frac{\partial E}{\partial Q}\right)_{A,J}\right)_{J,Q}. \tag{19}$$

Using eqs (14) and (16) we get

$$\left(\frac{\partial \kappa}{\partial Q}\right)_{A,J} = \frac{8\pi G}{c^2}\left(\frac{\partial \Phi}{\partial A}\right)_{J,Q}, \tag{20}$$

and finally,

$$\left(\frac{\partial}{\partial Q}\left(\frac{\partial E}{\partial J}\right)_{A,Q}\right)_{A,J} = \left(\frac{\partial}{\partial J}\left(\frac{\partial E}{\partial Q}\right)_{A,J}\right)_{A,Q}. \tag{21}$$

Using eqs (15) and (16) we have

$$\left(\frac{\partial \Omega}{\partial Q}\right)_{A,J} = \left(\frac{\partial \Phi}{\partial J}\right)_{A,Q}. \tag{22}$$

For any process in which the surface gravity κ is constant and by the zeroth law of black hole dynamics (which states in essence that *for a stationary axisymmetric black hole in a space time which is asymptotically flat, it is possible to give a general definition of the surface gravity κ such that κ is constant on the horizon*), for a stationary axisymmetric black hole, we once again consider eq. (12),

$$\frac{\kappa c^2}{8\pi G}\mathrm{d}A = \mathrm{d}\left(\frac{\kappa c^2}{8\pi G}A\right). \tag{23}$$

Therefore,

$$\mathrm{d}\left[E - \frac{\kappa c^2}{8\pi G}A\right] = \Omega \mathrm{d}J + \Phi \mathrm{d}Q \tag{24}$$

or

$$\mathrm{d}K = \Omega \mathrm{d}J + \Phi \mathrm{d}Q. \tag{25}$$

$K = E - \frac{\kappa c^2}{8\pi G}A$ is the additional available energy. Now black holes have negative specific heat. Thus K must observe the condition $K < \frac{1}{4}E$ in order that the black holes be in a state of stable thermal equilibrium.

Now from eq. (25),

$$\left(\frac{\partial K}{\partial J}\right)_Q = \Omega, \tag{26}$$

$$\left(\frac{\partial K}{\partial Q}\right)_J = \Phi. \tag{27}$$

Since dK is a perfect differential,

$$\left(\frac{\partial}{\partial Q}\left(\frac{\partial K}{\partial J}\right)_Q\right)_J = \left(\frac{\partial}{\partial J}\left(\frac{\partial K}{\partial Q}\right)_J\right)_Q \tag{28}$$

or using eqs (26) and (27) we have

$$\left(\frac{\partial \Omega}{\partial Q}\right)_J = \left(\frac{\partial \Phi}{\partial J}\right)_Q. \tag{29}$$

Equation (29) is a more flexible version of eq. (22), since the constancy of A is not required and yet the equation still holds good.

We still have three relations: eqs (18), (20) and (22). Now since in devouring matter and emitting thermal radiation, heat is involved and there is net absorption of heat in the former case and rejection of heat in the latter case, we conclude that a black hole has enthalpy corresponding to a Hawking temperature T_{H} given by eq. (1). We define the black hole enthalpy mathematically as

$$H = E - \Omega J - \Phi Q. \tag{30}$$

Therefore,

$$\mathrm{d}H = \mathrm{d}E - \Omega\,\mathrm{d}J - \Phi\,\mathrm{d}Q - J\,\mathrm{d}\Omega - Q\,\mathrm{d}\Phi \tag{31}$$

or using eq. (12) we have

$$\mathrm{d}H = \frac{\kappa c^2}{8\pi G}\mathrm{d}A - J\,\mathrm{d}\Omega - Q\,\mathrm{d}\Phi. \tag{32}$$

Hence,

$$\left(\frac{\partial H}{\partial A}\right)_{\Omega,\Phi} = \frac{\kappa c^2}{8\pi G}, \tag{33}$$

$$\left(\frac{\partial H}{\partial \Omega}\right)_{A,\Phi} = -J, \tag{34}$$

$$\left(\frac{\partial H}{\partial \Phi}\right)_{A,\Omega} = -Q. \tag{35}$$

Since dH is a perfect differential,

$$\left(\frac{\partial}{\partial \Omega}\left(\frac{\partial H}{\partial A}\right)_{\Omega,\Phi}\right)_{A,\Phi} = \left(\frac{\partial}{\partial A}\left(\frac{\partial H}{\partial \Omega}\right)_{A,\Phi}\right)_{\Omega,\Phi}. \tag{36}$$

