The Stellar-Dynamical Oeuvre

James Binney
Theoretical Physics, Oxford University, Keble Road, Oxford OX1 3NP, UK.

1. Introduction

Chandrasekhar was active in stellar dynamics only during the five years 1939 – 1944. Over this period the focus of his attention varied systematically, so that a chronological ordering of his papers corresponds fairly exactly with an ordering by topic. In sections 2 and 3 I review in chronological order all the stellar-dynamical papers that he published in refereed journals. Section 4 summarizes the content of his book The Principles of Stellar Dynamics. Section 5 attempts to assess the impact of his writings on the further development of stellar dynamics.

2. The ellipsoidal hypothesis

Chandrasekhar entered the field of stellar dynamics with two monumental papers on the ellipsoidal hypothesis. The first paper [1] (154 pages of the Astrophysical Journal) dealt with steady-state models, while the second paper (202 pages of the Astrophysical Journal) extended the theory to time-dependent models.

The fundamental equation of the theory of collisionless stellar systems is the collisionless Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{\partial \Phi}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0. \tag{1}$$

Eddington (Eddington 1915) first sought solutions to this equation in which the distribution function $f$ depends on the stellar coordinates $x$ and velocities $v$ only through the function

$$Q(x, v) \equiv \mathbf{v}' \cdot \mathbf{M} \cdot \mathbf{v}' + \sigma \tag{2}$$

where $v' \equiv v - v_0(x)$ is a star’s residual velocity, $\mathbf{M}(x)$ is a symmetric matrix and $\sigma(x)$ is a scalar function. That is, Eddington assumed $f$ to be a function of $Q$. This hypothesis is called the ellipsoidal hypothesis because $Q$ is constant on ellipsoids in velocity space. The principal axes of these velocity ellipsoids are specified by $\mathbf{M}$ and they are centred on the mean-streaming velocity at $x$, namely $v_0(x)$. The principal velocity dispersions are inversely proportional to the square roots of the eigenvalues of $\mathbf{M}$. The variation of the stellar density from point to point is largely controlled by $\sigma$. In [2] Chandrasekhar remarks that if several different models can be found that share a common gravitational potential, a more complex
model that does not satisfy the ellipsoidal hypothesis can be formed by taking any weighted sum of the distribution functions of the basic models. He calls this idea the principle of superposition in stellar dynamics. Since the phenomenon of Oort’s ‘high-velocity’ stars’ informs us that the stellar velocity distribution at the Sun does not even approximately satisfy the ellipsoidal hypothesis, this principle is essential if the ellipsoidal hypothesis is to offer any hope of casting light on the dynamics of galaxies such as the Milky Way.

The ellipsoidal hypothesis requires that \( Q + \sigma \) be a constant of stellar motion. So the question arises: which potentials admit several integrals that are quadratic in the velocities? This problem had been addressed by Whittaker (Whittaker 1936) who concluded that the potentials were those already identified by Stackel (Stäckel 1883) as giving rise to separable Hamilton-Jacobi equations.

Strangely, in [1] Chandrasekhar does not proceed from Whittaker’s work but formulates the problem anew and comes to the conclusion that if a model contains differential star-streaming, it must possess at least helical symmetry. Such a system can be spatially finite only if it is axially symmetric.

This result is surprising because Stackel’s potentials are not all axisymmetric. It is the more surprising in that Chandrasekhar argues that the ellipsoidal hypothesis allows a much larger range of solutions than Eddington had supposed possible, and Eddington had concluded that the allowed potentials were just Stackel’s potentials. Eddington did certainly err in assuming that the directions of the principal axes of the velocity ellipsoids at different points within the model would define a system of coordinate planes. As Chandrasekhar pointed out, the directions that a given principal axis takes at different points define a global vector field \( n(x) \), but an integrability condition must be satisfied if it is to be possible to solve the partial differential equation \( n \nabla \phi = 0 \) for the function \( \phi(X) \) which would define Eddington’s coordinate surfaces through \( \phi = \) constant.

Chandrasekhar sought solutions to his sets of equations by recasting them in all the usual coordinate systems. In this way he was able to characterize completely all planar and spherical solutions. For the planar case he derived an interesting model in which the mean azimuthal streaming velocity is given by

\[
\overline{v}_\phi(R) = \frac{g}{\sqrt{s^2 + R^2}},
\]

(3)

and the principal velocity dispersions satisfy the Oort-Lindblad relation \( \sigma^2/\sigma^2_R = -B/(A - B) \), where \( A \) and \( B \) are the Oort constants.

