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An Exact Solution of the Gamma Ray Burst Arrival Time Analysis Problem

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Abstract. An analytical solution of the GRB arrival time analysis is presented. The errors in the position of the GRB resulting from timing and position errors of different satellites are calculated. A simple method of cross-correlating gamma ray burst time-histories is discussed.

Key words. Gamma Ray Bursts—triangulation.

1. Introduction

Most GRB detectors (with the exception of BATSE) have practically no angular resolution one can speak of. The positions of GRB sources in the celestial sphere are determined accurately (a few arc minute error box) from the arrival times of the GRB photon front at widely spaced detectors onboard several different satellites that constitute what is known as an inter-planetary network (IPN). *Numerical* solutions of this Arrival Time Analysis problem have been developed and are in use since many years (K. Hurley, pvt. comm.). In the present work an *analytical* solution of this arrival time analysis (triangulation) problem of GRBs is described. The error in the determined position of the GRB source that results from position and timing errors of different satellites is also calculated. A simple method of cross-correlating two given GRB time-histories is discussed.

2. Method of calculation

If the positions of the satellites and the timings are exactly known, the GRB position may be determined exactly, i.e., it will be a point on the celestial sphere. In practice, however, there are uncertainties both in the positions of the satellites and in the timings. These uncertainties result in a corresponding uncertainty in the position of the GRB. Hence, the GRB source can be located only within an error box, the size of which is determined by the uncertainties in satellite positions and the timings. Using the timing information from two widely separated spacecrafts an annulus in the sky may be obtained. Using a third satellite two different annuli are obtained that intersect each other and result in two error boxes. If there is a fourth non-coplanar satellite in the network, then, using this information the ambiguity between the two error boxes may be resolved and a unique positional error box is obtained for the source of the detected GRB.

Sometimes multiple GRB monitors are flown on a single satellite. Using the relative intensities of detection, a coarse position is derived. This itself may remove the degeneracy between the two error boxes. Otherwise, earth occultation of one of the two directions also might be used for the same purpose. In that case the fourth satellite is not absolutely essential. On the other hand, having a network of a large number of widely separated satellites with on-board GRB monitors with good timing accuracy helps in determining very accurate source positions.

3. Determination of the mean radii of the annular rings

Let us consider three different satellites with their mean positions given in the geocentric equatorial co-ordinate frame, namely, r_i, α_i, δ_i where $i = 1, 2, 3$. r_i is the radial distance of the ith satellite from the geocenter, α_i is the right ascension and δ_i is the declination. Let t_i be the mean arrival time of the burst at the ith satellite. We transform r_i, α_i, δ_i by the following prescription, $r_i \Rightarrow r_i, \alpha_i \Rightarrow \phi_i, (90^o - \delta_i) \Rightarrow \theta_i$ (θ_i is the North Polar distance of the satellite). This coordinate system will be referred to as the original (O) system. It is a geocentric reference system in which the X-axis is the line $O\gamma$ where γ is the first point of Aries and the Z-axis points towards the celestial North pole.

$$\mathbf{r_{12}} = \mathbf{r_1} - \mathbf{r_2} \tag{1}$$

where $\mathbf{r_{12}}$ is the vector from the satellite which saw the burst second to the satellite which saw the burst first. The X-component of $\mathbf{r_{12}}$ is

$$r_{12x} = r_1 \sin\theta_1 \cos\phi_1 - r_2 \sin\theta_2 \cos\phi_2. \tag{2}$$

The Y- and Z-components of $\mathbf{r_{12}}$ are defined likewise. The three components of $\mathbf{r_{12}}$ in this spherical polar coordinate system are:

$$r_{12} = sqrt(r_{12x}^2 + r_{12y}^2 + r_{12z}^2), \tag{3}$$

$$\theta_{12} = \cos^{-1}(r_{12z}/r_{12}), \tag{4}$$

$$\phi_{12} = \tan^{-1}(r_{12y}/r_{12x}). \tag{5}$$

The relative time delay between the arrivals of the GRB front at the two satellites is

