J. Astrophys. Astr. (1997) **18,** 23–32

Gravitational Potential Energy of Interpenetrating Spherical Galaxies in Hernquist's Model

K. S. V. S. Narasimhan, K. S. Sastry* & Saleh Mohammed Alladin
Inter University, Centre for Astronomy and Astrophysics, Ganeshkhind, Pune 411007, India
** Permanent Address: Department of Astronomy, Osmania University, Hyderabad 500007.*

Received 1996 February 15; accepted 1996 September 19

Abstract. Hernquist's (1990) mass model for spherical galaxies and bulges described by the deVaucouleur's profile gives analytical expressions for the density profile and the potential. These have been used to derive a simple and exact analytical expression for the gravitational potential energy of a pair of interpenetrating spherical galaxies represented by this model. The results are compared with those for polytropic and Plummer models of galaxis.

Keywords: Stellar systems: dynamics—galaxies: interactions.

1. Introduction

The force of attraction between two galaxies when they interpenetrate is weaker than the force which one would expect from the inverse square law. The departure of the intergalactic force from the inverse square force can be accounted for by writing the mutual potential energy of interaction of two spherical galaxies of masses M_1 and M_2 separated by a distance r in the form

$$W(r) = -\frac{GM_1M_2}{r}\psi \tag{1}$$

(Alladin 1965). The correction factor ψ is a function of the separation r and the density distributions $\rho(r_1)$ and $\rho(r_2)$ of the galaxies, ψ, derived from potential theory corrects for the fact that one is actually dealing with extended configurations and not mass points.

The forces due to overlap of galaxies can also be taken into account by introducing a softening parameter ε and writing,

$$W(r) = -\frac{GM_1M_2}{r}\left(1+\frac{\varepsilon^2}{r^2}\right)^{-0.5} \tag{2}$$

(Aarseth 1966) which gives

$$\psi_{AA} = \frac{r}{(r^2+\varepsilon^2)^{0.5}}. \tag{3}$$

In this paper, we derive an analytical expression for the interaction potential energy of two overlapping spherical galaxies, represented by Hernquist's (1990) model, not necessarily having the same scale lengths and compare the results with those obtained by earlier workers, mostly by numerical methods.

The salient features of the model used in the present analysis are given in section 3.

2. A review of the earlier works

Using the analysis of spherically symmetric matter by Limber (1961), Alladin (1965) determined $W(r)$ of two galaxies treating them as the superposition of polytropes of integral indices $n = 0$ through 5 having a common radius R and writing;

$$\psi_{AL} = \psi(n_1, n_2, r/R), \tag{4}$$

r being the intergalactic separation and n_1 and n_2, the indices of the polytropes. He has tabulated ψ for two galaxies having the same size. Potdar & Ballabh (1974) extended his work to penetrating galaxies of unequal dimension, using

$$\psi_{PB} = \psi(k, n_1, n_2, s), \tag{5}$$

where $k = R_1/R_2$,the ratio of the radii of the two galaxies, and $s = r/R_2$. Numerical estimates for $k = 1,2,5$ and 10 have been made by them. Estimates for ψ for two disk galaxies have been given by Ballabh (1973) and for disk-sphere galaxies by Ballabh (1975).

Alladin & Narasimhan (1982) have pointed out that equation (3) agrees very closely with equation (4), if

$$\varepsilon = [(\psi/r)_{r=0}]^{-1}. \tag{6}$$

Detailed discussion on this is given in Zafarullah, Narasimhan & Sastry (1983). The physical significance of ε can be seen from the fact that

$$\varepsilon = \overline{R_{12}(0)} \equiv [\langle 1/r_{12}(0) \rangle]^{-1}, \tag{7}$$

where $r_{12}(0)$ is the distance between a star in the galaxy of mass M_1 and a star in the galaxy of mass M_2 when the centres of the galaxies coincide and the separation r between the galaxies is measured in units of $\overline{R_{12}(0)}$. (Narasimha Rao, Alladin & Narasimhan 1994)—The use of ε adequately takes into account the dependence of potential energy on density distribution.