Chandrasekhar’s treatment of three-dimensional systems was less complete, but did turn up two remarkable homogeneous systems with spheroidal bounding surfaces within which the stellar streaming velocity has a component perpendicular to the equatorial plane. He speculated that the existence of one of these systems might be connected with the fact that the apparently flattest elliptical galaxies have axis ratio \( a:c \approx 3:1 \).

In his second paper on the ellipsoidal hypothesis Chandrasekhar allowed the quantities \( v_0 \), \( M \) and \( \sigma \) to become functions of time as well as of position. This
enabled him to demonstrate the instability of some of his earlier models, as well as to investigate models that display spiral structure.

His fundamental set of governing equations comprised a set of ten coupled partial differential equations in space that must be satisfied by the components of $M$, six further equations connecting the time derivative of $M$ to $v_0 (x, t)$, and six integrability equations. He investigated solutions to this extremely complex system of equations under various assumptions regarding the form of the gravitational potential $\Phi(x, t)$. For an axially symmetric system the latter has to be of the form

$$\Phi(r, \theta) = \Phi_0(t) - \frac{s}{2s} r^2 + \frac{1}{s^2} \Phi_1(r/s, \theta),$$

where $\Phi_0 , \Phi_1$ and $s(t)$ are arbitrary functions. Chandrasekhar used this formalism to study the dynamics of homogeneous spheroidal systems in some detail. He found that such systems have both stable and unstable modes which move them between configurations that satisfy the ellipsoidal hypothesis. The unstable modes corresponded to uniform contraction or expansion of the system.

The last part of this massive paper discusses systems with isotropic distributions of residual velocities, so that the distribution function is of the form

$$f = f(s^2|v - v_0|^2 + \sigma).$$

The structure of the model is determined by $s(t)$. If $s > 0$, the spiral arms are leading and the system is expanding, and conversely if $s < 0$. The azimuthal streaming velocity in these models is approximately of solid-body form.

3. The relaxation time

After his exhausting if not entirely exhaustive discussion of the ellipsoidal hypothesis Chandrasekhar turned to the problem of stellar relaxation. From the pioneering work of Jeans it was known that to lowest order it is possible to imagine that stars move in galaxies and globular clusters in the smoothed out potential that one obtains if one replaces the actual stellar distribution by the underlying probability density. In the next order of approximation one must allow for the deflections of stars from their zeroth-order orbits that are caused by the graininess of the actual potential. In [3] Chandrasekhar estimated the time $t_r$ required for the cumulative effect of these deflections to become appreciable by calculating the time average of $\Sigma_v(\Delta E)^2$, where $\Delta E$ is the change in energy that a star suffers during a binary encounter with another star. This quantity had been earlier incorrectly evaluated by Eddington, K. Schwarzschild and Rosseland. The intricacy of the calculation arises because distant, weak encounters cause the sum to diverge logarithmically with the impact parameter $b_{max}$ of the most distant encounter considered. Chandrasekhar argued that $b_{max}$ should be taken to be the mean distance between stars. Under this assumption he evaluated the relaxation time $t_r$ in the Milky Way near the Sun and
in a typical globular cluster, finding that $t_r$ was $\sim 10^{14}$y in the former case and $\sim 10^{10}$y in the latter case.

In [4] Williamson & Chandrasekhar employed a new criterion to estimate the relaxation time $t_r$: they evaluated the time average of $\Sigma \sin^2 \phi$, where $\phi$ is the angle through which a star is deflected during a binary encounter. The resulting values of $t_r$ were in good agreement with those obtained by consideration of $\Sigma (\Delta E)^2$ unless the test star was moving very much faster than the rms velocity of the field stars.

In [5] Chandrasekhar expresses profound dissatisfaction with the approach of [3] and [4]. This was brought to a head by an attempt to calculate the time average of $\Sigma \Delta E$. Since $\Delta E$ has a fluctuating sign, it is to be expected that the sum in question is the small difference of large terms. Mathematically, one has to evaluate an integral over the parameters of binary orbits that is not absolutely convergent. Worse still, to obtain a finite result for this integral, an upper limit $b_{\text{max}}$ has to be set to the range of the impact parameter. Chandrasekhar feared that terms that depend strongly on $b_{\text{max}}$ could not realistically be taken to cancel to the accuracy required if one is to obtain a physically plausible result. Since the problem was associated with the consideration of binary encounters at large impact parameters, he suspected that the underlying problem lay with the fundamental assumption that it was possible to treat the effects of graininess in the gravitational potential as the sum of a large number of entirely independent binary encounters.

