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Analytical Solution for Stellar Density in Globular Clusters

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Abstract. In this paper, four parameters analytical solution will be established for the stellar density function in globular clusters. The solution could be used for any arbitrary order of outward decrease of the cluster's density.

Key words. Stellar density in globular clusters—theoretical astrophysics—statistical astronomy.

1. Introduction

A globular cluster is a spherical collection of stars that orbits a galactic core as a satellite. They are generally composed of hundreds of thousands of low-metal, old stars. The types of stars found in a globular cluster are similar to those in the bulge of a spiral galaxy but confined to a volume of only a few cubic parsecs. Observations of globular clusters show that these stellar formations arise primarily in regions of efficient star formation, and where the interstellar medium is at a higher density than in normal star-forming regions. Globular clusters are fairly common; there are about 158 (Frommert & Hartmut 2007) currently known globular clusters in the Milky Way, with perhaps 10–20 more undiscovered. Andromeda, for instance, may have as many as 500 globular cluster (Barmby & Huchra 2001); whereas some giant elliptical galaxies, such as M87 (Strom *et al.* 1981), may have as many as 10,000 globular clusters. These globular clusters orbit the galaxy out to large radii, 40 kiloparsecs (approximately 131 thousand light-years) or more.

Galactic globular clusters, which are ancient building blocks of our Galaxy, represent a very interesting family of stellar systems in which some fundamental dynamical processes have taken place on time scale shorter than the age of the universe. For example, horizontal branch (HB) stars in globular clusters offer an investigation of the mass loss mechanisms taking place in red giants (Valcarce & Catelan 2008). Moreover, it was proposed to use the HB to infer which is today the relative number fraction of 'normal' and anomalous stars in clusters (D'Antona & Caloi 2008). In contrast with galaxies, it was known since the last twenty years that globular clusters represent unique laboratories for learning about two-body relaxation, mass segregation from equipartition of energy (Spitzer 1987), stellar collisions (Binney & Tremaine 1987), stellar mergers, and core collapse.

From the photographs of a globular cluster, one can notice how the stars in this projection on a plane, are clustering in much closer packing near the center, while the outer parts are much looser. The distribution law, describing this clustering, is of highest interest for understanding the internal dynamics of this system and of its origin. However, the distribution in space must be a function of the radius *r*, different from the distribution in projection on a plane.

In fact, those numerical methods provide very accurate solutions in general. But certainly if full analytical formulae are utilized via symbol manipulating digital computer programs, they definitely become invaluable for obtaining solutions of any desired accuracy. Moreover, symbolic computing algorithms for scientific problems, in general, represent a new branch of numerical methods that we may call 'algorithmization' of problems. Following this line of recent researches and also due to the important role of the space distribution in understanding the dynamical evolution of globular clusters (Shin *et al.* 2008), the present paper is developed to establish analytical solution for the space density distribution of globular clusters. This solution depends on four parameters that can be obtained from star counts, and can be used to fit any order of the outward decrease of the cluster density.

2. Basic formulations

2.1 *Assumptions*

The distribution of stars in globular clusters is inferred from counts of stellar images on photographic plates. Although several of the clusters appear to be somewhat ellipsoidal in form (van den Bergh & Sidney 2008) we consider only the case of spherical symmetry; the analytical results can then be applied to spherical clusters and to those whose ellipticity is small. When the departure from the spherical form is considerable, the general problem of stellar distribution in such clusters, in practice, is almost intractable.

2.2 *Relation between the cluster density and plate density function*

Let $\Phi(r)$ be the density function of the cluster at a distance r from its center. Let $f(x)$ is the plate density of the cluster's stars at a distance x from the center of cluster as shown on the plate. If $\sigma(x)$ denotes the number of images counted on the plate within a circle of radius x, then $f(x)$ is related to $\sigma(x)$ by:

$$f(x) = -\frac{1}{2\pi x}\frac{\mathrm{d}\sigma(x)}{\mathrm{d}x}. \qquad (2.1)$$

Equation (2.1) will be used for obtaining $f(x)$ from counts of stars on photographic plates (note that the negative sign is due to the fact that $\sigma(x)$ is monotonic decreasing function as discussed in section 3).

Finally, $\Phi(r)$ and $f(x)$ are related by integral equation (e.g., Spitzer 1987), of the form

$$\Phi(r) = \frac{1}{\pi}\int_r^R \left(x^2 - r^2\right)^{1/2} \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{x}\frac{\mathrm{d}f}{\mathrm{d}x}\right\}\mathrm{d}x. \qquad (2.2)$$

3. Empirical formula for $\sigma(x)$

Observations of globular clusters show very smooth spherical distribution of brightness. If we assume that the amount of light we measure is proportional to the number of stars giving rise to this light, we can determine the run of the star density as a function of distance from the cluster center. We can measure the brightness of globular cluster in successive rings concentric with the cluster center. This may be done by direct star counts on a photographic plate, or by photographic photometry in which circular apertures of progressively large size are employed. The radius of the cluster image is taken as unity, then, what we can determine from such observations is the number of star images $\sigma_i \equiv \sigma(x_i)$ counted on the plate within circles of radii x_i for some $i = 1, 2, \ldots, N$ (say) and $x_i \leq 1$.

Now, since the stars are clustering in much closer packing near the center, while the outer parts are much looser, i.e., $\sigma(x)$ is monotonic decreasing function of x. The orders of the outward decrease differ from one cluster to another and also differ from one region to another of the same cluster. In order to account for these variations so as to suit many applications, we fit $\sigma(x)$ to the curve

$$\sigma(x) = \frac{a + bx}{A + Bx}, \tag{3.1}$$

where a, b, A and B are constants to be determined.