$$\Delta t_{12} = t_2 - t_1. \tag{6}$$

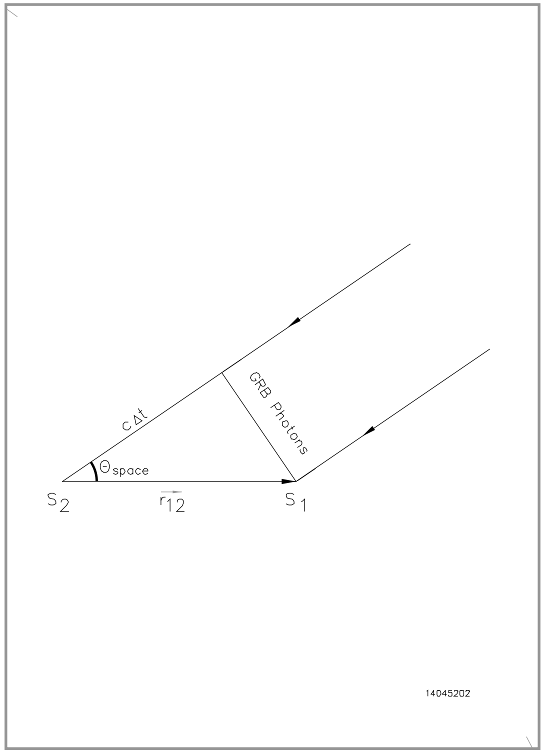
The mean radius of the annulus around $\mathbf{r_{12}}$ in which the GRB source is located is given by

$$\theta_{sp12} = \cos^{-1}(c * \Delta t_{12}/r_{12}) \tag{7}$$

(refer to Fig. 1a) where c is the velocity of light. Similarly, the mean radius of the annulus around $\mathbf{r_{13}}$ in which the GRB source is located is (when satellites 1 and 3 are considered),

$$\theta_{sp13} = \cos^{-1}(c * \Delta t_{13}/r_{13}) \tag{8}$$

where Δt_{13} is the time delay observed between detectors 1 and 3.