Narasimha Rao, Alladin & Narasimhan (1994) wrote:

$$\psi_{NAN} = \frac{r}{(r^2 + \varepsilon^2)^{0.5}}, \quad \varepsilon = \overline{R_{12}(0)}, \tag{8}$$

where ε was not obtained analytically, but was adopted from comparison with the results of the polytropic model obtained numerically. The values of $\overline{R_{12}(0)}$ for polytropes of different indices have been tabulated by Alladin (1965); those for the Plummer model galaxies with different scale lengths are given in Narasimhan and Alladin (1986).

In their numerical simulations of galactic collisions Aarseth and Fall (1980) have used

$$\psi_{AF} = \frac{r}{(r^2 + \alpha_1^2 + \alpha_2^2)^{0.5}}, \tag{9}$$

where α_1 and α_2 are the scale lengths of two galaxies represented by the Plummer model. This is a convenient analytic expression but it has not been obtained rigorously from the basic equation for $W(r)$ given in section 4 (equation 20).

The relationship between equations (4), (8) and (9) has been discussed by Narasimha Rao *et al* (1994). It is shown in this paper that Hernquist's (1990) model which has the advantage of representing properly deVaucouleur's intensity profile for elliptical galaxies gives simple and exact analytical expression for $W(r)$.

3. Hernquist's model: Basic equations

An analytic mass model for spherical galaxies and bulges described by the de Vaucouleur's (1948) $R^{1/4}$ profile for elliptical galaxies has been proposed by Hernquist (1990). He has shown that its intrinsic properties and projected distribution lend themselves to analytical treatment. This has been the motivation for the analysis in the next section. This model is a special case in the family of models for spherical stellar systems developed by Tremaine *et al.* (1994) and is one of the most successful analytical models for elliptical galaxies and the bulges of spiral galaxies. It has as $\rho\alpha r^{-4}$ profile in its outer parts and a central density cusp of strength $\rho\alpha r^{-1}$

The density profile is given by

$$\rho(r) = \frac{M\alpha}{2\pi r}\frac{1}{(r+\alpha)^3}, \tag{10}$$

where M is the total mass and α, the scale length. It follows that the mass interior to r and the potential are given respectively by

$$M(r) = M\frac{r^2}{(r+\alpha)^2} \tag{11}$$

and

$$V(r) = -\frac{GM}{r+\alpha}. \tag{12}$$

By defining

$$\bar{\rho} = \frac{2\pi\alpha^3}{M}\rho(r) = \frac{\alpha^4}{r(r+\alpha)^3} \tag{13}$$

and

$$\bar{\psi} = -\frac{\alpha}{GM}V(r) = \frac{\alpha}{r+\alpha}, \tag{14}$$

we get a simple relationship

$$\bar{\rho} = \bar{\psi}^4/(1-\bar{\psi}) \tag{15}$$

between the density and the potential (Hernquist 1990)

The self gravitational potential energy is given by

$$|\Omega| = \frac{GM^2}{6\alpha_H}. \tag{16}$$

Hence the dynamical radius is $\bar{R} = 3\alpha_H$.

We note from equation (11) that the mass is finite, even though the galaxy extends to infinity. The half-mass radius or the median radius R_h is given by

$$R_h = (\sqrt{2} + 1)\alpha. \tag{17}$$

About 90% of the mass lies within 18α.

Fish (1964) obtained the following empirical relationship between the potential energy Ω and the mass M of elliptical galaxies:

$$|\Omega| = 9.6 \times 10^{-8} M^{1.5} \text{ (C.G.S. units)}. \tag{18}$$

Using this, in conjunction with equation (16), we find that the mass M of a Hernquist's model galaxy is related to the scale length α by

$$M = 73.9\,\alpha^2 \text{ (C.G.S. units)}. \tag{19}$$

This gives an approximate dimension associated with a given mass.

In the next section, we derive an analytical expression for $W(r)$ for galaxies represented by this model.

4. Gravitational potential energy of interpenetrating galaxies in Hernquisfs model

Let two galaxies of masses M_1 and M_2, with centres at O_1 and O_2 separated by a distance r, slightly overlap each other. The gravitational potential energy $W(r)$ of the galaxies arising from only their mutual attraction on each other or the interaction potential energy is given by

$$W(r) = \int_{M_2} V(r_1)\,\mathrm{d}M_2, \tag{20}$$

where $V(r_1)$ is the potential due to the galaxy of mass M_1 at a distance r_1 from O_1, and $\mathrm{d}M_2$ is the element of the galaxy of mass M_2 lying at the distance r_1. The integration is carried out over the entire mass M_2.

