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Some Remarks on the State of Matter in the Interior of Stars.

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With 3 figures. (Received September 28, 1932.)

It is shown that for *all* stars for which the radiation-pressure is greater than a tenth of the total pressure, an appeal to the FERMI-DIRAC statistics to avoid the central singularity which arises in the discussions of the centrally condensed and the collapsed stars cannot be made. The bearing of this result on the possible state of matter in the interior of stars is indicated.

Since the publication of MILNE'S memoir on the "Analysis of Stellar Structure" in the Monthly Notices for November 1930¹⁾ a great deal of Work has been done to consider "composite" stellar models. But the following simple considerations seem to have escaped notice and it seems worth while to state them explicitly.

§ 1. *The Surfaces of Demarcation.* As we approach the centre of a centrally-condensed or a collapsed star, we change over to the equation of state $P = K_1 \rho^{5/3}$ if the perfect gas law breaks down. If the perfect gas-law breaks down at all, the actual transition from the perfect-gas envelope to the degenerate core must occupy a certain zone, but we could for the sake of convenience consider a definite surface of demarcation defined as the surface at which the two equations of state give the same gas-pressure.

Now in the perfect gas envelope the *total* pressure is given by

$$P = \left[\left(\frac{k}{\mu} \right)^4 \frac{3}{a} \left(\frac{1 - \beta}{\beta^4} \right) \right]^{1/3} \rho^{4/3}, \quad (1)$$

where

$$\beta = 1 - \frac{\kappa L}{4\pi c G M}, \quad (2)$$

κ = the opacity coefficient, L = luminosity in ergs cm⁻³, M = mass in grams, k = BOLTZMANN'S Constant, μ = molecular weight = αm_H (say), m_H = mass of the hydrogen atom. Since for the standard model the gas pressure p is given by

$$p = \beta P, \quad (3)$$

we have

$$p = C \rho^{4/3}, \quad (4)$$

where

$$C = \left[\left(\frac{k}{\mu} \right)^4 \frac{3}{a} \frac{1 - \beta}{\beta} \right]^{1/3} = \frac{2.632 \cdot 10^{16}}{\alpha^{4/3}} \left[\frac{1 - \beta}{\beta} \right]^{1/3}. \quad (4')$$

¹⁾ Referred to as 1. c.

The equation of state in the degenerate zone is

$$p = K_1 \varrho^{5/3}, \quad (5)$$

where

$$K_1 = \frac{1}{20} \left(\frac{3}{\pi} \right)^{2/3} \frac{h^2}{m \mu^{5/3}} = \frac{9.890 \cdot 10^{12}}{\alpha^{5/3}}. \quad (6)$$

At the first surface of demarcation which we will call S_1 , the density ϱ_1 is given by

$$C \varrho_1^{4/3} = K_1 \varrho_1^{5/3},$$

or

$$\varrho_1 = \left(\frac{C}{K_1} \right)^3. \quad (7)$$

Now, it is well known that the equation of state (5) changes over again into the relativistic-degenerate-equation of state

$$p = K_2 \varrho^{4/3}, \quad (8)$$

$$K_2 = \frac{hc}{8 \mu^{4/3}} \left(\frac{3}{\pi} \right)^{1/3} = \frac{1.228 \cdot 10^{15}}{\alpha^{4/3}}. \quad (8')$$

Hence if *circumstances* permit we have to consider a second surface of demarcation, S_2 , where the density ϱ_2 is given by

$$\varrho_2 = \left(\frac{K_2}{K_1} \right)^3. \quad (9)$$

Hence we have *two* surfaces of demarcation if and only if

$$\varrho_2 > \varrho_1,$$

or

$$\left(\frac{K_2}{K_1} \right)^3 > \left(\frac{C}{K_1} \right)^3, \quad (10)$$

i. e. only when [cf. equations (4'), (6), (8)']

$$\frac{1 - \beta}{\beta} < \frac{h^3 c^3 a}{512 \pi k^4} = 0.1015,$$

or

$$\beta > 0.9079. \quad (11)$$

It may be remarked in passing that the above value for β is independent of the assumed molecular weight. *It depends only on the mass, luminosity and opacity in the gaseous envelope.* It is also independent of whether we consider the same opacity for the degenerate zone and the gaseous envelope, or different opacities in the two regions.