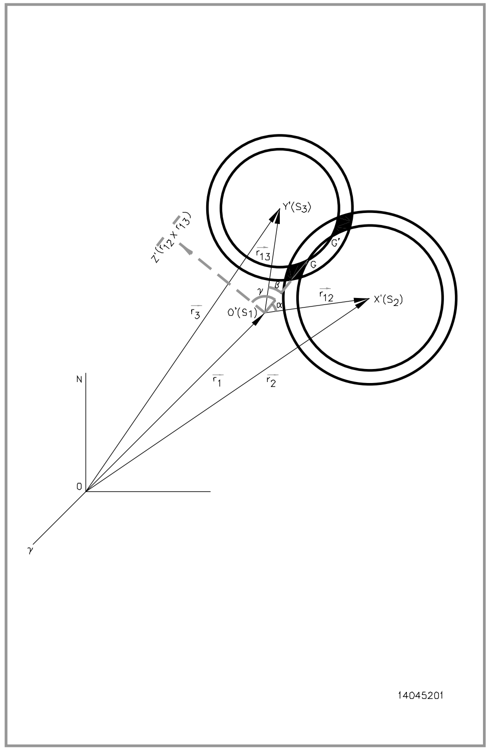


Figure 1. **(a)** Determination of the mean radius (θ_{space}) of the annular ring from the relative time delay of GRB arrivals at two satellites S_1 and S_2 separated by a distance r_{12}. **(b)** $O\gamma N$ is the geocentric coordinate system (O is the geocentre, γ is the first point of Aries and N is the celestial north pole). $O'X'Y'Z'$ is the oblique coordinate system, where $\mathbf{OO'}$, $\mathbf{OX'}$ and $\mathbf{OY'}$ are the position vectors of the three satellites 1, 2 and 3 respectively. $\mathbf{O'Z'}$ points along the vector ($\mathbf{r_{12}Xr_{13}}$). The two circles (having radii θ_{sp12} and θ_{sp13} respectively) intersect at two points G and G'. The GRB source is located in one of the two shaded error boxes (the degeneracy is removed by using a fourth non coplanar satellite). The vector $\mathbf{OG}$ makes angles α, β and γ with the three axes ($\mathbf{r_{12}}$, $\mathbf{r_{13}}$, and $\mathbf{r_{12}Xr_{13}}$) of the oblique coordinate system.

4. Determination of the widths of the annular rings

The error in θ_{sp} (the radius of the annulus) resulting from the uncertainty in time of arrival of the GRB ($\delta\Delta t_{ij}$) is given by

$$\mathrm{d}\theta_{spT} = (c * \delta(\Delta t_{ij}))/(r_{ij} * \sin(\theta_{sp})). \tag{9}$$

The magnitude of the error in the position vector $\mathbf{r_i}$, $\mathrm{d}\mathbf{r_i}$ is

$$\mathrm{mod}(\mathrm{d}\mathbf{r_i}) = sqrt(\mathrm{d}r_i^2 + r_i^2 \mathrm{d}\alpha_i^2 \cos^2\delta_i + r_i^2 \mathrm{d}\delta_i^2) \tag{10}$$

where $r_i = \mathrm{mod}(\mathbf{r_i})$. The maximum error in position is given by $\mathrm{d}\mathbf{r} = \mathrm{d}\mathbf{r_i} + \mathrm{d}\mathbf{r_j}$. The error in the radius of the annulus resulting from this position error of the two satellites is given by

$$\delta\theta_{spPOS} = (\mathrm{d}r_i + \mathrm{d}r_j)/(r_{ij} * \sin(\theta_{spij})). \tag{11}$$

The total width of the annulus (resulting from both timing as well as position uncertainties) is, therefore,

$$\delta\theta_{sp} = 2 * (\delta\theta_{spPOS} + \delta\theta_{spT}). \tag{12}$$

The two boundaries of the annulus are given by $\theta_{\mathrm{spin}} = \theta_{sp} - \delta\theta_{sp}$ and $\theta_{\mathrm{spout}} = \theta_{sp} + \delta\theta_{sp}$.

5. Calculation of the points of intersection

The two annuli intersect at totally eight points (two sets of four points each). Each set of four points constitute the four corners of an error box. The GRB source is located in one of these two error boxes. An oblique co-ordinate system (Fig. 1b) is chosen in which the three axes are along $\mathbf{r_{12}}(O'X')$, $\mathbf{r_{13}}(O'Y')$ and $\mathbf{r_{12}} X \mathbf{r_{13}}(O'Z')$, (the three satellites S_1, S_2 and S_3 are located at the points O', X' and Y' respectively). The angles $Y'O'Z' = \lambda = Z'O'X' = \mu = 90^o$ and $X'O'Y' = \nu$ where $\cos\nu$ may be calculated as

$$\sin\theta_{12}\cos\phi_{12}\sin\theta_{13}\cos\phi_{13} + \sin\theta_{12}\sin\phi_{12}\sin\theta_{13}\sin\phi_{13} + \cos\theta_{12}\cos\theta_{13} = \cos\nu. \tag{13}$$