Using eqs (33) and (34) we have

$$\left(\frac{\partial \kappa}{\partial \Omega}\right)_{A,\Phi} = -\frac{8\pi G}{c^2}\left(\frac{\partial J}{\partial A}\right)_{\Omega,\Phi}. \tag{37}$$

Consider the next pair,

$$\left(\frac{\partial}{\partial \Phi}\left(\frac{\partial H}{\partial A}\right)_{\Omega,\Phi}\right)_{A,\Omega} = \left(\frac{\partial}{\partial A}\left(\frac{\partial H}{\partial \Phi}\right)_{A,\Omega}\right)_{\Omega,\Phi}. \tag{38}$$

Using eqs (33) and (35)

$$\left(\frac{\partial \kappa}{\partial \Phi}\right)_{A,\Omega} = -\frac{8\pi G}{c^2}\left(\frac{\partial Q}{\partial A}\right)_{\Omega,\Phi}. \tag{39}$$

Then consider the final pair,

$$\left(\frac{\partial}{\partial \Omega}\left(\frac{\partial H}{\partial \Phi}\right)_{A,\Omega}\right)_{A,\Phi} = \left(\frac{\partial}{\partial \Phi}\left(\frac{\partial H}{\partial \Omega}\right)_{A,\Phi}\right)_{A,\Omega}. \tag{40}$$

Using eqs (34) and (35),

$$\left(\frac{\partial Q}{\partial \Omega}\right)_{A,\Phi} = \left(\frac{\partial J}{\partial \Phi}\right)_{A,\Omega}. \tag{41}$$

Equation (41) may not be considered as a fundamental relation, as Q under normal circumstances does not relate to Ω, and the same goes for J and Φ; yet we shall consider it as a fundamental relation in case in the future someone proves a relation between these quantities which was hitherto unknown due to the unobservable nature of the internal configurations of the black hole and the quantum nature of singularity as well as that in the framework of supergravity (Bais 1983).

Now consider eq. (32). For constancy of surface gravity κ, angular frequency of rotation Ω, and potential Φ of the event horizon, simultaneously,

$$\mathrm{d}\left[H - \frac{\kappa c^2}{8\pi G}A\right] = 0. \tag{42}$$

We define an auxiliary free energy F as

$$F = H - \frac{\kappa c^2}{8\pi G}A. \tag{43}$$

Therefore,

$$\mathrm{d}F = 0. \tag{44}$$

Using eq. (30) in eq. (43) and differentiating we have

$$\mathrm{d}F = \mathrm{d}E - \Omega \mathrm{d}J - \Phi \mathrm{d}Q - J\,\mathrm{d}\Omega - Q\,\mathrm{d}\Phi - \frac{\kappa c^2}{8\pi G}\mathrm{d}A - \frac{c^2 A}{8\pi G}\mathrm{d}\kappa. \tag{45}$$

Substituting for dE from eq. (12) in eq. (45) we have

$$\mathrm{d}F = -\frac{c^2 A}{8\pi G}\mathrm{d}\kappa - J\,\mathrm{d}\Omega - Q\,\mathrm{d}\Phi. \tag{46}$$

Hence, we get

$$\left(\frac{\partial F}{\partial \kappa}\right)_{\Omega,\Phi} = -\frac{c^2 A}{8\pi G}, \tag{47}$$

$$\left(\frac{\partial F}{\partial \Omega}\right)_{\kappa,\Phi} = -J, \tag{48}$$

$$\left(\frac{\partial F}{\partial \Phi}\right)_{\kappa,\Omega} = -Q. \tag{49}$$

Since dF is a perfect differential,

$$\left(\frac{\partial}{\partial \Omega}\left(\frac{\partial F}{\partial \kappa}\right)_{\Omega,\Phi}\right)_{\kappa,\Phi} = \left(\frac{\partial}{\partial \kappa}\left(\frac{\partial F}{\partial \Phi}\right)_{\kappa,\Phi}\right)_{\Omega,\Phi}. \tag{50}$$

