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Gravitational Clustering of Galaxies in an Expanding Universe

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Abstract. We inquire the phenomena of clustering of galaxies in an expanding universe from a theoretical point of view on the basis of thermodynamics and correlation functions. The partial differential equation is developed both for the point mass and extended mass structures of a two-point correlation function by using thermodynamic equations in combination with the equation of state taking gravitational interaction between particles into consideration. The unique solution physically satisfies a set of boundary conditions for correlated systems and provides a new insight into the gravitational clustering problem.

Key words. Gravitational clustering—thermodynamics—correlation function: structure of universe.

1. Introduction

Galaxies interact gravitationally and the characterization of this clustering is a problem of current interest. The gravitational interaction between galaxies and galaxy clusters have played an important role in the evolution of the observed universe. Various theories of the cosmological many body problem have been developed mainly from a thermodynamic point of view. We have made use of the equations of state along with the correlation functions for the development of a theoretical model. This can be done by solving a system of Liouville's equation or BBGKY-hierarchy equations and have been discussed by workers like Saslaw (1972) and Peebles (1980). But BBGKY hierarchy equations are too complicated to handle for higher order correlation functions. However, the lowest order two-point correlation function is useful for discussing the phenomenon of gravitational clustering of galaxy clusters which contain information on all the higher n-particle correlations in the full BBGKY-hierarchy (Peebles 1980; Zhan & Dyer 1989; Hamilton 1993). The physical validity of the application of thermodynamics in the clustering of galaxies and galaxy clusters has been discussed on the basis of N-body computer simulation results (Itoh *et al.* 1993). In gravitational thermodynamics the value of b which is the ratio of gravitational correlation energy to twice kinetic energy measures two-point correlation function ξ_2 and depends on the average number density $\bar{n}$, temperature T and the interparticle distance r. Thus it is valuable to understand the functional form of ξ_2 which depends upon the value of $b(n, T)$. The gravitational galaxy clustering carried out by Ahmad *et al.* (2002, 2006) ensures a more fundamental statistical mechanical description of the cosmological many-body problem.

The two partial differential equations developed for the galaxy clusters (with point mass and extended mass structures) in an expanding universe provide a new approach for understanding the phenomena.

2. Approaches

2.1 *Development of differential equation for two-point correlation function*

Taking the lead of Saslaw & Hamilton (1984), we start with a general pair of equations of state for the internal energy U and the pressure P:

$$U = \frac{3}{2}NT(1-2b), \tag{1}$$

$$P = \frac{NT}{V}(1-b). \tag{2}$$

These two equations represent the equations of state (e.g., Hill 1956), where the dimensionless

$$b = \frac{2\pi G m^2 \bar{n}}{3T}\int_V \xi_2(\bar{n}, T, r) r dr \tag{3}$$

defines the measuring correlation parameter. Here $\bar{n} = N/V$ is the average number density of the system of particles each of mass m. T is the temperature, V the volume, G is the universal constant of gravitation, $\xi_2(\bar{n}, T, r)$ is the two-particle correlation function and r the distance. These expressions assume a large volume V for their validity.

The thermodynamic parameters P, V and T are related by

$$P = P(V, T) \quad \text{and} \quad U = U(V, T).$$

According to the first law of thermodynamics, for any system

$$dQ = dU + PdV \tag{4}$$

where dQ is the amount of heat flowing into the system, dU is the gain in its internal energy and PdV is the work done by the system as its volume increases by the infinitesimal amount dV under the internal pressure P.

For fixed number of particles N, the internal energy U is related to V and T as

$$dU = \left(\frac{\partial U}{\partial V}\right)_T dV + \left(\frac{\partial U}{\partial T}\right)_V dT. \tag{5}$$

Hence equation (4) becomes:

$$dQ = \left[\frac{\partial U}{\partial V} + P(V, T)\right] dV + \left(\frac{\partial U}{\partial T}\right) dT. \tag{6}$$

According to the second law of thermodynamics, if a system undergoes an infinitesimal quasi-static change as a result of which it gains an infinitesimal amount of heat dQ at the absolute temperature T, then there exists a perfect differential dS of a function S (entropy) defined by:

$$dS = \frac{dQ}{T}. \tag{7}$$

It must be understood that this definition applies only if the system undergoes an infinitesimal quasi-stationary change.

The combination of equation (7) with equation (6) leads to

$$dS = \frac{dQ}{T} = \frac{1}{T}(dU + PdV) = \frac{1}{T}\left[\frac{\partial U}{\partial V} + P(V,T)\right]dV + \left(\frac{\partial U}{\partial T}\right)\frac{1}{T}dT. \tag{8}$$

As dS is a perfect differential, we must have

$$\frac{\partial}{\partial T}\left[\frac{1}{T}\left(\frac{\partial U}{\partial V}\right) + \frac{P}{T}\right] = \frac{\partial}{\partial V}\left[\left(\frac{\partial U}{\partial T}\right)\frac{1}{T}\right] \tag{9}$$

which leads to:

$$\left(\frac{\partial U}{\partial V}\right)_{T,N} = T\left(\frac{\partial P}{\partial T}\right)_{V,N} - P. \tag{10}$$

