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Black Hole Dynamics: A Survey of Black Hole Physics from the Point of View of Perturbation Theory

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Abstract. This article is a brief survey of the contribution of perturbative studies to our understanding of black hole physics. For natural reasons, I will not be able to discuss all details required for an exhaustive understanding of a field that has been active for the last forty years. Neither will I be able to cover all problem areas where perturbation theory has been applied. My aim is simply to provide the interested reader with a few pointers that can serve as useful starting points for an odyssey through the literature.

1. A brief history of . . .

Speculations about black holes – objects that are so compact that not even light can escape their gravitational pull – date back to the late 1780s, when John Michell was the first to combine Newtonian gravity with the corpuscular theory of light and suggest that massive stars may be "invisible". The idea soon went out of fashion as the wave-theory of light ascended, but it was brought back into the arena (albeit in a slightly different guise) with Einstein's general theory of relativity. The first black hole solution to Einstein's field equations was, of course, discovered by Schwarzschild almost immediately after the publication of Einstein's original paper. Thus it became clear that, from a theoretical point of view, black holes may exist (see Israel (1987) for a detailed account of the development of the concept of black holes).

Even though it may not have been generally appreciated at the time, several discoveries in the 1930s made a strong case for the actual existence of these exotic objects. Firstly, Chandrasekhar's 1931 proof that there is an upper limit on the mass of white dwarfs ($M \leq 1.4M_\odot$). Secondly, Chadwick's 1932 discovery of the neutron and the subsequent idea — due to Baade and Zwicky — that entire stars made up of these particles may exist. Such neutron stars would be limited to a mass less than something like $3M_\odot$. Finally, the seminal work on gravitational collapse by Oppenheimer and Snyder from 1939, that provided the first demonstration of how the implosion of a star forms a black hole.

Still, there were no observational evidence for the actual existence of black holes in the universe. Indeed, it was not clear how one would go about trying to observe an, in principle, invisible object. The first indications that there are black holes out there followed the launch of the UHURU satellite, and the true opening of the X-ray window, in 1970. During its three year lifetime the satellite discovered several hundred discrete X-ray sources in the sky. Among them was the first, and still one of the strongest candidates for a black hole: Cygnus X1. The lower limit on the mass of the invisible companion in this binary system is estimated to be at least 6 $M_\odot$. Thus, it is unlikely to be a neutron star, and by implication it can only be a black hole.

Recent observations, as described in other chapters of this volume, have provided further detailed evidence for black holes. Among these, the stunning pictures of the core of M87 taken by the Hubble Space Telescope that indicate a central mass of three billion solar masses, the observation of water masers and a disklike structure in NGC 4258 and the monitoring of stars close to the centre of our own galaxy immediately come to mind. The data indicate that most galaxies harbor a gigantic black hole at the centre. In a similar way, the case for solar-mass black holes have continued to strengthen with detailed observation of low-mass X-ray binaries. Given the recent observational advances there is every reason to expect that we will soon have a much improved understanding of astrophysical black holes. And within a few years a new generation of gravitational-wave detectors should come on-line. These promise to open up a new window to the universe. This could well lead to future astronomers being able to monitor black holes more or less daily. It is a fascinating prospect.

2. Stability

The relevance of the collapse work of Oppenheimer & Snyder (1939) was not immediately realized. One can easily identify two reasons for this. The most obvious one is the break out of the second world war. For several years, the world was in deep turmoil and not focused on basic (somewhat ethereal) research. A second reason is that the scientific community was not yet prepared to accept the inevitability of gravitational collapse. For example, the true non-singular nature of the event horizon had not yet been understood.

Oppenheimer & Snyder had considered a very idealized scenario — the collapse of a dust cloud: spherical, nonspinning, with no internal pressure or shocks etcetera. In this case the result indicated that the formation of a black hole was inevitable. But what about a less idealized – more realistic – case? What happens when a non-ideal star collapses? This question prompted John Wheeler and his students to pick up the torch after it had been left smouldering for almost twenty years. Their investigations led to what Kip Thorne has branded the "golden age of black hole physics" (Thorne 1994), and the answer to the question of non-ideal collapse was succinctly summarized by Richard Price in 1972: *Whatever can be radiated is radiated!* A collapsing star always settles down to a black hole without "hair", i.e. a black hole that can be fully described in terms of its mass, electric charge and angular momentum.