[6] contains Chandrasekhar's first attempt at a better theory of fluctuations in stellar systems. Holtsmark had already evaluated the probability distribution of the different values of the electric field at a given point in a plasma. Moreover Smoluchowski had argued that the strength of a fluctuating random variable could be taken to relax exponentially back towards its equilibrium value after each upward fluctuation. The characteristic relaxation time would in general be a function $\tau(F)$ of the magnitude $F$ of the original fluctuation. Chandrasekhar reasoned that he could use these results to calculate the relaxation time in a stellar system if he could calculate Smoluchowski’s relaxation time $\tau(F)$.

Exasperatingly, this approach was still bedeviled by divergencies - this time associated with large field strengths due to close encounters. Chandrasekhar recognized that a satisfactory treatment of the problem of high field strengths lay beyond his present approach and simply imposed a lower cutoff on the interstellar distance. This cutoff was taken to be velocity dependent, being larger for lower relative velocities of stars.

Since Chandrasekhar could show that nearest neighbours dominate Holtsmark’s probability distribution, he could estimate the lifetime of a given field strength as the time required by the perturbing neighbour to move significantly further away or nearer.

Chandrasekhar's two papers [7] and [8] with von Neumann worked out in a rigorous way the ideas that lay behind [6]. In each paper the principal task was to
evaluate the joint probability distribution

\[
\frac{\mathrm{d}^6 P(F, \dot{F})}{\mathrm{d}^3 F \, \mathrm{d}^3 \dot{F}}
\]

that a test star of mass \( M \) experiences a force \( F \) that changes at rate \( \dot{F} \). Paper [7] assumes an isotropic distribution of random velocities of perturbing stars relative to the test star. Since motion of the test star relative to the rest frame of a cluster as a whole will be reflected in an anisotropic distribution of velocities relative to that star, paper [8] evaluates \( P(F, \dot{F}) \) for the anisotropic case.

\( F \) is a function of the positions of all the stars: if \( x_i \) is the position of the \( i^{th} \) field star, we have

\[
F(x, \{x_i\}) = \sum_i \frac{GM_m(x-x_i)}{|x-x_i|^3} \equiv \sum_i F_i.
\]  

(7)

Differentiating (7) with respect to \( t \) one easily obtains a similar expression for \( F(x, v, \{x_i, v_i\}) \).

The probability \( P \) of finding our \( N \) field stars in any given region \( \tau \) of \( 6N \)-dimensional phase space is

\[
P = \int \prod_{i=1}^{N} \left( \frac{1}{V} \, \mathrm{d}^3 x_i \, \mathrm{d}^3 v_i f(x_i, v_i) \right),
\]

(8)

where \( f \) has been normalized such that

\[
\int \mathrm{d}^3 x \, \mathrm{d}^3 v \, f = V.
\]

(9)

Then from the standard properties of the Dirac \( \delta \)-function it follows that

\[
W(F_0, \dot{F}_0) \equiv \frac{\mathrm{d}^6 P(F_0, \dot{F}_0)}{\mathrm{d}^3 F_0 \, \mathrm{d}^3 \dot{F}_0} = \int \prod_{i=1}^{N} \left( \frac{1}{V} \, \mathrm{d}^3 x_i \, \mathrm{d}^3 v_i f(x_i, v_i) \right) \delta^3(F - F_0) \delta^3(\dot{F} - \dot{F}_0).
\]

(10)

On replacing each \( \delta \)-function by

\[
\delta(x) = \int \frac{dk}{2\pi} e^{ixk}
\]

(11)

we quickly find that the Fourier transform of \( W \) is

\[
\tilde{W}(k, k') = \left( \frac{1}{V} \int \mathrm{d}^3 x_i \, \mathrm{d}^3 v_i f(x_i, v_i) e^{i(k F_i + k' F_i)} \right)^N,
\]

(12)
where $f$ is the system’s distribution function. Since we are interested in the limit $N \to \infty$, $N/V = \text{constant} \equiv n$, we exploit equation (9) to write

$$
\bar{W}(k, k') = \left(1 - \frac{1}{V} C(k, k')\right)^{nV} = e^{-nC},
$$

(13)

Where

$$
C(k, k') \equiv \int d^3x_i d^3v_i f(x_i, v_i) \left\{1 - e^{i(k \cdot x_i + k' \cdot x'_i)}\right\}.
$$

(14)

In a tour de force Chandrasekhar and von Neumann did the six-dimensional integral in (14) by changing integration variables from $(x_i, v_i)$ to $(F_i, F_i)$. They could then obtain $W$ by inverse Fourier transformation and finally evaluate expectation values of interest.