Choosing O_2 as origin, the polar axis in the direction of O_1 and $\mathrm{d}M_2$ in spherical polar coordinates r_2, θ_2, ϕ_2, equation (20) reduces to

$$W(r) = \int_0^{2\pi}\int_0^{\infty}\int_0^{\pi} V(r_1)\rho(r_2) r_2^2 \sin\theta_2\,\mathrm{d}\theta_2\,\mathrm{d}r_2\,\mathrm{d}\phi_2, \tag{21}$$

where r_2 is the distance of d M_2 from O_2, $\rho(r_2)$ is the mass density of M_2 at the distance r_2 and θ_2 is the polar angle. r_1 and r_2 are connected by the relation

$$r_1^2 = r^2 + r_2^2 - 2 r r_2 \cos\theta_2.$$

Integrating over the azimuthal angle and using equations (10) and (12), we get

$$|W(r)| = G M_1 M_2 \alpha_2 \int_0^{\infty}\int_0^{\pi} \frac{r_2 \sin\theta_2\,\mathrm{d}r_2\,\mathrm{d}\theta_2}{(r_2+\alpha_2)^3\left[(r^2+r_2^2-2rr_2\cos\theta_2)^{0.5}+\alpha_1\right]}. \tag{22}$$

Integration over θ_2 then yields

$$|W(r)| = \frac{G M_1 M_2 \alpha_2}{r}\int_0^{\infty}\left[\frac{2r_2}{(r_2+\alpha_2)^3} - \alpha_1\frac{\ln(r+r_2+\alpha_1)}{(r_2+\alpha_2)^3} + \alpha_1\frac{\ln(r-r_2+\alpha_1)}{(r_2+\alpha_2)^3}\right]\mathrm{d}r_2$$

$$= \frac{GM_1M_2}{r}\left[2\alpha_2 I_1 - \alpha_1\alpha_2 I_2 + \alpha_1\alpha_2 I_3\right], \tag{23}$$

where

$$I_1 = \frac{1}{2\alpha_2},$$

$$I_2 = \frac{1}{2}\left[\frac{\ln(r+\alpha_1)}{\alpha_2^2} + \frac{1}{\alpha_2(r+\alpha_1-\alpha_2)} - \frac{\ln\{(r+\alpha_1)/\alpha_2\}}{(r+\alpha_1-\alpha_2)^2}\right],$$

$$I_3 = \frac{1}{2}\left[\frac{\ln(r+\alpha_1)}{\alpha_2^2} - \frac{1}{\alpha_2(r+\alpha_1+\alpha_2)} - \frac{\ln\{(r+\alpha_1)/\alpha_2\}}{(r+\alpha_1+\alpha_2)^2}\right]. \tag{24}$$

From equations (23) and (24), after some algebra, we get

$$|W(r)| = \frac{GM_1M_2}{r}\left[1 - \frac{s_1+1}{(s_1+1)^2-\alpha_{21}^2} + 2\alpha_{21}^2\frac{s_1+1}{[(s_1+1)^2-\alpha_{21}^2]^2}\ln\left(\frac{s_1+1}{\alpha_{21}}\right)\right], \tag{25}$$

where $s_1 \equiv \frac{r}{\alpha_1}$ and $\alpha_{21} = \alpha_2/\alpha_1$

If $\alpha_2 \ll \alpha_1$, then

$$|W(r)| = \frac{GM_1M_2}{r+\alpha_1}. \tag{26}$$

When $M_1 = M_2$, a $\alpha_1 = \alpha_2$, equation (20) reduces to

$$|W(s)| = \frac{GM_1M_2}{\bar{R}s}\left[1 - \frac{3s+1}{(3s+1)^2-1} + 2.\frac{3s+1}{[(3s+1)^2-1]^2}\ln(3s+1)\right], \tag{27}$$

where $s = r/R$.