The probability for finding a galaxy in volume dV_1 and also one in volume dV_2 is related to two-point correlation function $\xi_2(r)$ by:

$$P_{12} = \bar{n}^2[1 + \xi_2(r)]dV_1 dV_2. \tag{11}$$

In general ξ_2 will depend on the absolute positions $\mathbf{r}_1$ and $\mathbf{r}_2$ of two volume elements. However, if we average overall positions and all directions, then ξ_2 will become a function only of the separation $\mathbf{r} = |\mathbf{r}_1 - \mathbf{r}_2|$. For $\xi_2(r)$ to provide a good description, the system must be statistically homogeneous so that on an average overall, the galaxies in a given volume $V\left(= \frac{4}{3}\pi r^3\right)$ will be the same as for any sufficiently large subset of galaxies in the same volume. In the grand canonical ensemble, the two-point correlation function ξ_2 depends on $\bar{n}$, T as well as on r. Thus we may write

$$\xi_2 = \xi_2(\bar{n}, T, r), \tag{12}$$

$$d\xi_2 = \frac{\partial \xi_2}{\partial \bar{n}}d\bar{n} + \frac{\partial \xi_2}{\partial T}dT + \frac{\partial \xi_2}{\partial r}dr, \tag{13}$$

$$\frac{d\xi_2}{dV} = \frac{\partial \xi_2}{\partial \bar{n}}\frac{d\bar{n}}{dV} + \frac{\partial \xi_2}{\partial T}\frac{dT}{dV} + \frac{\partial \xi_2}{\partial r}\frac{dr}{dV}. \tag{14}$$

Using the condition

$$\frac{dT}{dV} = 0. \tag{15}$$

Using equations (1) to (3) in equation (10), we have:

$$3\bar{n}\frac{\partial \xi_2}{\partial \bar{n}} + T\frac{\partial \xi_2}{\partial T} - r\frac{\partial \xi_2}{\partial r} = 0. \tag{16}$$

Equation (16) is a first order partial differential equation for two-point correlation function. It is characterized by number density $\bar{n}$, temperature T and the interparticle distance r, therefore two-point correlation function ξ_2 will depend on the values of $\bar{n}$ and T for the ensemble as well as on the spatial co-ordinate r in a statistically homogeneous distribution of galaxies clustering gravitationally in an expanding universe.

The galaxies in an expanding universe are treated as point mass particles, but actually they have extended structures and therefore make use of a softening parameter ϵ.

Hence for extended mass distribution, the internal energy U and pressure P can be written as,

$$U_{\text{ext}} = \frac{3}{2}NT - \frac{2\pi G m^2 N^2}{3V^2}\int_V \xi_2(\bar{n}, T, r)\left[1 + \frac{\epsilon^2}{r^2}\right]^{\frac{-1}{2}} \frac{dV}{4\pi r}, \tag{17}$$

$$P_{\text{ext}} = \frac{NT}{V} - \frac{2\pi G m^2 N^2}{3V^2}\int_V \xi_2(\bar{n}, T, r)\left[1 + \frac{\epsilon^2}{r^2}\right]^{\frac{-3}{2}} \frac{dV}{4\pi r}. \tag{18}$$

Again repeating the same procedure the new derived equation is,

$$3\bar{n}\frac{\partial \xi_2}{\partial \bar{n}} + T\frac{r^2}{(r^2+\epsilon^2)}\frac{\partial \xi_2}{\partial T} - r\frac{\partial \xi_2}{\partial r} = 0. \tag{19}$$

Equation (19) is a first order partial differential equation for a two-point correlation function of galaxies with extended structures clustering gravitationally. For extended mass structures, the value of ϵ is taken between 0.01 and 0.05 (in the units of total radius). The thermodynamic description of a two-point correlation function for describing the gravitational interaction of galaxies can be defined by the physical behaviour of equations (16) and (19) respectively.

2.2 *Functional form of two-point correlation function*

The partial differential equations developed in (16) and (19) can have a variety of solutions, but we are interested in a solution that is physically valid. This can be verified when a set of boundary conditions assigned for two-point correlation function are satisfied.

- The gravitational clustering of galaxies in a homogeneous universe requires ξ_2 to have a positive value which obviously depends upon the limiting values of $\bar{n}$, T and r.
- When $\bar{n}$, T and r are very small (approximately tending to zero), the two-particle correlation function ξ_2 will increase except for the number density $\bar{n}$. Similarly when $\bar{n}$, T and r are very large, the corresponding correlation function ξ_2 will decrease except for the number density $\bar{n}$.
- When two-particle correlation function increases, the clustering of galaxies becomes dominant because of virial equilibrium, which suggests that at low

temperatures and high densities more and more clusters are formed. Thus we can write this boundary condition as:

When $\bar{n}T^{-3}$ is very large (approximately tending to infinity), the two-particle correlation function ξ_2 will increase and the measuring correlation parameter b approaches to 1. The reverse will happen for $\bar{n}T^{-3}$ to be very small (approximately tending to zero).