In their first paper they tabulated the lifetimes of states with given values of $F$, where the lifetime was defined to be $T \equiv \int \frac{F}{\sqrt{\langle F^2 \rangle_F}}$. In their second paper they derived the formula

$$
\left\langle F \right\rangle_{F, v} = \frac{2}{3} \pi GmnB(F, m) \left(\frac{3 \cdot v \cdot F}{F^2} - v\right).
$$

(15)

Here $m$ is the average mass of the field stars and $B$ is a given positive function. Dotting through by $F$ we find

$$
\left\langle \frac{dF^2}{dt} \right\rangle_{F, v} = \frac{8}{3} \pi GmnB F \cdot v.
$$

(16)

Hence $F$ tends to increase in magnitude when the star is moving in the direction of $F$, and decrease in magnitude in the opposite case. Chandrasekhar & von Neumann argue that the existence of dynamical friction follows from this result. There would seem to be a more elementary and convincing physical interpretation of equation (16): $v$ is in the same direction as $F$ when a star is falling into a potential well. The potential of a point particle is such that then $F$ is increasing. Thus equation (16) merely reflects the obvious kinematics of motion through a cloud of point masses.

In the first part of [9] Chandrasekhar showed that the existence of dynamical friction follows from the ability of a system to come to thermal equilibrium. Specifically, if a force of magnitude $F$ acts for a time $T$ the star’s velocity changes by an amount $|\delta v| = FT$. After time $t$ the star will have experienced $N = t/T$ such random velocity increments, and in velocity space will have diffused a distance of order $\Delta v = FT\sqrt{N} = F\sqrt{T}$. If the system is to come into thermal equilibrium, the diffusion to ever higher velocities has to be resisted by a frictional force. If, following the work of Ornstein, Uhlenbeck and others on Brownian motion, one assumes that the resistive force is proportional to $v$, one concludes that its coefficient must be

$$
\eta = \frac{\left\langle F^2 \right\rangle T}{2\langle v^2 \rangle}.
$$

(17)
In the second part of [9] Chandrasekhar presented the now conventional derivation of dynamical friction: one averages over all encounters the component of the change in a star’s velocity that is parallel to its peculiar velocity. Reassuringly, the resulting value of $\eta$ satisfies equation (17).

In paper [10] Chandrasekhar estimates the rate at which stars escape from a star cluster. This he does by considering the diffusion of a star in velocity space. At $t = 0$ the probability density $f(v)$ of the star having speed $v$ is $\delta(v - V_0)$ and at all times $f$ is assumed to vanish at and above the escape velocity $v_e$. It is a simple matter to solve the relevant diffusion equation subject to these boundary conditions. Then the rate of escape at time $t$ is just the integral of the probability current around the sphere of radius $v_e$.

In this way Chandrasekhar estimated the times required for galactic clusters to evaporate both when dynamical friction was included and when it was ignored. The inclusion of dynamical friction lengthened the time required for a cluster to evaporate by a factor of between 15 and 50 depending on the cluster’s central concentration. In the specific case of the Pleiades cluster, the inclusion of dynamical friction extended the expected lifetime of the cluster from $\sim 3 \times 10^7$ y to $\sim 5 \times 10^8$ y.

In the final section of [10] Chandrasekhar derives the collisional term of the full Boltzmann equation for use in further studies. It is

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} - \frac{\partial \Phi}{\partial x} \cdot \frac{\partial f}{\partial v} = \frac{\partial}{\partial v} \left( q \frac{\partial f}{\partial v} + \eta f v \right).$$

(Paper [11] upgrades the analysis of paper [10] by allowing for the velocity-dependence of the coefficients $q$ and $\eta$ in equation (18), which had been taken to be constants in paper [10]. This upgrade, which necessitated significant numerical work, increased the predicted lifetimes of clusters such as the Pleiades to values of order $3 \times 10^9$ y.