Using eqs (48) and (49), we have

$$\left(\frac{\partial A}{\partial \Omega}\right)_{\kappa,\Phi} = \frac{8\pi G}{c^2}\left(\frac{\partial J}{\partial \kappa}\right)_{\Omega,\Phi}. \tag{51}$$

For the next pair, we have

$$\left(\frac{\partial}{\partial \Phi}\left(\frac{\partial F}{\partial \kappa}\right)_{\Omega,\Phi}\right)_{\kappa,\Omega} = \left(\frac{\partial}{\partial \kappa}\left(\frac{\partial F}{\partial \Phi}\right)_{\kappa,\Omega}\right)_{\Omega,\Phi}. \tag{52}$$

Using eqs (48) and (50), we have

$$\left(\frac{\partial A}{\partial \Phi}\right)_{\kappa,\Omega} = \frac{8\pi G}{c^2}\left(\frac{\partial Q}{\partial \kappa}\right)_{\Omega,\Phi}. \tag{53}$$

Finally, for the last pair,

$$\left(\frac{\partial}{\partial \Omega}\left(\frac{\partial F}{\partial \Phi}\right)_{\kappa,\Omega}\right)_{\kappa,\Phi} = \left(\frac{\partial}{\partial \Phi}\left(\frac{\partial F}{\partial \Omega}\right)_{\kappa,\Phi}\right)_{\kappa,\Omega}. \tag{54}$$

Using eqs (49) and (50), we have

$$\left(\frac{\partial Q}{\partial \Omega}\right)_{\kappa,\Phi} = \left(\frac{\partial J}{\partial \Phi}\right)_{\kappa,\Omega}. \tag{55}$$

This completes our quest for the nine fundamental black hole dynamic relations given by eqs (18), (20), (22) or (29), (37), (39), (41), (51), (53) and (55). Finally, let us compare eq. (14) and eq. (33); we then have an auxiliary relation:

$$\left(\frac{\partial E}{\partial A}\right)_{J,Q} = \left(\frac{\partial H}{\partial A}\right)_{\Omega,\Phi}. \tag{56}$$

We now proceed to deduce an equation involving a $\frac{\kappa c^2}{8\pi G}\mathrm{d}A$ term, dubbed as the $\kappa - \mathrm{d}A$ equation in short. We have

$$\mathrm{d}A = \left(\frac{\partial A}{\partial \kappa}\right)_{\Omega,\Phi} \mathrm{d}\kappa + \left(\frac{\partial A}{\partial \Omega}\right)_{\kappa,\Phi} \mathrm{d}\Omega + \left(\frac{\partial A}{\partial \Phi}\right)_{\kappa,\Omega} \mathrm{d}\Phi, \tag{57}$$

$$\frac{\kappa c^2}{8\pi G}\mathrm{d}A = \frac{\kappa c^2}{8\pi G}\left(\frac{\partial A}{\partial \kappa}\right)_{\Omega,\Phi} \mathrm{d}\kappa + \frac{\kappa c^2}{8\pi G}\left(\frac{\partial A}{\partial \Omega}\right)_{\kappa,\Phi} \mathrm{d}\Omega + \frac{\kappa c^2}{8\pi G}\left(\frac{\partial A}{\partial \Phi}\right)_{\kappa,\Omega} \mathrm{d}\Phi. \tag{58}$$

Now we define the specific heat of the black hole at constant Ω, Φ as

$$C_{\Omega,\Phi} = \frac{\kappa c^2}{8\pi G}\left(\frac{\partial A}{\partial \kappa}\right)_{\Omega,\Phi} \tag{59}$$

and using two of the above nine fundamental relations viz., eqs (51) and (53) we have

$$\frac{\kappa c^2}{8\pi G}\mathrm{d}A = C_{\Omega,\Phi}\,\mathrm{d}\kappa + \kappa\left[\left(\frac{\partial J}{\partial \kappa}\right)_{\Phi} \mathrm{d}\Omega + \left(\frac{\partial Q}{\partial \kappa}\right)_{\Omega} \mathrm{d}\Phi\right]. \tag{60}$$