Equation (13) is obtained by forming the dot product of the two unit vectors along $\mathbf{r_{12}}$ and $\mathbf{r_{13}}$. The vectors $\mathbf{r_{12}}$ and $\mathbf{r_{13}}$ are completely known in terms of the original $(O\gamma N)$ co-ordinate system. Let us consider one of the points of intersection, viz. that between the two inner circles. The direction cosines of the unit vector that joins the origin to this point of intersection (with respect to the oblique axes, $O'X'Y'Z'$) are $\cos\theta_{\mathrm{spin}}(\theta_{sp12in} = \theta_{sp12} - \delta\theta_{sp12})$, $\cos\theta_{sp13in}(\theta_{sp13in} = \theta_{sp13} - \delta\theta_{sp13})$ and X (unknown) respectively. Let us denote these three direction cosines by $\cos\alpha$ (α to be distinguished from right ascension), $\cos\beta$ and $\cos\gamma$ respectively. $\cos\gamma$ is unknown and may be calculated from the following relation (Bell 1960)

$$\begin{vmatrix} 1 & \cos\nu & \cos\mu & \cos\alpha \\ \cos\nu & 1 & \cos\lambda & \cos\beta \\ \cos\mu & \cos\lambda & 1 & \cos\gamma \\ \cos\alpha & \cos\beta & \cos\gamma & 1 \end{vmatrix} = 0 \tag{14}$$

which is equivalent to

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma\sin^2\nu - 2\cos\nu\cos\alpha\cos\beta = 1 - \cos^2\nu \tag{15}$$

(under the assumption $\lambda = \mu = 90^\circ$, i.e., $\cos\lambda = \cos\mu = 0$, $\sin\lambda = \sin\mu = 1$). This is the relation satisfied by the direction cosines of any line. The two distinct values of $\cos\gamma$ obtained from this relation correspond to the two different points of intersection between the two inner circles. Hence all the three angles, i.e., all the direction cosines are known in terms of the oblique co-ordinate system $(O'X'Y'Z')$. Next, a transformation is necessary to calculate the direction cosines, i.e. the angles Θ, Φ with respect to the original $(O\gamma N)$ co-ordinate system from the direction cosines with respect to the oblique co-ordinate system. This is done in the following manner.

Let the unit vector in the original $(O\gamma N)$ co-ordinate frame be denoted by $\mathbf{V}$. Its three components in the original co-ordinate system are $\sin\Theta\cos\Phi$, $\sin\Theta\sin\Phi$ and $\cos\Theta$ respectively. The matrix $\mathbf{A}$ that transforms this unit vector $\mathbf{V}$ (in the reference frame O) to the unit vector $\mathbf{V}'$ (say), in the oblique reference frame O' is given by

$$\begin{pmatrix} \sin\theta_{12}\cos\phi_{12} & \sin\theta_{12}\sin\phi_{12} & \cos\theta_{12} \\ \sin\theta_{13}\cos\phi_{13} & \sin\theta_{13}\sin\phi_{13} & \cos\theta_{13} \\ \sin\theta_{12}\sin\phi_{12}\cos\theta_{13} & \sin\theta_{13}\cos\phi_{13}\cos\theta_{12} & \sin\theta_{12}\cos\phi_{12}\sin\theta_{13}\sin\phi_{13} \\ -\sin\theta_{13}\sin\phi_{13}\cos\theta_{12} & -\sin\theta_{12}\cos\phi_{12}\cos\theta_{13} & -\sin\theta_{12}\sin\phi_{12}\sin\theta_{13}\cos\phi_{13} \end{pmatrix}.$$

The three elements of the last row are to be normalised by dividing each element by $sqrt(\Sigma_{j=1,3}A^2_{3j}$. This is necessary since the norm of the vector $\mathbf{r_{12}}$ X $\mathbf{r_{13}}$ is not unity. In this case, the vector $\mathbf{V}'$ is known and the components of the vector $\mathbf{V}$ are to be determined. This is done by the inverse transformation

$$\mathbf{V} = \mathbf{A^{-1}}\mathbf{V}'. \tag{16}$$

The inverse of the transformation matrix $\mathbf{A}$, $\mathbf{A^{-1}}$ is calculated. The three components of the vector $\mathbf{V}$ are determined from the three components of the vector $\mathbf{V}'$ (in the oblique co-ordinate system) using the following relationships.