In [12] Chandrasekhar applied the methodology developed in his papers with von Neumann to the calculation of the correlation function of the gravitational field. That is, he calculated several projections of the tensor-valued expectation value $(F^{(1)}_i F^{(2)}_j)$, where $F^{(i)}$ is the force at $x^{(i)}$. The key quantity to be calculated is the characteristic function $C(k, k')$ that determines the probability density $W(F^{(1)}, F^{(2)})$ through equation (13). It is given by [cf. (14)]

$$C(k, k', \Delta) = N \int d^3 x d^3 v f(x, v) \left( 1 - e^{i[k \cdot F(x) + k' \cdot F(x + \Delta)]} \right).$$

(19)

After evaluating the integral in (19) Chandrasekhar was able to show that the mean value of the field at $x + \Delta$ dotted with the unit vector in the direction of the field at $x$ has a Taylor series of the form

$$\langle \hat{F}^{(1)} \cdot \hat{F}^{(2)} \rangle = a_0 - a_1 \Delta + a_4 \Delta^4 + \cdots,$$

(20)

where the $a_i$ are positive quantities that he evaluated.

Paper [13] calculates the temporal autocorrelation $\langle F(t_1) F(t_2) \rangle$ of the force
experienced by a test star at times $t_1$ and $t_2$. This problem is assumed to be identical with that solved in [12] upon substitution of $\Delta (t_2 - t_1)$ for $\Delta$ in equation (19). That is, Chandrasekhar neglects the explicit dependence of $\mathbf{F}(\mathbf{x}, t)$ on time.

In [14] Chandrasekhar applies the equal-time autocorrelation of the field evaluated in [12] to the dissolution of wide binaries. His reasoning is as follows. He focuses on the relative acceleration of the two stars in the direction of the background force that acts on star 1. After a time $t$ this has changed the relative velocities of the two stars by an amount

$$\Delta \mathbf{v} = \left[ \mathbf{F}(\mathbf{x}^{(1)}) - \mathbf{F}(\mathbf{x}^{(2)}) \right] \cdot \hat{n}(\mathbf{x}^{(1)}) t,$$

where the hat indicates a unit vector. On taking the expectation of both sides of the equation, results proved in [12] allow one to show that the right-hand equals $4\pi Gmn a t$, where $m$ and $n$ are the mass and number density of the background stars and $a$ is the separation of the binary components. Chandrasekhar argues that the binary will be dissolved once $4\pi Gmn a t$ has become comparable to the orbital velocity of the binary, namely $[G(m_1 + m_2)/a]^{1/2}$. This analysis suggests that binaries with $a \lesssim 2000$ AU will be dissolved within the age of the Milky Way.

4. The book

The book *Principles of Stellar Dynamics* appeared in 1942, around the time of his first paper with von Neumann. After a brief survey of what was then known observationally about the Milky Way, clusters and external galaxies, the book discusses the relaxation time along the lines of papers [3–5]. The next two chapters contain highly condensed and significantly clarified versions of papers [1] and [2] before presenting Lindblad’s ideas about spiral structure. The fifth and final chapter concerns star clusters. Global conservation theorems, including the virial theorem, are derived. The relaxation time and the rate of stellar evaporation are estimated. The effects of tides on the stability of a homogeneous ellipsoidal cluster that moves on a circular orbit around the galactic centre are studied. Finally the equilibria of spherical clusters are considered from the point of view of the isothermal sphere.

5. The influence of Chandrasekhar's papers

5.1 The ellipsoidal hypothesis

The impact of Chandrasekhar's two papers on the ellipsoidal systems has necessarily been limited by the extraordinary length of these highly mathematical papers. The discussion is throughout of a mathematical rather than a physical nature.

The equilibrium of a stellar system is determined by the interplay of dynamics, which requires that the distribution function be a constant of stellar motion, and Poisson's equation. The remarkable thing about papers [1] and [2] is the infrequency with which Poisson's equation is mentioned. Implicitly it appears a few times as the
underpinning for the quadratic nature of the potential of a homogeneous ellipsoid. But the analysis is dominated by the implications of $f$ being a constant of motion.