But we have jumped far ahead in the story. In 1957 John Wheeler and Tullio Regge published what can be considered the first paper on black hole perturbation theory (Regge & Wheeler 1957). Their motivation was to investigate whether a black hole was stable to external perturbations or whether it "would explode if an ant sneezed in it's vicinity" (as Vishveshwara so aptly has described it (Vishveshwara 1998). To address the stability issue Regge and Wheeler derived the equations that describe a slightly deformed black hole. This is done by assuming linear perturbations

$$g_{\mu\nu} = g_{\mu\nu}^{\text{background}} + h_{\mu\nu}, \qquad \text{where } |h_{\mu\nu}| \ll 1. \tag{1}$$

The perturbations of a Schwarzschild black hole can then be divided into two classes. The first induces inertial frame-dragging (rotation) and is often referred to as axial (or odd parity). The second class corresponds to perturbations that remain unchanged after sign of φ. These are called polar (or even parity) perturbations. In the case of non-rotating black holes these two classes decouple and can be studied separately.

Axial perturbations are governed by what is now known as the Regge-Wheeler equation

$$\frac{\partial^2 \psi}{\partial r_*^2} - \frac{\partial^2 \psi}{\partial t^2} - V\psi = 0, \tag{2}$$

where

$$V = \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} - \frac{2M(1-s^2)}{r^3}\right]. \tag{3}$$

Here s is the spin-weight of the perturbing field, l is the integer index of the spherical harmonic used to describe the angular properties of the perturbation, and the tortoise coordinate is defined as

$$\frac{\partial}{\partial r_*} = \left(1 - \frac{2M}{r}\right)\frac{\partial}{\partial r}. \tag{4}$$

This translates the part of spacetime accessible to a causal observer into the range $-\infty \leq r_* \leq \infty$. (In other words, the event horizon is pushed all the way to $r_* = -\infty$ A similar equation (the Zerilli equation (Zerilli 1970)) can be derived for polar perturbations. In other words, the description of perturbed black holes involves solving a wave equation with a deceivingly simple effective potential.

Given the above differential equation we can address the main question for black hole stability: Will a perturbation of a black hole become unbounded if evolved according to the linear equations? A satisfactory answer to this question was first provided by Vishveshwara (1970a). He showed how one can derive (by multiplying (2) with its complex conjugate and integrating) the following "energy integral"

$$\int_{r_*=-\infty}^{r_*=+\infty}\left[\left|\frac{\partial \psi}{\partial t}\right|^2 + \left|\frac{\partial \psi}{\partial r_*}\right|^2 + V|\psi|^2\right] \mathrm{d}r_* = \text{constant}. \tag{5}$$

Since V is positive definite this bounds $\partial\psi/\partial t$ and excludes exponentially growing solutions. This implies that there are no unstable modes of a nonrotating black hole.

However, the simple argument above leaves a few loopholes through which an instability might sneak in. For example, perturbations that grow linearly (or slower) with t are not ruled out. Also, we have only provided a bound for integrals of'. ψ. The perturbation may still blow up in an ever narrowing spatial region. For non-rotating black holes these gaps were filled by Kay & Wald (1987), who proved that ψ) remains pointwise bounded when evolved from smooth, bounded initial data. In other words, Schwarzschild black holes are stable for initial data that has compact support on the Kruskal extension.

Studies of the stability of rotating black holes (which is the most relevant case from an astrophysical point of view) are not as straightforward. Because of the nature of the perturbation equations (see an example later) one cannot readily derive an energy integral like equation (5). That it is possible to do this, and hence prove mode-stability of Kerr black holes, was shown by Bernard Whiting (1989) using an intricate set of coordinate transformations. A complete proof of the stability of Kerr black holes (a la Kay & Wald (1987)) is still outstanding.

3. Quasinormal modes

It was soon realized that the perturbation equation can also be used to provide information about how a black hole interacts with its environment. One can, for example, study how a black hole reacts if one were to "kick" it in some way. Work in this direction was pioneered by Vishveshwara (1970b). As he has subsequently described it (Vishveshwara 1998): The question was "*how do you observe a solitary black hole? To me the answer seemed obvious. It had to be through scattering of radiation, provided the black hole left its fingerprint on the scattered wave …. So, I started pelting the black hole with Gaussian wave packets. If the wave packet was spatially wide, the scattered one was affected very little. It was like a big wave washing over a small pebble. But when the Gaussian became sharper, maxima and minima started emerging, finally levelling off to a set pattern when the width of the Gaussian became comparable to or less than the size of the black hole. The final outcome was a very characteristic decaying mode, to be christened later as the quasinormal mode. The whole experiment was extraordinarily exciting.*"

