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XLVIII. *The Density of White Dwarf Stars.*

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1. THE first application of the Fermi-Dirac statistics to stellar problems was by Fowler † in connexion with the well-known problem of the companion of Sirius. This idea has lately been taken up by Stoner ‡ and others to calculate the limiting density of white dwarf stars. In this paper another way of arriving at the order of magnitude of the density of white dwarfs from different considerations is given.

2. Let p_r denote the radiation pressure and p_G the gas pressure, and the total pressure P is then given by

$$P = p_r + p_G. \qquad (1)$$

We introduce the constant β, such that

$$\left.\begin{aligned} p_r &= (1-\beta)P, \\ p_G &= \beta P. \end{aligned}\right\} \qquad (2)$$

We will make the assumption that β = 1 approximately, *i.e.*, we leave the radiation pressure out of account. We are dealing therefore with *ideal* conditions which can perhaps exist only in stars which are much higher in the white dwarf stage than even O_2 Eridani B.

* Communicated by R. H. Fowler, F.R.S.

† R. H. Fowler. Month. Not. Roy. A. S. lxxxvii. p. 114 (1926)

‡ E. C. Stoner, Phil. Mag. vii. p. 63 (1929): ix. p. 944 (1930).

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Now for a fully degenerate electron gas (in the Sommerfeld sense) the pressure is given by

$$p_e=\frac{\pi}{60}\frac{h^2}{m}\left(\frac{3n}{\pi}\right)^{5/3}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

We assume that it is this electron pressure which is by far the greatest contribution to the gas pressure, und therefore to the total pressure. Further, if ρ is the density of the stellar material, the number of electrons is given by

$$n=\frac{\rho}{\mu\mathrm{H}(1+f)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

where f is the ratio of the number of ions to the number of electrons (we can in practice neglect f), H the mass of the hydrogen atom, and μ the molecular weight. For a fully ionized material of the type we are dealing with μ = 2·5 nearly. We will use this value Later. We have therefore

$$p_G=\frac{\pi}{60}\frac{h^2}{m}\left(\frac{3}{\pi\mathrm{H}}\right)^{5/3}\frac{\rho^{5/3}}{\mu^{5/3}(1+f)^{5/3}}$$

$$=9{\cdot}845\times10^{12}\left[\frac{\rho}{\mu(1+f)}\right]^{5/3}, \quad . \quad . \quad . \quad . \qquad (5)$$

(The values used for *h*, *m*, etc. are those given in. A. S. Eddington's 'Internal Constitution of Stars,' Appendix (1).)

Putting

$$\mathrm{K}=\frac{9{\cdot}845\times10^{12}}{\mu^{5/3}(1+f)^{5/3}}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

we have for the total pressure

$$\mathrm{P}=\mathrm{K}\rho^{5/3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

We can now straightway apply the theory of the polytropic gas spheres, where for the exponent γ we have

$$\gamma=5/3 \text{ or } 1+\frac{1}{n}=5/3,$$

giving $$n=3/2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

We have therefore*

$$\left(\frac{\mathrm{GM}}{\mathrm{M}'}\right)^{+1/2}\left(\frac{\mathrm{R}'}{\mathrm{R}}\right)^{-3/2}=\frac{[5/2\mathrm{K}]^{3/2}}{4\pi\mathrm{G}}, \quad . \quad . \quad . \quad . \qquad (9)$$

* A. S. Eddington, 'Internal Constitution of Stars,' p. 83 *et seq*. The notation is the same as that used in his book and now generally adopted. The particular equation (9) follows from the second of the equations (57.3)

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Or
$$\frac{GM}{M'} = \frac{125 \times 9{\cdot}845^3 \times 10^{36}}{128\pi^2 G^2} \cdot \frac{1}{\mu^{0}(1+f)^5}\left(\frac{R'}{R}\right)^3 . \quad . \quad (10)$$

The values o£ R' and M' can be obtained from the extensive tables given by Emden in his 'Gas-Kugeln,' and are (page 79, tabbelle 4)

$$\left.\begin{aligned} R' &= 3{\cdot}6571, \\ M' &= 2{\cdot}7176. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Using these values in (10), and expressing the mass in terms of that of the Sun (= 1·985 × 10^{33} grams), we get the result

$$(M/\odot)R^3 = \frac{2{\cdot}14 \times 10^{28}}{\mu^5}(=2{\cdot}192 \times 10^{26}). \quad . \quad (12)$$