The ellipsoidal hypothesis now seems a confusing amalgam of two logically distinct ideas. First Jeans' theorem states that the distribution function depends on $(x, v)$ only through constants of motion. Second we ask, which potentials have isolating integrals that are up to quadratic functions of the velocities? Energy is always such an integral. Angular momentum is often another. For a century it has been known that the general steady-state potential with three global quadratic isolating integrals is a Stackel potential. This knowledge did not make its full impact on stellar dynamics until 1985 when de Zeeuw showed (de Zeeuw 1985) that some remarkably realistic model galaxies have Stäckel’s potentials. Eddington's investigation of the ellipsoidal hypothesis brought him closer to the discovery of the astronomical importance of Stäckel potentials than did Chandrasekhar’s.

Most of Chandrasekhar's work on the ellipsoidal hypothesis was concerned with time-dependent models. There are still extremely few results in this field. In fact, other than Chandrasekhar's, the only exact time-dependent models of which I am aware are those of Freeman (Freeman 1966) and Sridhar (Sridhar 1989). Both of these were constructed by first identifying a non-trivial isolating integral of the potential. In many respects Chandrasekhar's work foreshadows these models, but does not seem to have directly influenced them.

5.2 Discreteness noise in the gravitational field

It is surely Chandrasekhar's papers on the effects of discreteness noise in the gravitational field that have been the most influential of his stellar-dynamical papers. The idea that stellar systems are nearly collisionless but should slowly relax as a result of discreteness effects had been understood for many years before Chandrasekhar entered the field. His contribution was to calculate the diffusion coefficients $[\Delta E]$ and $[(\Delta E)^2]$ accurately, to state the collisional Boltzmann equation clearly and, above all, to identify the action of dynamical friction.

With hindsight his path to dynamical friction seems tortuous. He first lost confidence in what we now regard as the standard way to calculate diffusion coefficients, including the coefficient of dynamical friction. Then with von Neumann he developed an approach to the study of discreteness noise that avoids the concept of a binary encounter, which Chandrasekhar had identified as the source of his unease with standard methodology. Dynamical friction makes its first appearance in stellar dynamics at the end of Chandrasekhar's second paper with von Neumann. This fact is remarkable, as I think it is clear that dynamical friction lies beyond the reach of the Chandrasekhar-von Neumann approach. Indeed, as Mulder (Mulder 1983) has so elegantly described, the frictional drag that a body experiences as it moves through a stellar system arises from the attraction that the body experiences for the region of enhanced density that tails behind it like a wake behind a ship. By contrast, a basic assumption of the Chandrasekhar-von Neumann method is that
stars are randomly distributed in real space. So there is no way that the frictional force could emerge from the calculations of paper [8].

The conclusion is inescapable, that by the time [8] was being finished Chandrasekhar had deduced the existence of dynamical friction by the arguments of [9] and looked retrospectively at his calculations with von Neumann for evidence of the phenomenon.

Two things are notable about Chandrasekhar's discussion of dynamical friction. First, there is no mention that it is caused by each particle being attracted backwards by its gravitational wake. Second, there is no mention of its implications for the dynamics of massive bodies. In the 1970s dynamical friction would be seen to play a large role in several interesting astronomical phenomena by opposing the motion of galaxies, star clusters and interstellar gas clouds. But Chandrasekhar is so focused on the role that friction plays in the establishment of thermal equilibrium amongst bodies of comparable mass that he makes light of the remarkable fact that a body's deceleration is proportional to its mass.

Chandrasekhar's work on evaporation from star clusters remained fiducial until Hénon's work in the 1960s (e.g., Hénon 1960). The main respect in which it required refinement was its disregard of the increase in the periods of stars as their energies creep up towards the escape energy. Although Chandrasekhar was probably the first person to write down the collisional Boltzmann equation for a stellar system, his treatment of cluster dynamics was confined to velocity rather than phase space. Hence it could not take into account variations in the time stars spend in a cluster's collision-dominated core.

The effects of stochastic variations in astronomical gravitational fields remain an important area of research. The core of the field has grown directly out of the part of Chandrasekhar's work that rested on the contributions of Jeans and K. Schwarzschild. By contrast the part of Chandrasekhar's work that was inspired by the work of Holtsmark has achieved little resonance. I think the reason is that while it provides a wealth of information regarding the spatial structure of the field, it provides at best very limited information about the temporal structure of the field. This is unfortunate because an elementary calculation shows that the rate of change of a star's energy is the integral along its path of $\frac{\partial \Phi}{\partial t}$. Thus classical relaxation is entirely determined by temporal variations in $F = -\nabla \Phi$ and a technique such as that developed by Chandrasekhar & von Neumann that provides at most the expectation of the first time derivative of the field, is not promising.