During the 1970s the perturbation equations were used to study black holes in many dynamical situations, such as small bodies falling into (or being scattered by) a black hole (Davis *et al.* 1971), and slightly nonspherical gravitational collapse (Cunningham, Price & Moncrief 1979). It was found that the emerging radiation shows similar features in all cases (cf. Fig. 1). The initial response consists of a broad-band burst, followed by the quasinormal-mode ringing and finally, at late times, a power-law fall-off. Remarkably, the last two features are independent of the nature of the perturbing agent. They reflect the detailed nature of a black hole spacetime.

Since their serendipitous discovery, the quasinormal modes have attracted considerable attention. Still, the quasinormal-mode spectra of various black holes were not completely unveiled until rather recently (and there are still some outstanding questions). This reflects the fact that the mode-problem is not trivial. The reason for the underlying difficulty is, however, easily explained. The effective potential V is of short range, and corresponds to a single potential barrier. This means that the black hole problem is in many ways similar to one of potential scattering in quantum mechanics. The quasinormal modes are solutions to the equation that do not depend on the character of waves falling onto the black hole. In effect, they must be solutions to (2) that satisfy the causal condition of purely ingoing waves crossing the event horizon, while at the same time behaving as purely outgoing waves reaching spatial infinity (the quasinormal modes are analogous to the resonances in quantum scattering). Assuming a time-dependence e^{-iwt}, a general causal solution to (2) is prescribed by the asymptotic behaviour

$$\psi \sim \begin{cases} e^{-i\omega r_*} & \text{as } r \to 2M, \\ A_{\text{out}} e^{i\omega r_*} + A_{\text{in}} e^{-i\omega r_*} & \text{as } r \to +\infty. \end{cases} \tag{6}$$

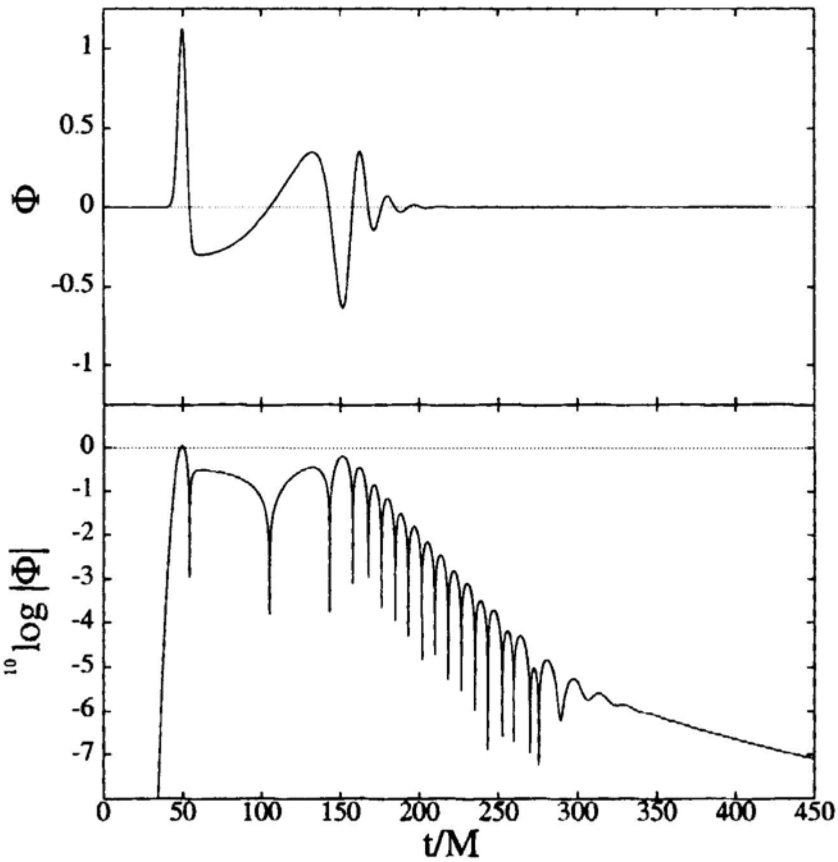

Figure 1. A recreation of Vishveshwara's scattering experiment: The response of a Schwarzschild black hole as a Gaussian wavepacket of scalar waves impinges upon it. The first bump (at $t = 50\ M$) is the initial Gaussian passing by the observer on its way towards the black hole. Quasinormal-mode ringing clearly dominates the signal after $t \approx 150\ M$. At very late times (after $t \approx 300\ M$) the signal is dominated by a power-law fall-off with time.