§ 2. The meaning of the fundamental inequality (11) is made clear by the following.

In the following graph I plot $\log p$ against $\log \varrho$.

For numerical calculations I use $\alpha = 2$. The straight line ABK represents the equation of state $p = K_1 \varrho^{5/3}$ and BC the equation of state $p = K_2 \varrho^{4/3}$. These two intersect at B where the density is that which corresponds to the *second* surface of demarcation, namely ϱ_2 . ABC gives roughly the equation of state of a degenerate gas.

Let us consider a star for which $\beta = 0.98$. By (4) we get

$$\log p = 14.455 + \frac{4}{3} \log \varrho. \tag{12}$$

DE represents this equation. It intersects the degenerate equation of state AB , C at E . The point E corresponds to the first surface of demarcation S_1 . Hence for all stars for which $\beta = 0.98$, we first traverse a perfect gas envelope with an equation of state represented by DE . Then we traverse a degenerate zone corresponding to EB and finally (if we have not yet reached the centre) a relativistically degenerate zone.

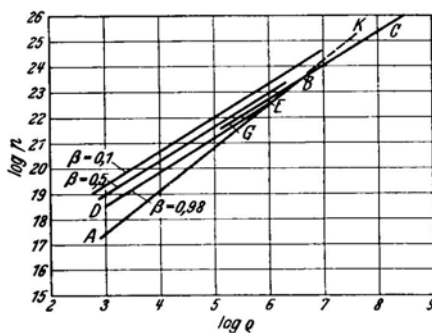


Fig. 1.

Now, if $\beta = 0.9079$, then GB represents the perfect gas equation of state and the degenerate zone reduces to a single layer, and the relativistically degenerate zone is described equally well by the perfect gas equation.

Now if $\beta < 0.9079$ ¹⁾ the perfect gas equation of state has *no* intersections with ABC and this means that however high the density may become the temperature rises sufficiently rapidly to prevent the matter from becoming degenerate.

In this connection it will have to be remembered that considerations of relativity do not affect the equation of state of a perfect gas. $p = N k T$, is true independent of relativity.

§3. *Centrally-Condensed Stars.* Now, for each mass M there is a unique luminosity L_0 — the "EDDINGTON luminosity" which makes the star

¹⁾ The radiation pressure is *greater* than a tenth of the total pressure if $\beta < 0.9079$.

a perfect gas sphere, with a polytropic index 3. This L_0 characterizes a unique β_0 which is in fact related to M by means of EDDINGTON'S quartic equation:

$$1 - \beta = 0.00809 \left(\frac{M}{\odot} \right)^2 \alpha^4 \beta^4. \quad (13)$$

Now from the definition of a centrally-condensed and a collapsed star, it is clear that

$$\left. \begin{array}{l} \beta_{c.c.} < \beta_0 \\ \beta_{col.} > \beta_0 \end{array} \right\} \quad (14)$$

Consider first the mass \mathfrak{M} for which $\beta_0 = 0.9079$. By (13) we have

$$\mathfrak{M}/\odot = 6,623 \alpha^{-2}. \quad (15)$$

If we assume $\alpha = 2$,

$$\mathfrak{M}/\odot = 1.656. \quad (15')$$

Now consider a centrally-condensed star of mass M greater than (or equal to) \mathfrak{M} . Then we obviously have

$$\left. \begin{array}{l} M\beta_0 < \mathfrak{M}\beta_0 = 0.908. \\ M\beta_{c.c.} < \mathfrak{M}\beta_0 < 0.908. \end{array} \right\} \quad (16)$$