Similarly,

$$\frac{\kappa c^2}{8\pi G}\mathrm{d}A = \frac{\kappa c^2}{8\pi G}\left(\frac{\partial A}{\partial \kappa}\right)_{J,Q} \mathrm{d}\kappa + \frac{\kappa c^2}{8\pi G}\left(\frac{\partial A}{\partial \Omega}\right)_{J,Q} \mathrm{d}\Omega + \frac{\kappa c^2}{8\pi G}\left(\frac{\partial A}{\partial \Phi}\right)_{J,Q} \mathrm{d}\Phi. \tag{61}$$

We define the specific heat at constant J, Q as

$$C_{J,Q} = \frac{\kappa c^2}{8\pi G}\left(\frac{\partial A}{\partial \kappa}\right)_{J,Q} \tag{62}$$

and using another pair of the nine fundamental relations, we have

$$\frac{\kappa c^2}{8\pi G}\mathrm{d}A = C_{J,Q}\,\mathrm{d}\kappa + \kappa\left[\left(\frac{\partial J}{\partial \kappa}\right)_{A,Q} \mathrm{d}\Omega + \left(\frac{\partial Q}{\partial \kappa}\right)_{A,J} \mathrm{d}\Phi\right]. \tag{63}$$

As mentioned earlier, $C_{\Omega,\Phi} < 0$ and $C_{J,Q} < 0$.

The energy $E(=Mc^2)$ of the black hole is in principle the sum of the additional available energy (or the first free energy function as we have called it) K and $\frac{\kappa c^2}{8\pi G}\mathrm{d}A$. The latter is what is called the bound energy or the energy that is not available for work. This makes K, the energy which is available for work in time-reversible processes (white holes) observing constancy of surface gravity. Since the area of the event horizon always tends to increase, it is clear that the bound energy of the black hole always tends to increase, with the result that K, the additional available energy for work, tends to decrease. This decrease in K tends the black holes to approach a state of stable thermal equilibrium. Hence, the first free energy function K is called the additional available energy.

Let us now consider what happens in the case of a time-reversible process observing constancy of surface gravity. For an infinitesimal time-reversible process,

$$\mathrm{d}K = \mathrm{d}E - \mathrm{d}\left(\frac{\kappa c^2}{8\pi G}A\right) \tag{64}$$

or

$$\mathrm{d}K = \mathrm{d}E - \frac{\kappa c^2}{8\pi G}\mathrm{d}A - \frac{c^2 A}{8\pi G}\mathrm{d}\kappa. \tag{65}$$

Using eq. (12), we have

$$\mathrm{d}K = -\frac{c^2 A}{8\pi G}\mathrm{d}\kappa + \Omega\,\mathrm{d}J + \Phi\,\mathrm{d}Q. \tag{66}$$

Since the surface gravity is constant $\mathrm{d}\kappa = 0$ and we have back our eq. (25), viz.,

$$\mathrm{d}K = \Omega\,\mathrm{d}J + \Phi\,\mathrm{d}Q \tag{67}$$

or

$$K_2 - K_1 = \int_1^2 \Omega\,\mathrm{d}J \;+\; \int_1^2 \Phi\,\mathrm{d}Q, \tag{68}$$

i.e., the change in the additional available energy or the first free energy function of a black hole. During a time-reversible process, observing constancy of surface gravity is equal to the work done upon the black hole system. In other words, the entire work in such a process is done at the cost of additional available energy of the hole.

Just as a black hole performs work at the expense of its potential energy, so also, does a black hole undergoing a time-reversible isothermal process performs work at the expense of its additional available energy. This of course is a hypothetical situation unless the black hole is stationary (or at least quasi-stationary) and axisymmetric throughout any such time-reversible process that it undergoes.

Given the entropy of a system as a function of the energy E of the system and various other macroscopic parameters, one can define the temperature as $\frac{1}{T} = \frac{\partial S}{\partial E}$. Thus, one can define the temperature of the black hole to be

$$\frac{1}{T_{\mathrm{H}}} = \left(\frac{\partial S_{\mathrm{H}}}{\partial E}\right)_{J,Q}. \tag{69}$$