In this description, the quasinormal modes correspond to $A_{\text{in}} = 0$. To identify a quasinormal-mode solution we must be able to determine a solution that behaves as $e^{i\omega r_*}$ as $r_* \to \infty$, with no admixture of ingoing waves. We require that the mode is damped according to an observer at a fixed location (since we have already proved that the black hole is stable — no unstable mode-solutions exist). This means that Im $\omega_n < 0$. Thus, the problem involves identifying solutions that diverge exponent-tially as $r_* \to \infty$, on a constant t hypersurface. A similar problem arises at the horizon. There are by now several accurate methods for handling this difficulty and unveiling the entire spectrum of quasinormal modes (Leaver 1985; Nollert & Schmidt 1992; Andersson 1997).

Although we are not going to discuss the details of the mode-spectrum here, it is worthwhile giving a flavour of the astrophysically most important modes. First of all, the detection of a mode-signal from a black hole in anything other than gravitational waves is extremely unlikely. The reason for this is that, at the frequencies of a typical quasinormal mode an electromagnetic wave is not expected to travel far in the interstellar medium. So we focus our attention on gravitational perturbations. For a quadrupole perturbation of a Schwarzschild black hole, the fundamental gravitational-wave quasinormal mode has frequency

$$f \approx 12\,\mathrm{kHz}\left(\frac{\mathrm{M}_\odot}{M}\right), \tag{7}$$

while the associated e-folding time is

$$\tau \approx 0.05\,\mathrm{ms}\left(\frac{M}{\mathrm{M}_\odot}\right). \tag{8}$$

The various overtones all have shorter e-folding times. Hence, the quasinormal modes of a black hole are very short lived. We can compare a black hole to other resonant systems in nature by defining a quality factor

$$Q \approx \frac{1}{2}\left|\frac{\mathrm{Re}\,\omega_n}{\mathrm{Im}\,\omega_n}\right|. \tag{9}$$

For the quasinormal modes we then find $Q \approx l$. This should be compared to the result for the fundamental fluid pulsation mode of a neutron star: $Q \sim 1000$, or the typical value for an atom: $Q \sim 10^6$. The Schwarzschild black hole is clearly a very poor oscillator in comparison to these other systems.

It is worthwhile mentioning a few outstanding problems regarding quasinormal modes. The calculation of quasinormal-mode frequencies is no longer a challenge. But there are still facets of the mode problem that require a deeper understanding. There have been a few studies of the excitation of quasinormal modes, i.e. to what extent the various modes are excited by given initial data (Leaver 1986; Sun & Price 1988; Andersson 1995), but this work needs to be extended considerably if we are to achieve a complete understanding. This is particularly important for rapidly spinning black holes, for which some quasinormal modes become almost undamped (Leaver 1985). Such modes would seem to be ideal for gravitational-wave detection (Finn 1992), but this will only be the case if they are actually excited to an appreciable level. At the present time this has not been demonstrated to be the case. Also, recent work has uncovered some peculiarities in the behaviour of the quasinormal mode frequencies as the black hole gets near to either maximal rotation or maximal electric charge (Andersson & Onozowa 1996; Onozowa 1997). Specifically, the overtone modes seem to spiral onto a limiting complex frequency as the black hole becomes extreme. It is interesting to speculate about the reason for this behaviour. It may be that this is an issue of little physical relevance, but it may also be that the underlying physics dictates this strange behaviour and that we can learn something new by studying it further (Kallosh *et al.* 1998).