The second value for (M/⊙) R^3, given in brackets, we get by using the value 2·5 for μ. We can express (12) differently, as follows:

$$R^6\rho = \frac{1{\cdot}014 \times 10^{61}}{\mu^5}(=1{\cdot}039 \times 10^{59}), \quad . \quad . \quad (13)$$

$$\rho = 2{\cdot}162 \times 10^6 (M/\odot)^2. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

We will apply the above equations to the case of the companion of Sirius. The mass of it, as determined from the double star orbit, is trustworthy, and equals ·85⊙· The computed radius = 1·8 × 10^7. (But we cannot use this value in (13) to calculate the density, as it is based on formulae which may not be applicable to this case.) From the mass we can derive the radius und equals 6·361 × 10^8 (about thirty times the accepted value). For the density of the companion of Sirius we get from (14), *provided* it were completely degenerate (which, however, is extremely unlikely),

$$\rho c._{\text{Sirius}} = 1{\cdot}562 \times 10^6 \text{ grams per cm.}^3 \quad . \quad (15)$$

The mean density assumed is ·5 × 10^5, being thus thirty times smaller than that given by (15). We can, however, take the value given by (15) as indicating the *maximum* density which a stellar material having a mass equal to that of the companion of Sirius can have. A similar calculation can be made for O_2 Eridani B and Procyon B, and the calculated values are collected in a table below. The calculations for the *limiting density* on Stoner's theory give different values, and they are also given for comparison. We discuss the cause of the difference below.

We further note (i.) that the radius of a white dwarf is inversely proportional to the cube root of the radius, (ii.) the density is proportional to the square of the mass, (iii.) the central density would be six times the mean density ρ.

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3. Stoner (*loc. cit.*) arrives at a formula for the *limiting* density for a material composed of completely ionized atoms on the following argument:—

The density increases as the sphere shrinks, and the limit is reached when the gravitational energy released just supplies the "energy required to squeeze the electrons closer together." The limiting condition would then be given by

$$\frac{d}{dn}(E_G + E_K) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

E_G being the gravitational energy and E_K the kinetic energy, for which, of course, the Fermi formula is used. The formula he gets is (without his latter relativity-mass correction)

$$\rho_{max.} = 3 \cdot 977 \times 10^6 (M/\odot)^2, \quad . \quad . \quad . \quad . \qquad (17)$$

which is exactly the same as our (14) with a difference in the

Star.	Mass.	Radius.	Density.		
			As calc. by (14).	Accepted value.	By Stoner's formula (17).
O_2 Eridani B.	$\cdot 44 \odot$	$7 \cdot 927 \times 10^8$	$4 \cdot 186 \times 10^5$	$\cdot 98 \times 10^5$	$7 \cdot 8 \times 10^5$
Procyon B.	$\cdot 37 \odot$	$8 \cdot 399 \times 10^8$	$2 \cdot 960 \times 10^5$	—	$5 \cdot 445 \times 10^5$
Companion of Sirius.	$\cdot 85 \odot$	$6 \cdot 361 \times 10^8$	$1 \cdot 562 \times 10^6$	$\cdot 5 \times 10^5$	$2 \cdot 872 \times 10^6$

numerical factor only, the discrepancy being about 1:2. The difference in the two is obviously due to the fact that our value for ρ is not the " limiting density" in the sense in which Stoner uses the term ; but our calculation gives us a much nearer approximation to the conditions actually existent in white dwarfs than Stoner's calculation does. At any rate, it brings out clearly that the *order of magnitude* of the density which one can on purely theoretical considerations attribute to a white dwarf is the same.

Our results (see table) agree with Stoner's in showing that O_2 Eridani B is much nearer the ideal dwarf-star stage than the companion of Sirius, but indicate also that neither of them is so far from the ideal stage as Stoner's calculation would seem to indicate.

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Summary.

The density of the white dwarf stars is reconsidered from the point of view of the theory of the poly tropic gas spheres, and gives for the *mean density* of a white dwarf (under ideal conditions) the formula

$$\rho = 2{\cdot}162 \times 10^6 \times (M/\odot)^2.$$

The above formula is derived on considerations which are a much nearer approximation to the conditions *actually existent* in a white dwarf than the previous calculations of Stoner based on uniform density distribution in the star and which gave for the limiting density the formula

$$\rho = 3{\cdot}977 \times 10^6 \times (M/\odot)^2.$$