A special case is that when two galaxies overlap with their centres coinciding, then $r_1 = r_2$ and $r = 0$, and we get:

$$|W(0)| = \frac{GM_1M_2}{\alpha_1}H(\alpha_{12}) \equiv \frac{GM_1M_2}{\overline{R_{12}(0)}}, \tag{28}$$

where

$$H(\alpha_{12}) = \frac{\alpha_{12}}{(\alpha_{12}-1)^3}[\alpha_{12}^2 - 2\alpha_{12}\ln(\alpha_{12}) - 1]. \tag{29}$$

When $\alpha_2 \ll \alpha_1$, we get

$$|W(0)| = \frac{GM_1M_2}{\alpha_1}. \tag{30}$$

For identical galaxies ($M_1 = M_2 = M$, $\alpha_1 = \alpha_2 = \alpha$), we get:

$$|W(0)| = \frac{GM^2}{3\alpha}. \tag{31}$$

In Table 1, we give the values of $H(\alpha_{12})$ and $\overline{R_{12}(0)}/\alpha_2$ for a few values of α_{12} as obtained from the present analysis.

Thus the potential density pair of Hernquist's model leads to a simple and completely analytical expression for the gravitational potential energy of a pair of

Table 1. $\overline{R_{12}(0)}$ as a function of α_{12}.

α_{12}	$H(\alpha_{12})$	$\overline{R_{12}(0)}/\alpha_2$
1·0	0·3333	3·000
1·5	0·4032	3·720
2·0	0·4548	4·398
5·0	0·6176	8·096
10	0·7263	13·77

interpenetrating galaxies represented by this model. Further, the result is general in the sense that it is applicable to identical as well as non-identical galaxies.

5. Results and discussion

In analytical studies of galactic collisions and in numerical simulations of encounters between galaxies, a number of workers have either used the Plummer model or polytropic model to represent galaxies. We therefore compare the results of the present analysis with those obtained for galaxies represented by these models.

The density distribution of the Plummer model (also known as the polytropic sphere of index $n = 5$) is given by

$$\rho(r) = \frac{3M}{4\pi} \frac{\alpha_p^2}{(r^2 + \alpha_p^2)^{5/2}}, \tag{32}$$

where M is the total mass and α_p is the scale length. It follows that the mass interior to r and the potential are given by

$$M(r) = M \frac{r^3}{(r^2 + \alpha_p^2)^{3/2}}, \tag{33}$$

and

$$V(r) = -\frac{GM}{(r^2 + \alpha_p^2)^{1/2}}. \tag{34}$$

Also

$$|\Omega| = \frac{3\pi}{32} \frac{GM^2}{\alpha_p}, \tag{35}$$

and hence the dynamical radius $\overline{R} = 16\alpha_p/3\pi$.

The density, in this model, falls off as r^{-5} at large radius. Such a rapid falloff, in general, is not compatible with the observed brightness distribution of galaxies. The galactic brightness distribution decays somewhat more slowly as r^{-3} to r^{-4} (Tremaine & Lee 1987). It may be noted that in Hernquist's model $\rho(r) \sim r^{-4}$.

In Fig. 1 we compare mass distribution of the Plummer model and Hernquist's model galaxies. We find that both the models agree very closely in mass distribution up to $r = 0.8\overline{R}$ i.e., up to the sphere containing about 50 per cent of the mass of the galaxy.

A comparison of equations (32), (33) and (34) with equations (10), (11), and (12) pertaining to Hernquist's model shows that the denominators in Hernquist's