4. Late-time tails

As already mentioned, the response of a black hole to any external perturbation is dominated by a power-law fall-off at very late times (see Fig. 2). This feature was first noticed by Richard Price (1972) in his studies of gravitational collapse. He deduced that a perturbation corresponding to a certain multipole l will fall off as an

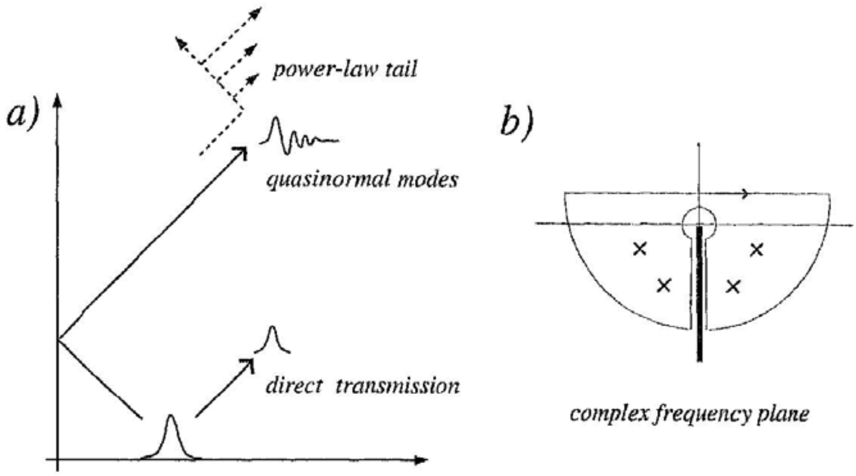

Figure 2. (a) Schematic description of a black hole's response to initial data of compact support. The directly transmitted wave (from a source point y) arrives at a distant observer (at r_*) roughly at $t - r_* + y = 0$. The black holes response, that is dominated by quasinormal mode ringing, reaches the observer at roughly $t - r_* - y = 0$. At very late times the signal falls off as an inverse power of time. This power-law tail arises because of multiple backscattering off the spacetime curvature, (**b**) Integration contours in the complex frequency plane. The original inversion contour for the Green's function lies above the real frequency axis. When analytically continued in the complex plane this contour can be replaced by the sum of 1) the quasinormal modes [the singularities of the Green's function; the first few are represented by crosses in the figure] 2) an integral along the branch cut (a thick line along the negative imaginary ω axis in the figure), that leads to the power law tail, and 3) high frequency arcs (that one would expect vanish at most times, but they should also lead to roughly "flat space propagators" at early times).

inverse power of time

$$\psi \sim t^{-2l-3} \qquad \text{as } t \to +\infty \text{ for constant } r. \tag{10}$$

This power-law tail arises because of backscattering off the slightly curved spacetime far away from the central object (Ching *et al.* 1995). The tail is generic and, in fact, independent of the existence of a horizon (Gundlach *et al.* 1994). Similar tails will arise in the spacetime exterior to a star, a black hole or an imploding/exploding shell of matter as long as the mass involved is the same. Furthermore, since the behaviour in the far zone dictates the nature of the tail, Kerr black holes must have tails similar to the Schwarzschild one (Andersson 1997).

It should also be made clear that the late-time tail is radiative: It exists both at the future horizon and at future infinity (Leaver 1986; Gundlach *et al.* 1994).

$$\psi \sim \begin{cases} u^{-l-2} & \text{as } u \to +\infty, \\ v^{-2l-3} & \text{as } v \to +\infty. \end{cases} \tag{11}$$

It is instructive to discuss the way that the tail and the quasinormal modes arise in a dynamical scenario in a little bit more detail. Suppose we are given some (perturbative) scalar field on a spacelike hypersurface $t = 0$ (say), and that we want to predict the future behaviour of the field. That is, we want to solve

$$\Box\Phi = 0, \tag{12}$$

for a specific set of initial data. In spherical symmetry it is useful to introduce

$$\Phi_{\ell m} = \frac{u_\ell(r_*, t)}{r} Y_{\ell m}(\theta, \varphi). \tag{13}$$

The function $u_\ell(r_*,t)$ then solves the Regge-Wheeler equation (with $s = 0$), and the future evolution of a field given at some initial time ($t = 0$) follows from

$$u_\ell(r_*, t) = \int G(r_*, y, t)\partial_t u_\ell(y, 0)\mathrm{d}y + \int \partial_t G(r_*, y, t)u_\ell(y, 0)\mathrm{d}y. \tag{14}$$

Where G is the appropriate (retarded) Green's function, and $G(r_*, y, t) = 0$ for $t \leq 0$.