The generalized second law of black hole dynamics as given by Bekenstein is then equivalent to the requirement that heat should not run uphill from a cooler system to a warmer one. With context to the radiation communicated by black holes (Hawking radiation and other forms of quantum radiation decreasing the area of event horizon of the black hole) this law is given in a more convenient statement (Kabe 2010). We define the area of event horizon A of a black hole as the rate of change of the additional available energy with the surface gravity at constant angular momentum and charge, i.e,

$$A = -\frac{8\pi\, G}{c^2}\left(\frac{\partial K}{\partial \kappa}\right)_{J,Q} \tag{70}$$

and we have from eqs (24) and (25),

$$E = K + \frac{\kappa c^2}{8\pi G} A \tag{71}$$

or

$$E = K - \left(\kappa \frac{\partial K}{\partial \kappa}\right)_{J,Q}. \tag{72}$$

The main importance of the additional available black hole energy lies in the field of statistical mechanics of black holes. The second free energy function F is the black hole dynamic potential at constant angular frequency of rotation and constant potential of the black hole. We have seen that F remains constant during a process in which the angular frequency of rotation Ω and potential Φ of the event horizon remain constant. It is also apparent that the black hole enthalpy may be expressed as the sum of the second free energy function and the bound energy.

Lastly, we consider the homogeneous fluid approximation, which holds for homogeneous gravitational fields in black hole space times or de Sitter black holes.

In this case A is a function of surface gravity, angular momentum and charge. This is apparent from eq. (12), or from eqs (2), (9) and (10),

$$\mathrm{d}A = \left(\frac{\partial A}{\partial \kappa}\right)_{J,Q} \mathrm{d}\kappa + \left(\frac{\partial A}{\partial J}\right)_{\kappa,\Phi} \mathrm{d}\Omega + \left(\frac{\partial A}{\partial Q}\right)_{\kappa,\Omega} \mathrm{d}\Phi. \tag{73}$$

Therefore,

$$\begin{aligned}\left(\frac{\partial A}{\partial \kappa}\right)_{\Omega,\Phi} = & \left(\frac{\partial A}{\partial \kappa}\right)_{J,Q} + \left(\frac{\partial \kappa}{\partial \kappa}\right)_{J,Q} + \left(\frac{\partial A}{\partial \Omega}\right)_{\kappa,\Phi} \left(\frac{\partial \Omega}{\partial \kappa}\right)_{J,Q} \\ & + \left(\frac{\partial A}{\partial \Phi}\right)_{\kappa,\Omega} \left(\frac{\partial \Phi}{\partial \kappa}\right)_{J,Q}.\end{aligned} \tag{74}$$

Using two of the fundamental relations viz., eqs (51) and (53), we conceive

$$\begin{aligned}\frac{\kappa c^2}{8\pi G}\left(\frac{\partial A}{\partial \kappa}\right)_{\Omega,\Phi} = & \frac{\kappa c^2}{8\pi G}\left(\frac{\partial A}{\partial \kappa}\right)_{J,Q} \\ & + \kappa\left[\left(\frac{\partial J}{\partial \kappa}\right)_{\Omega,\Phi}\left(\frac{\partial \Omega}{\partial \kappa}\right)_{J,Q} + \left(\frac{\partial Q}{\partial \kappa}\right)_{\Omega,\Phi}\left(\frac{\partial \Phi}{\partial \kappa}\right)_{J,Q}\right],\end{aligned} \tag{75}$$

or using eqs (59) and (62) we have

$$C_{\Omega,\Phi} - C_{J,Q} = \kappa\left[\left(\frac{\partial J}{\partial \kappa}\right)_{\Omega,\Phi}\left(\frac{\partial \Omega}{\partial \kappa}\right)_{J,Q} + \left(\frac{\partial Q}{\partial \kappa}\right)_{\Omega,\Phi}\left(\frac{\partial \Phi}{\partial \kappa}\right)_{J,Q}\right]. \tag{76}$$

This is the approximation that we needed. Wherever the difference of specific heats is required, one has to use eq. (76) and induct the value given by the right-hand side of eq. (76). Since $C_{\Omega,\Phi}$ and $C_{J,Q}$ are both negative, the right-hand side will be positive if and only if $C_{J,Q}$ is more negative than $C_{\Omega,\Phi}$. On the other hand, if $C_{\Omega,\Phi}$ is relatively more negative, then the right-hand side will be negative. For a Schwarzschild black hole, both the specific heats are one and the same.

The nine fundamental relations deduced above might find applications in a wide range of problems in black hole thermodynamics, especially in the physics of the accretion tori or accretion disks around black holes and in the physics of radiating and primordial black holes wherein temperatures are significant. These and many other applications in various areas are plausible.

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