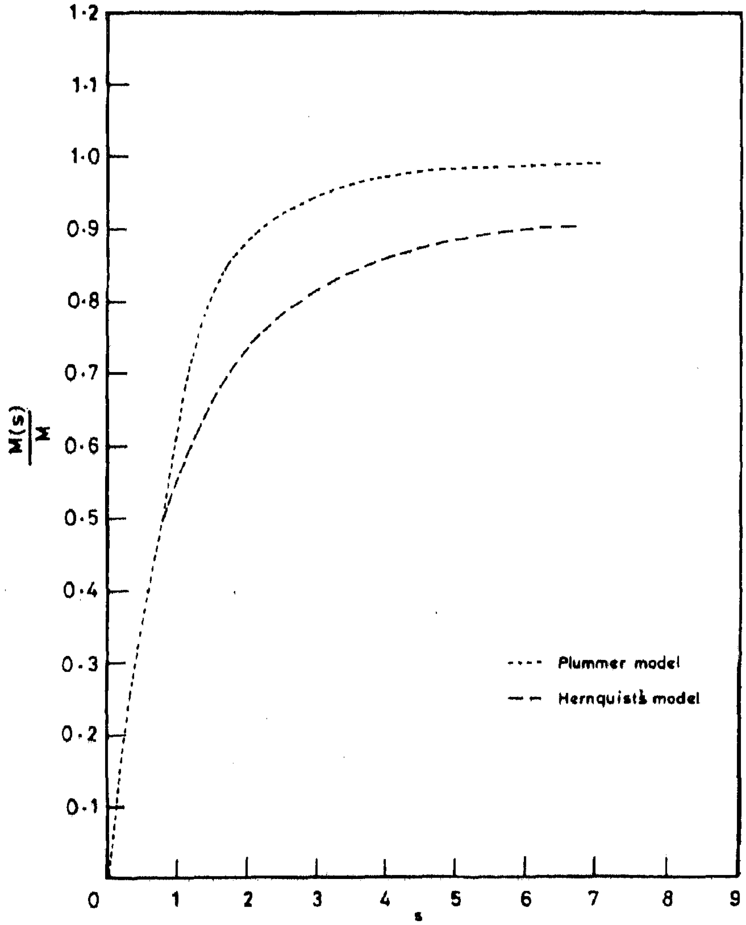

Figure 1. Comparison of mass distribution.

model have integral exponents, while in the Plummer model, the exponents in the denominator are fractional. Thus simple analytical results could be obtained in Hernquist's model.

Using the basic equations, for two galaxies of equal mass M and the scale length α represented by the Plummer model, Toomre (1977) obtained

$$|W(0)| = \frac{3\pi}{16}\frac{GM^2}{\alpha}. \tag{36}$$

For galaxies of differing mass and scale length, Ahmed's (1979) analysis yields

$$|W(0)| = \frac{GM_1M_2}{\alpha_1}A(\alpha_{12}), \tag{37}$$

where

$$A(\alpha_{12}) = \frac{\alpha_{12}^2}{(\alpha_{12}^2-1)^2}\left[(\alpha_{12}^2+1)E\left(\frac{\pi}{2},q\right) - 2F\left(\frac{\pi}{2},q\right)\right]. \tag{38}$$

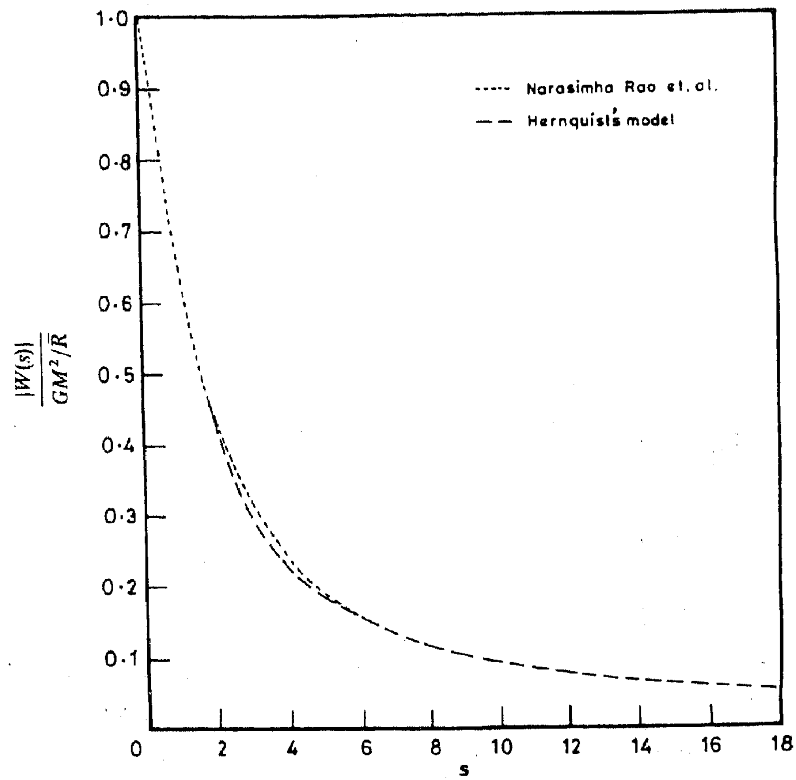

Figure 2. Comparison of interaction potential energy for identical galaxies.