The above problem is usually analyzed in the frequency domain (after Fourier decomposition). Then we can use complex frequencies to deduce the character of the Green's function. This way we find that the quasinormal modes are the poles of the Green's function, and we can account for them by means of the residue theorem (Leaver 1986; Andersson 1997). In order to do this quantitatively, we need more information than the mode-frequency itself. In the general case we need also the exact form of the mode-eigenfunctions, and subsequently we must evaluate integrals of products of these functions. This is a truly difficult task, and even though it may lead to an accurate representation of parts of the black holes response (Leaver 1986; Sun & Price 1988), it is not very instructive. It is usually better to proceed via approximations. One such approach is based on assuming that (**i**) the observer is situated far away from the black hole, (**ii**) the initial data has considerable support only far away from the black hole, and (**iii**) the initial data has no support outside the observer. These assumptions facilitate an analytic approximation of quasinormal-mode excitation. Specifically we find that the mode-contribution to the Green's function is (Andersson 1997)

$$G^{QNM}(r_*, y, t) = \mathrm{Re}\left[\sum_{n=0}^{\infty} \frac{A_{\mathrm{out}}(\omega_n)}{\omega_n \alpha_n} e^{-i\omega_n(t-r_*-y)}\right], \tag{15}$$

where

$$A_{\mathrm{in}}(\omega) \approx (\omega - \omega_n)\alpha_n, \tag{16}$$

and the sum is over all modes in the fourth quadrant of the complex ω-plane. This approximate result is quite useful. First of all, we can now readily estimate the mode-excitation in situations where our underlying assumptions are relevant. We can also gain some insight in the convergence of the mode-sum at different times (Andersson 1997). For example, based on the above result it would seem natural to introduce the concept of a "dynamical" mode excitation. To ensure causality G should vanish for $t - r_* + y > 0$. This translates into a lower limit of integration $y = r_* - t$. Similarly, the contribution from the high-frequency arcs in the lower half of the ω-plane will diverge unless $t - r_* - y > 0$. This introduces an upper limit of integration $y = t - r_*$. Once these limits are used, we find that the mode-contribution converges at all times, and represents the main part of the signal very well (Andersson 1997).

In the complex-frequency picture, the late-time tail is associated with a branch cut (usually taken along the negative imaginary axis) in G(r_*, y, t). Analysis of this branch cut contribution leads to (Leaver 1986; Andersson 1997)

$$G^{\text{tail}}(r_*,y,t) = (-1)^{\ell+1}\frac{(2\ell+2)!}{[(2\ell+1)!!]^2}\frac{4M(r_*y)^{\ell+1}}{t^{2\ell+3}} \quad (17)$$

to leading order. But this result is only accurate for very late times. To study the regime where quasinormal ringing gives way to the power-law fall-off we must include several higher order terms.

5. The coalescence of spinning black holes

With the advent of gravitational-wave astronomy not more than a few years away, an enormous effort is focused on predictions of waveforms for what are anticipated to be the most relevant scenarios. Without accurate theoretical templates one will not be able to dig out weak signals from a typically noisy datastream, and may miss out on many interesting events. Perturbation theory is playing a relevant role in this modeling. It has for example been used to address issues related to the convergence of the post-Newtonian expansion used to describe the inspiral phase of a binary system (see Sasaki's contribution to this volume). Somewhat surprisingly, the perturbation approach has also provided relevant results for the eventual coalescence of two black holes.

As was realized by Richard Price & Jorge Pullin (1994) a few years ago, the final stages of the collision of two black holes can be approximated using perturbation theory. The idea behind what is now commonly referred to as the "close-limit" approximation is very simple. Assume that the two black holes are surrounded by a common horizon. If so, they can be considered as a single perturbed black hole. For example, the standard Misner initial data set can be viewed as representing a "Schwarzschild background + something else". In this picture, the full problem is reduced to an initial-value problem for the Zerilli equation. This problem can readily be solved, and the results compare favourably with fully nonlinear numerical relativity simulations (Anninos *et al.* 1995). Why is this, seemingly naive, approximation such a success? A reasonable explanation (corroborated by the full nonlinear calculation) is that the spacetime is only strongly distorted in the region close to the horizon. Because of the existence of the potential barrier outside the black hole most of this perturbation be scattered back onto the black hole. The waves that reach a distant observer mainly originate from the region outside the peak of the potential, where the initial perturbation is much smaller and linearized theory is a reasonable approximation.