Equation (16) can be integrated along with its characteristics as,

$$\frac{d\bar{n}}{3\bar{n}} = \frac{dT}{T} = \frac{dr}{-r}. \tag{20}$$

This equation can have a variety of solutions in different combinations like:

$$\xi_2 = f(\bar{n}T^{-3}, Tr) \quad \text{or} \quad f(Tr, \bar{n}r^3) \quad \text{or} \quad f(\bar{n}r^3, \bar{n}T^{-3}).$$

All these functions are scale invariant under the transformation

$$\bar{n} \to \lambda^{-3}\bar{n}, \qquad T \to \lambda^{-1}T \quad \text{and} \quad r \to \lambda r$$

where λ is a constant.

We are interested in a solution which satisfies the above-mentioned boundary conditions.

After trying many combinations, we choose the solution as

$$\xi_2(\bar{n}, T, r) = \left[\frac{C_1\bar{n}T^{-3}}{1 + C_1\bar{n}T^{-3}}\right]^2 \frac{1}{C_1\bar{n}T^{-3}} \frac{1}{(C_2Tr)^2} \tag{21}$$

and

$$\xi_2(\bar{n}, T, r) = \frac{b(1-b)}{C(Tr)^2} \tag{22}$$

C, C_1 and C_2 are constants. On testifying the boundary conditions, we see that the two-particle correlation function defines its exact behaviour. The unique solutions shown in equations (21) and (22) show the dependence of ξ_2 on $\bar{n}T^{-3}$, Tr and b. When we consider other combinations shown in the ξ_2 solutions, two-particle correlation function again shows its dependence on $\bar{n}$, T and r. The unique solution of equation (16) can also be verified by using the value of the parameter b defined by Saslaw & Hamilton (1984) which is described by equation (3). Rewriting equation (3) as,

$$b = \frac{2\pi Gm^2\bar{n}}{3T} \int_V \xi_2(\bar{n}, T, r) \frac{dV}{4\pi r}. \tag{23}$$

On substituting equation (21) in (23), the result leads to

$$b = \frac{2\pi Gm^2}{9C_1C_2^2} \frac{C_1\bar{n}T^{-3}}{1 + C_1\bar{n}T^{-3}} = \frac{\beta\bar{n}T^{-3}}{1 + \beta\bar{n}T^{-3}}. \tag{24}$$

This is the same equation as used earlier by Saslaw & Hamilton (1984) for the thermodynamic dependence of b. The self evaluation of b clarifies some important consequences:

- The unique solution chosen in equation (21) of a two-particle correlation function gets immediately confirmed.
- The thermodynamic parameter b shows a complete dependence on the combination $\bar{n}T^{-3}$.

2.3 *Power law for* ξ_2

The two-point correlation function in a gravitational galaxy clustering obeys power law Peebles (1980), which has also been confirmed from N-body computer simulations (Itoh *et al.* 1993) and is written as

$$\xi_2 = r^{-1.8}. \tag{25}$$

From equation (3), we can write

$$\frac{\partial \bar{n}}{\partial V}\left[\frac{\partial b}{\partial \bar{n}} - \frac{b}{\bar{n}}\right] = \frac{2\pi G m^2 \bar{n}}{3T}\frac{\xi_2(\bar{n}, T, r)}{4\pi r}. \tag{26}$$

The combination of physical limits and mathematical simplicity made by Saslaw & Hamilton (1984) suggests that the simple form of b is described by equation (24). β is a positive dimensionless constant and is independent of $\bar{n}$, T and r.

$$\frac{\partial b}{\partial \bar{n}} = \frac{b(1-b)}{\bar{n}}, \qquad \frac{\partial \bar{n}}{\partial V} = \frac{-\bar{n}}{V}. \tag{27}$$

The substitution of equation (27) in (26) gives the modified form of equation (25) as:

$$\xi_2 = r^{-2.0} \tag{28}$$

which shows that the power law for a two-point correlation function developed is in close agreement with Peeebles law, and hence confirms the applicability of gravitational quasi-equilibrium thermodynamics used to study the two-point correlation function for understanding gravitational clustering of galaxies in an expanding universe.

3. Discussion

Many theories have been discussed so far for understanding the gravitational clustering of galaxies in an expanding universe, but in our apporoach we have made use of basic thermodynamics, correlation functions and have found their applicability in clarifying many important results. The development of equations (16) and (19) indicates the dependence of ξ_2 on $\bar{n}$, T and r. The results of two-point correlation function ξ_2 can be extended to higher order correlations also. The functional form of ξ_2 has been obtained from various physical and boundary conditions. Our results also indicate the dependence of b on the specific combination $\bar{n}T^{-3}$ and therefore clarifies the earlier result of Saslaw & Hamilton (1984). Till date, all the theories on gravitational interaction of correlated systems (e.g., galaxies, clusters of galaxies, etc.) are confined to point mass systems only. However, the theory so developed is not only applicable to point mass systems but can also be applicable to extended structures, thus indicating the dependence on one more parameter called softening parameter ϵ on ξ_2. A theoretical analysis of a thermodynamic model along with the correlation functions used will lead to a new direction for understanding a large scale structure of the universe.

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