Here $E(\pi/2, q)$ and $F(\pi/2, q)$ are the complete first and second elliptic integrals and

$$q = (1 - \alpha_{21}^2)^{1/2} \tag{39}$$

Equations (37) and (38) may be compared with equations (28) and (29) obtained in the present analysis. Unlike the function $A(\alpha_{12})$ which involves elliptic integrals, the function $H(\alpha_{12})$ is exact and can be evaluated with greater ease.

An analytic expression for $W(r)$ for any separation r between the galaxies represented by the Plummer model is given in Ahmed (1984). It involves a number of elliptic integrals. $W(r)$ obtained in the present analysis (equation 25) is completely analytic, exact and simple.

Narasimha Rao, Alladin & Narasimhan (1994) have pointed out that the values of $W(r)$ obtained from equation (3) agree closely with those obtained from equation (1) for the Plummer model galaxies and those obtained by Alladin (1965) for a pair of interpenetrating galaxies represented by polytropes of common radius R and indices (4–4) and (4–2) and that the differences between the values turn out to be statistically insignificant. We therefore restrict ourselves to comparing the results obtained from the present analysis with those obtained by Narasimha Rao *et al.*

In Fig. 2 we compare $W(r)$ obtained from Narasimha Rao *et al* (1994) with that obtained from the present work (Hernquist's model). We find that the values of $W(r)$ are in close agreement.

If we equate Ω and M as given by equations (35) and (16), i.e., if we keep the mass and dynamical radius same in both models, the scale length α_p of the Plummer model gravitational potential energy of penetrating spherical galaxies. To our knowledge, such simple, exact analytical expressions for these were not obtained earlier for spherical models of galaxies. $W(r)$ obtained from Hernquist's model closely agrees with those obtained from Plummer and polytropic models for galaxies of the same mass and dynamical radius.

Acknowledgement

S.M.A. thanks UGC, New Delhi for financial support.

References

Aarseth, S. J. 1966, *Mon. Not. R. astr.* Soc., **132,** 35.
Aarseth, S. J., Fall, S. M. 1980, *Astrophys. J.* **236,** 43.
Ahmed, F. 1979, *Astrophys. Space Sci.* **60,** 493.
Ahmed, F. 1984, *Clusters and Groups of Galaxies,* eds. F. Mardirossian, G. Giuricin & M. Merretti (Dordrecht, Holland; D. Reidel) p. 515.
Alladin, S. M. 1965, *Astrophys. J.* **141,** 768.
Alladin, S. M., Narasimhan, K. S. V. S. 1982, *Phys. Reports.* **92,** 339.
Ballabh, G. M. 1973, *Astrophys. Space. Sci.* **17,** 292.
Ballabh, G. M. 1975, *Astrophys. Space Sci.* **38,** 407.
Chandrasekhar, S. 1939, An Introduction to the Study of Stellar Structure (New York: Dover) p. 101.
de Vaucouleur, G. 1948, Ann. *d'Ap.,* **11,** 247.
Fish, R. A. 1964, *Astrophys. J.* **135,** 284.
Hernquist, L. 1990, *Astrophys. J.* **356,** 359.
Limber, D. N. 1961, *Astrophys. J.* **134,** 537.
Narasimhan, K. S. V. S., Alladin, S. M. 1986, *Astrophys. Space Sci.* **128,** 307.
Narasimha Rao, M., Alladin, S. M., Narasimhan, K. S. V. S. 1994, *Cel. Mech. and Dyn. Astr.* **58,** 65.
Potdar, A., Ballabh, G. M. 1974, *Astrophys. Space Sci.* **26,** 353.
Toomre, A. 1977, *The Evolution of Galaxies and Stellar Populations,* eds. B. M. Tinsley & R. B. Larson (Yale University Observatory) p. 401.
Termaine, S., Lee, H. M. 1987, *Dark Matter in the Universe,* eds. J. Bahcall, T. Piran & S. Weinberg, (New Jersey: World Scientific) **4,** 103.
Tremaine, S., Richstone D. O., Yong –Ik Byun., Dressler, A., Faber, S. M., Grillmair, C., Kormendy, J., Lauer, T. R. 1994, *Astr. J.* **107,** 634.
Zafarullah, S. N., Narasimhan, K. S. V. S., Sastry, K. S. 1983, *Bull. Astr. Soc. India* **11**, 145.