The close-limit approach was first used to study head-on collisions of two non-rotating black holes. It has subsequently been used to investigate other cases, such as (**i**) boosted (radially or perpendicularly) holes and (**ii**) the aptly named "cosmic screw" (two colliding black holes with opposite spins) (Nollert 1996). All these cases can be viewed as perturbations of a final Schwarzschild black hole. This is, however, not the generic case since one would expect that the coalescence of two black holes forms a (rapidly) spinning black hole. Hence, one would like to be able to formulate a close-limit approximation based on perturbations of a final Kerr black hole. Several groups are presently working towards this goal. Several pieces are required before the answer can be puzzled together. First one must formulate the general initial-value problem and translate it into "Kerr + perturbations". This is not trivial. One main difficulty is associated with the fact that for Kerr black holes in Boyer-Lindquist coordinates, the spacelike slices are not conformally flat. This means that the standard initial-value formulation fails, and one must come up with a workable alternative (Gleiser *et al.* 1998; Campanelli *et al.* 1998; Krivan & Price 1998). A second part needed for the close-limit calculation is an evolution scheme for perturbations of Kerr black holes. Such a scheme has recently been put together, and it has been tested in several different ways.

Perturbations of rotating black holes are described by an equation first derived by Saul Teukolsky (1972). For a scalar field this equation can be written (using Boyer-Lindquist coordinates)

$$\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\varphi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\varphi^2} \tag{18}$$

$$-\frac{\partial}{\partial r}\left(\Delta\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) = 0, \tag{19}$$

were $\Delta \equiv r^2 - 2Mr + a^2$. Here it should be noted that: (**i**) We can always separate the azimuthal angle in the standard way ($\psi \sim e^{im\varphi}$), but the separation of the radius and the polar angle lead to a frequency-dependent separation constant. This means that time evolutions are best done using two space dimensions, i.e. coordinates (t, r, θ). (**ii**) The reason why the "energy integral" approach to stability fails becomes clear. It fails because of the existence of the ergosphere (due to the change of sign of the coefficient of $\partial^2\psi/\partial\varphi^2$). Inside

$$r_{es} = M + \sqrt{M^2 - a^2\cos^2\theta}, \tag{20}$$

observers cannot be stationary, and energy can be negative (as viewed from infinity).

The recent codes for evolving the Teukolsky equation (for scalar and gravitational perturbations) were written by William Krivan, Pablo Laguna & Philip Papadopoulos (1996) (with the present author as some kind of partially interacting external observer). These codes have to date been used to revisit problems that had previously been approached in the frequency domain. This served to test the codes and also give a new perspective on somewhat familiar results. This way it has been verified that the Kerr late-time tail is identical to Schwarzschild, but different multipoles are mixed due to (**i**) rotational effects, and (**ii**) imperfect initial data (Krivan, Laguna & Papadopoulos 1996). It was also demonstrated that there will typically, for a given m, be two distinct regimes of mode-ringing in the Kerr case (Krivan *et al.* 1997). This happens because the quasinormal mode frequencies of a Kerr black hole are no longer symmetrically placed relative to the Imω axis as in the case of Schwarzschild. Instead, if ω is a mode corresponding to l and m the conjugate $-\bar{\omega}$ will be a mode for l and $-m$. As $a \to M$ some quasinormal modes become very slowly damped. We have verified that these modes lead to very long-lived mode ringing for rapidly rotating black holes. This is an important step towards assessing the relevance of these modes for gravitational-wave detection. Finally, we have demonstrated how a slight amplification due to superradiance can be extracted (Andersson, Laguna & Papadopoulos 1998), but also that it demands very special initial data for this effect to be observable in a time-evolution situation. Taken together, these results show that the Teukolsky codes provide reliable tools that can be used to study many different problems in black hole physics.

6. Final words

In this short article I have tried to describe some of the ways in which perturbation theory has been used to shed light on the physics of black holes. I hope I have managed to show that one can learn a lot about black holes simply by studying their dynamics at the perturbative level. While I have mainly reviewed past efforts it seems logical to end this contribution with a look to the future. To me, it seems likely that the perturbative approach will continue to play a relevant role. First of all, fully nonlinear black hole calculations will always require benchmark tests to prove their reliability. In the appropriate limit the ideal code test is a comparison to perturbative results. There are also several ways in which perturbative methods can be used within a nonlinear calculations, e.g. to extract waves in the far zone. These are important applications, but perturbation theory is too powerful to serve as a simple slave to numerical relativity. A perturbative analysis has a considerable predictive power. If we have learned any lessons from the past – from the unveiling of the relevance of the quasinormal modes in most situations to the success of the close-limit approximation for black-hole collisions – it should be that the perturbation approach tends to be more accurate than one has any reason to expect. I would not be at all surprised if future results continue to bring home this message.

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