$$\mathbf{V_i} = \Sigma_{j=1,3}\mathbf{A^{-1}}_{ij}\mathbf{V'_j}. \tag{17}$$

Here $\mathbf{V'_1} = \cos\alpha$, $\mathbf{V'_2} = \cos\beta$ and $\mathbf{V'_3} = \cos\gamma$. The calculated values of $\mathbf{V_i}$ are transformed to α (right ascension) and δ (declination) as follows: $\delta = 90.0^\circ - \Theta$ where $\Theta = \cos^{-1}(\mathbf{V_3})$. and $\Phi = \tan^{-1}(\mathbf{V_2}/\mathbf{V_1})$. If Φ becomes less than 0 than $\Phi = \Phi + 360.0^o$; $\alpha = \Phi$. Thus the right ascension and declination of the intersection point is known in the original $(O\gamma N)$ co-ordinate system. In this manner the co-ordinates of each corner of the error box may be determined. It is to be noted that these errors are one standard deviation errors.

6. Determination of timing uncertainties: Cross-correlations of GRB time-histories

Since a given time interval Δt is equivalent to a spatial interval of $\Delta l = c * \Delta t$, the timing uncertainties, in general, contribute more to the final GRB error box size.

Therefore, it is very crucial to determine the relative time delay between two spacecrafts as precisely and accurately as possible. The relative time delay and its error are determined by doing cross-correlations between the two GRB time-histories. Various methods of cross-correlations are used in practice (K. Hurley, pvt. comm.). We describe below the chi-square method. Generally, the integration time scales and the energy bands for any two GRB detectors are different. Rebinning of high resolution data is, therefore, necessary in order to make the integration times equal. The GRB time-histories are selected such that the energy bands are either the same or are very nearly so, since the time-history of a given GRB recorded by a given detector varies with the energy band selected. The counting rates for different detectors are different and they are either scaled up or down as needed. The χ^2 is calculated as

$$(\chi^2)_i = \Sigma_{j=1,m}(A_{i+j-1} - B_j)^2/(A_{i+j-1} + B_j) \tag{18}$$

where A_k and B_l represent the two time-histories (background subtracted). m is the number of integration time bins in one (say, the short one) of the time-histories. n is the number of time bins in the other (longer) time-history. The length (duration) of a GRB time-history depends on the sensitivity of the detector. Here $i = 1, 2, \ldots, n - k$. The degrees of freedom $\nu = m - 1$. The value of χ^2 is calculated by shifting one of the time-histories with respect to the other, one bin at a time. The relative time-delay obtained between the two time-histories when the χ^2 is minimum is the required quantity. The true minimum of the χ^2 and the corresponding time lag are determined by interpolation. The uncertainty (say, 3 sigma) in the time lag is calculated by determining the points on either side for which the $\chi^2 = \chi^2_{\min} + \chi^2_1(\alpha)$ where α is the significance.

7. Discussion

The present method of solving the arrival time analysis (triangulation) problem of GRBs is straightforward and conceptually simpler than the *numerical* methods. It should be possible to adapt this method to determine arrival directions of Extensive Air Showers (EAS) in Ultra High Energy (UHE) or in Very High Energy (VHE) gamma ray astronomy experiments that use the Atmospheric Cerenkov Wavefront Sampling Technique (Majumdar *et al.* 2002). Plane fronts of relativistic particles or visible cerenkov photons respectively are fitted using a rather complicated χ^2 -minimisation method to determine the arrival directions of EAS in these experiments.

8. Conclusion

An analytical method to determine the position of a GRB source from the timings of arrival of the GRB at different satellites and their position co-ordinates has been presented. The errors (one standard deviation) resulting in the esimated position of the GRB are calculated from the position and timing errors of different satellites. A simple method of cross-correlating gamma ray burst time-histories has been described. Usefulness of the present method of triangulation has been discussed.

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