Hence, we have the result that for *all centrally condensed stars of mass greater than \mathfrak{M} , the perfect gas equation of state does not break down, however high the density may become, and the matter does not become degenerate. An appeal to the Fermi-Dirac statistics to avoid the central singularity cannot be made.*

Since however we cannot allow the infinite density which the centrally condensed solution of EMDEN'S differential equation — index 8 — allows at the centre and in the absence of our knowledge of any equation of state governing the perfect gas other than that of degenerate matter, our only way out of the singularity is to assume that there exists a maximum density ρ_{\max} which matter is capable of. We have therefore to consider the “fit” of a gaseous envelope of the centrally condensed type on to a homogeneous core at the maximum density of matter. *If we insist on the density to be continuous at the interface* the equation of “fit” is found to be¹⁾

$$\frac{1}{8} \xi' \Theta'^3 = - \left(\frac{d\Theta}{d\xi} \right)_{\xi=\xi'}, \quad (17)$$

¹⁾ S. CHANDRASEKHAR, M. N. **91**, 456, 1931, equation (47).

where the polytropic equation describing the gaseous part of the star is

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\Theta}{d\xi} \right) = -\Theta^3, \tag{17'}$$

where ξ is the value of ζ at which ϱ_{\max} begins. In (17') the meaning of Θ and ζ are the following:

$$\varrho = \lambda_3 \Theta^3, \quad r = \xi \left[\frac{C}{\pi G \beta} \right]^{-1/3} \lambda_3^{-1/3}. \tag{17''}$$

(λ_3 is a homology, constant). But (17) has no solutions if Θ is of the EMDEN'S or of the centrally-condensed type. Hence the acceptance of a ϱ_{\max} does not

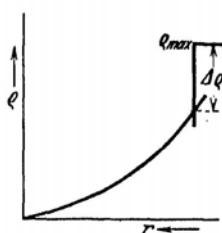


Fig. 2.

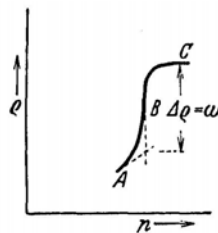


Fig. 3.

help us out of the difficulty if we insist on the density to be continuous at the interface. The procedure then to construct an equilibrium configuration would be to proceed along the centrally condensed solution until the mean density $\varrho_m(r)$ of the surviving mass $M(r)$ equals ϱ_{\max} which will occur at a determinate $r = r''$ (say) where

$$M(r'') = \frac{4}{3} \pi r''^3 \varrho_{\max}; \tag{18}$$

we then replace the material inside $r = r''$ by a sphere of incompressible matter at the density ϱ_{\max} . At r'' there will be a discontinuity of density (see Fig. 2).

Now the form of Θ as $\xi \rightarrow 0$ for a centrally condensed solution is (MILNE, 1. c) :

$$\Theta \sim \frac{1/\sqrt{2}}{\xi [\log(D/\xi)]^{1/2}}, \tag{19}$$

where D is a constant. D is fixed by the condition that the analytic continuation of (19) passes through $\xi = 1$ and $\Theta = 0$ and satisfies here the requisite boundary condition, namely

$$M = -\frac{4}{\pi^{1/2}} \left(\frac{C}{G\beta} \right)^{3/2} \left(\xi^2 \frac{d\Theta}{d\xi} \right)_0. \tag{20^1}$$

¹⁾ C is given by equation (4').

Hence we get the result that D is a function of L , M and x only and hence fixed. Since D is fixed by the boundary condition, it follows that the value of ζ'' at which $\Theta(\zeta'')$ becomes equal to Θ_{\max} (where cf. equation (17''))

$$\Theta_{\max} = \varrho_{\max}^{1/3} \lambda_3^{-1/3}, \quad (21)$$

is fixed as a function of λ_3 . In other words the discontinuity in Θ , $\Delta\Theta''$ at the interface ζ'' is a single-valued function of L , M , x and λ or

$$\Delta\Theta'' = F(L, M, \kappa; \lambda_3) \quad (21')$$

or by (21)

$$\Delta\varrho'' = f(L, M, \kappa; \lambda_3) \quad (22)$$

where $\Delta\varrho''$ is the discontinuity of density at the interface.