3.1 *Determination of a, b, A, and B*

In what follows, we shall develop an algorithm for the best rational approximation to $(a + bx)/(A + Bx)$ in the least-squares sense.

- Computational Algorithm 1
  - Purpose

    To determine the constants a, b, A and B.
  - Input

    $x_i;\ \sigma(x_i) \equiv \sigma_i;\quad i = 1, 2, \ldots, N.$
  - Computational sequence

1. For all $q = 1, 2, \ldots, N$,

   $P_0(x_q) = 0;\ Q_0(x_q) = 1$

2. For all $q = 1, 2, \ldots, N$,

   $f(x_q) = \sigma(x_q)\ Q_0(x_q) - P_0(x_q)$

   $\omega(x_q) = \dfrac{1}{Q_0^2(x_q)}$

   $g_1(x_q) = -\dfrac{P_0(x_q)}{Q_0(x_q)}$

   $g_2(x_q) = 1$

   $g_3(x_q) = x_q$

3. $\mathrm{S}_0 = \sum_{i=1}^{N} \left[\sigma(\mathrm{x}_l) - \frac{P_0(x_l)}{Q_0(x_l)} \right]^2.$

4. For $i = 1, 2, 3$; $j = i, \ldots, 3$ construct the symmetric matrix **H** and the vector **R**,

$$h_{ij} = \sum_{k=1}^{N} \omega(x_k) g_i(x_k) g_j(x_k)$$
$$r_i = \sum_{k=1}^{N} \omega(x_k) g_i(x_k) f(x_k).$$

5. Solve the linear system

   Hc = **R**, for c_1, c_2 and c_3.

6. For all $l = 1, 2, \ldots, N$,

$$\Delta Q_0(x_l) = c_1 x_l$$
$$\Delta P_0(x_l) = c_2 + c_3 x_l$$
$$P_0(x_l) = P_0(x_l) + \Delta P_0(x_l)$$
$$Q_0(x_l) = Q_0(x_l) + \Delta Q_0(x_l).$$

7. $S_1 = \sum_{i=1}^{N} \left[\sigma(x_l) - \frac{P_0(x_l)}{Q_0(x_l)} \right]^2.$

8. If $\left| \frac{S_0 - S_1}{S_1} \right| \geq 10^{-5}$, go to Step 2.

9. The required coefficients are:

   $a = c_2, \; b = c_3, \; A = 1, \; B = c_1.$

10. End.

4. Analytical expression of $\Phi(r)$

Having obtained the empirical formula for $\sigma(x)$ as

$$\sigma(x) = \frac{a + bx}{A + Bx},$$

the cluster density function $\Phi(r)$ could then be obtained analytically from equations (2.1) and (2.2) as listed in Appendix. Figure 1 is just a graphical illustration of our analytical solution for the case $a = 5$, $b = 8$, $A = 1$, $B = 9$.

In concluding the present paper, we stress that an efficient analytical solution was established for the solution of the stellar density distribution in globular clusters. The efficiency of the solution is due to its inclusion of only four parameters that can be obtained from star counts, and also due to its ability to fit any order of outward decease of the cluster density.

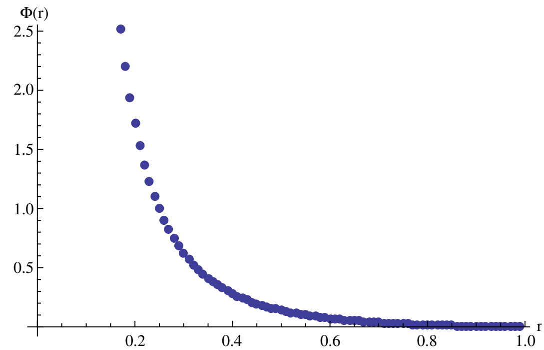

Figure 1.

Appendix

Analytical expression of the space density function of globular clusters

$$
\begin{aligned}
\Phi(r) = {} & \frac{1}{256A^{10}\pi(-1+r)} \\
& \times \Bigg((Ab-aB)(1-r)\Bigg(A\Bigg(2520B^7r^5\left(1-\frac{B^3r}{(A+B)^3}\right) \\
& -\frac{1260AB^9r^4(1+5r^2)}{(A+B)^3}+\frac{420A^2B^5r^3(6(A+B)^3-13B^3r-11B^3r^3)}{(A+B)^3} \\
& -\frac{210A^3B^7r^2(7+38r^2+3r^4)}{(A+B)^3} \\
& +\frac{2A^4B^3r(1012(A+B)^3-2685B^3r-2170B^3r^3+63B^3r^5)}{(A+B)^3} \\
& +\frac{A^{10}(-192+48r^2+8r^43r^6)}{(A+B)^3}+\frac{3A^9B(-192+48r^2+8r^4+3r^6)}{(A+B)^3} \\
& -\frac{6A^5B^5(220+1135r^2+70r^4+7r^6)}{(A+B)^3} \\
& +A^8B^2\left(-\frac{2024}{(A+B)^3}-\frac{80}{(A+Br)^3}+\frac{r^2(48+12r^2+5r^4)}{(A+B)^3}\right) \\
& +A^7\left(\frac{133}{r^2}+\frac{240B^2}{(A+Br)^2}-\frac{B^3(4920+240r^2+28r^4+9r^6)}{(A+B)^3}\right) \\
& +A^6\left(\frac{1400B^2}{A+Br}+\frac{2B^4(-2172-1615r^2+42r^4+9r^6)}{(A+B)^3}\right)\Bigg) \\
& +24B^2(16A^6+40A^4B^2r^2+70A^2B^4r^4+105B^6r^6) \\
& \times(\ln[A+B]+\ln[r]-\ln[A+Br])\Bigg)\Bigg).
\end{aligned}
$$

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