I am aware of one important problem in which much can be done from a knowledge of the spatial structure of $\Phi$ alone. This is the dynamics of centrifugally supported disks. Such disks tend to heat in the sense that stars diffuse away from nearly circular orbits that are narrowly confined to the equatorial plane, onto more eccentric and/or highly inclined orbits. Such diffusion can take place at constant energy, so that the equation $\dot{E} = \frac{\partial \Phi}{\partial t}$ does not constrain the essential phenomenon in an important way. For further discussion of this problem, which throws up several evident connections with Chandrasekhar's work, see Binney & Lacey (1988).
Was Chandrasekhar right to profoundly suspect the use of binary encounters in the calculation of diffusion coefficients? It is now clear that his fears were exaggerated. Indeed, Theuns (Theuns 1996) has shown that standard theory based on binary encounters allows an excellent quantitative understanding of stellar diffusion in \(N\)-body models. These simulations confirm Spitzer's conclusion (Spitzer 1987) that the upper limit \(b_{\text{max}}\) on the impact parameters that should be considered is not the mean interstellar distance as Chandrasekhar supposed, but the distance within which the stellar density is comparable to its local value.

Current interest in the field of fluctuating gravitational fields centres on the degree to which collective oscillations lead to larger fluctuations than one would expect from two-particle noise alone - see, e.g., Weinberg (1994).

5.3  Wide binaries

Chandrasekhar was attracted to the question of the dissolution of wide binaries as an application of results on the fluctuation of gravitational fields that he had derived following his papers with von Neumann. Unfortunately, his calculation is not persuasive. It makes sense to write \(\Delta v \sim Ft\) only if the force \(F\) acts in the same direction throughout the interval \((0, t)\). Now Chandrasekhar equates \(\Delta v\) to the product of time and the difference in the external forces on the two stars dotted with the unit vector in the direction of the force that acts on one of them. Therefore he is implicitly assuming that this direction is constant. For consistency he should have assumed that the direction of the force on the other star is also constant. In this case, the directions of the two external forces would be fixed in the rotating frame of the binary. This is implausible. In reality there are two cases to consider. Either the binary period is short compared to the characteristic time of fluctuations in the background field. Then each star will experience the average of the background field around its orbit, and the two stars will suffer very similar net accelerations. Mathematically, the orbital elements will be constant by virtue of their adiabatic invariance. In the opposite limit the binary period is long compared to the characteristic time of fluctuations. This is the limit in which the orbit can be disrupted, but it is also the limit which Chandrasekhar could not address because for this problem he did not even have the expectation value of the field’s first time derivative.

Subsequent work (e.g., Weinberg et al. 1985) has assumed that the dominant fluctuations arise from encounters between a binary and either a single star or a bound object such as a cluster or an interstellar cloud. With this assumption it is straightforward to evaluate the diffusion coefficients that are required to follow the evolution of any given initial population of binaries. This sort of analysis has the potential to place important constraints on the degree to which the Milky Way’s dark matter is concentrated into massive objects. The main difficulty at the present time is the observational determination of the numbers of binaries in each range of semi-major axes.
6. Summary

Chandrasekhar was essentially an applied mathematician rather than a physicist and one who was very 'productive' in the modern administrator's use of the word. Both of these characteristics tended to diminish his impact on the field of stellar dynamics. His mathematical orientation ensured that his papers are heavy going for a theoretical astrophysicist and completely impenetrable to the average astronomer. After pages of detailed calculations of particular integrals, integrability conditions and the roots of equations one longs for relief in the form of the description of the physical picture which emerges from the mathematics. Too often one longs in vain.

The essential difficulty of much of his mathematics was surely compounded by the speed with which he published. Several notations are frequently used for essentially the same quantity at different points in a paper, and determining the meaning of an equation can involve an exhausting backward chase through long chains of definitions. The potential for condensation and clarification is made evident by a comparison of chapters III and IV of The Principles of Stellar Dynamics with papers [1] and [2] from which they substantially derive.

Rightly scientists are remembered for the best rather than the worst things in their oeuvres. So it is proper that we should remember Chandrasekhar for his contributions to the theory of cluster evolution: for understanding how equilibrium between stochastic excitation and dynamical friction is attained, and for estimating cluster relaxation and evaporation times. In other fields he achieved more, but most would be happy to have attained as much in any field as he did in this.

References

Chandrasekhar’s papers


**Other papers**