But it has been suggested by LANDAU¹⁾ (among others) that the maximum density of matter will arise *after* some kind of *overcompressibility*, *the incompressibility setting in later* (see Fig. 8).

Further it has been suggested that 1) the pressure at which the overcompressibility sets in must be a physical property of the atomic nuclei and the electrons in the enclosure, and 2) the form of the curve $A B C$ is again an intrinsic physical property of matter. If we idealise the situation of Fig. 8, we see that $\Delta\varrho$ *ought to be a physical property of matter*. Let this $\Delta\varrho$ be ω . Then by (22) we have to so choose the homology constant λ_3 , that $\Delta\varrho''$ equals ω :

$$f(L, M, \kappa; \lambda_3) = \omega. \quad (23)$$

This fixes λ_3 and hence by (17'') fixes r_0 – the radius of the configuration. Hence *we are able to obtain equilibrium configurations for arbitrary mass, and arbitrary luminosity, the radius however being determinate in each case.*

§ 4. In the above section we have tried to construct the equilibrium configurations for all centrally-condensed stars of mass greater than \mathfrak{M}^2 , and found that the introduction of a homogeneous core at the maximum density of matter (ϱ_{\max}) with a discontinuity of density at the interface was necessary. We may now ask about the equilibrium configurations for centrally condensed stars with $\beta > 0.908$. Now the star has clearly a degenerate zone (see Fig. 1). A little consideration shows that if we come along a centrally-condensed solution in the perfect gas part of the star then at the interface S_1 (cf. § 1) we are compelled to choose a centrally-condensed solution for the polytropic equation of index " $3/2$ "

¹⁾ I am indebted to Dr. STRÖMGREN for advice on these matters.

²⁾ Or more generally, centrally-condensed stars with $\beta < 0.908$.

to describe the non-relativistic degenerate part of the star¹); also at the second surface demarcation S_2 we are again forced to choose a centrally-condensed solution for the polytropic equation of index "3". Hence in this case also we are unable to avoid the central singularity by appealing to the FERMI-DIRAC statistics alone. The star must have a homogeneous core with a discontinuity of density at the interface. The considerations of the previous section apply and we see that the centrally-condensed stars $\beta > 0.908$ differ from the centrally-condensed stars with $\beta < 0.908$ only in this, that while in the former type of stars we have to traverse a degenerate zone before reaching the homogeneous core, in the latter type, the stellar material continues to be a perfect gas till we reach the homogeneous core. Thus we find that *all centrally-condensed stars (on the standard model) must have a homogeneous core at the centre with a discontinuity of density at the interface.*

§ 5. *Collapsed-Stars.* Just a few remarks about collapsed stars may be permitted. A detailed analysis of *highly-collapsed* stars has been given elsewhere (CHANDRASEKHAR, 1. c).

Consider a collapsed star of mass greater than \mathfrak{M} and let further $\beta_0 < \beta_{\text{col.}} < 0.9078$. In other words the collapse" has not proceeded sufficiently far to increase β beyond 0.9078. In such a case the collapse can occur only on a homogeneous core. But if the collapse proceeds sufficiently far, such that $\beta_{\text{col.}} > 0.9078$ in spite of β_0 being less than 0.9078, the star will then possess a degenerate zone as well.

Conclusion: We may conclude that great progress in the analysis of stellar structure is not possible before we can answer the following fundamental question:

Given an enclosure containing electrons and atomic nuclei, (total charge zero) what happens if we go on compressing the material indefinitely?

¹) This is also true if we ascribe different opacities to the gaseous and the degenerate part of the star.