PLANCK'S LAW AND
THE LIGHT QUANTUM HYPOTHESIS

By *[Satyendranath] Bose*
Dacca University, India
Received by Zeitschrift fur Physik on 2 July 1924

The phase space of a light quantum in a given volume is subdivided into "cells" of magnitude h^3. The number of possible distributions of the light quanta of a macroscopically defined radiation over these cells gives the entropy and with it all thermodynamic properties of the radiation.

Planck's formula for the distribution of energy in blackbody radiation forms the starting point for the quantum theory which has been developed during the past twenty years and has yielded rich harvests in all fields of physics. Since its publication in the year 1901 many types of derivations of this law have been suggested. It is acknowledged that the fundamental assumptions of the quantum theory are inconsistent with the laws of classical electrodynamics. All existing derivations make use of the relation

$$\rho_\nu d\nu = \left(8\pi\nu^2 d\nu/c^3\right) E.$$

representing the relation between the radiation density and the mean energy of an oscillator, and they make assumptions concerning the number of degrees of freedoms of the ether as exemplified in the above equation (the first factor on the right-hand side). This factor, however, could be deduced only from the classical theory. This is the unsatisfactory point in all derivations, and it is not

surprising that again and again efforts are made which try to give a derivation free of this logical deficiency.

A remarkably elegant derivation has been given by Einstein. Recognizing the logical defect in the existing derivations, he attempted to deduce the formula independently of any classical theory. Starting with very simple assumptions about the energy exchange between molecules and the radiation field, he finds the relation

$$\rho_\nu = \frac{\alpha_{mn}}{\exp\left[(\epsilon_m - \epsilon_n)/kT\right] - 1}$$

However, in order to make this formula agree with that of Planck, he has to make use of Wien's displacement law and Bohr's correspondence principle. Wien's law is based on the classical theory; the correspondence principle assumes that the quantum theory agrees asymptotically with the classical theory in certain limiting cases.

In all cases it appears to me that the derivations have insufficient logical foundation. In contrast, the combining of the light quanta hypothesis with statistical mechanics in the form adjusted by Planck to the needs of the quantum theory does appear to be sufficient for the derivation of the law, independent of any classical theory. In the following I wish to sketch briefly the new method.

Let the radiation be enclosed in a volume V and its total energy be E. Let there be different species of quanta each characterized by the number N, and energy $h\nu_s(s = 0$ to $s = \infty)$. The total energy E is then

$$E = \sum_s N_s h\nu_s = V \int \rho_\nu d\nu \tag{1}$$

The solution of our problem requires then the determination of the numbers N_s which determine ρ_ν. If we can state the probability for any distribution characterized by an arbitrary set of N_s, then the solution is determined by the requirement that the probability be a maximum provided the auxiliary condition (1) is satisfied. It is this probability which we now intend to find.

The quantum has a moment of magnitude $h\nu_s\,/c$ in the direction of its forward motion. The instantaneous stateof the quantum is characterized by its coordinates $x,y,z,$ and the associated momenta p_x, p_y, p_z. These six quantities can be interpreted as point coordinates in a six-dimensional space; they satisfy the relation

$$p_x^2 + p_y^2 + p_z^2 = h^2 \nu^2 / c^2$$

by virtue of which the above-mentioned point is forced to remain on a cylindrical surface which is determined by the frequency of the quantum. In this sense the frequency domain dv_s is associated with the phase space domain

$$
\begin{aligned}
\int dx\ dy\ dz\ dp_x\ dp_y\ dp_z\ &=\ V 4\pi \left(h\nu/c\right)^2 h d\nu/c \\
&=\ 4\pi \left(h^3 \nu^2/c^3\right) V d\nu
\end{aligned}
$$

If we subdivide the total phase space volume into cells of magnitude h^3 , then the number of cells belonging to the frequency domain dv is $4\pi V\ (v^2/c^3)\ dv$. Concerning the kind of subdivision of this type, nothing definitive can be said. However, the total number of cells must be interpreted as the number of the possible arrangements of one quantum in the given volume. In order to take into account the polarization, it appears mandatory to multiply this number by the factor 2 so that the number of cells belonging to an interval dv becomes $8\pi V\ (v^2 dv/c^3)$.

It is now very simple to calculate the thermodynamic probability of a macroscopically defined state. Let N^s be the number of quanta belonging to the frequency domain dv^s. In how many different ways can we distribute these quanta over those cells which belong to the frequency interval dv^s? Let p_0^s be the number of vacant cells, P_1^s the number of those cells which contain one quantum, p_2^s the number of cells containing two quanta, etc.; then the number of different distributions is

$$\frac{A^s!}{p_0^s! \cdot p_1^s! \cdots}$$

where

$$A^s = \left(8\pi\nu^2/c^3\right) d\nu^s,$$

and

$$N^s = 0p_0^s + 1p_1^s + 2p_2^s + \dots$$

is the number of quanta belonging to the interval $d\nu^s$. The probability of the state which is defined by all the p_r^s is obviously

$$\prod_s \frac{A^s!}{p_0^s! p_1^s! \dots}$$

In view of the fact that we can look at the p_r^s as large numbers, we have

$$\ln W = \sum_s A^s \ln A^s - \sum_s \sum_r p_r^s \ln p_r^s,$$

where

$$A^s = \sum_r p_r^s.$$

This expression should be maximum satisfying the auxiliary condition

$$E = \sum_s N^s h\nu^s; \quad N^s = \sum_r r p_r^s.$$

Carrying out the variation gives the condition

$$\sum_s \sum_r \delta p_r^s \left(1 + \ln p_r^s\right) \doteq 0, \quad \sum_s \delta N^s h\nu^s = 0,$$

$$\sum_r \delta p_r^s = 0, \quad \delta N^s = \sum_r r \delta p_r^s$$

It follows that

$$\sum_s \sum_r \delta p_r^s \left(1 + \ln p_r^s + \lambda^s\right) + \frac{1}{\beta} \sum_s h\nu^s \sum_r r \delta p_r^s = 0$$

From this we get as the next step

$$p_r^s = B^s \exp\left(-rh\nu^s/\beta\right).$$

However, since

$$A^s = \sum_r B^s \exp\left(-\frac{rh\nu^s}{\beta}\right)$$

$$= B^s \left[1 - \exp\left(-\frac{h\nu^s}{\beta}\right)\right]^{-1},$$

We have

$$B_s = A^s \left[1 - \exp\left(-h\nu^s/\beta\right)\right].$$

Furthermore, we have the relation

$$N^s = \sum_r r p_r^s$$

$$= \sum_r r A^s \left[1 - \exp\left(-\frac{h\nu^s}{\beta}\right)\right] \exp\left(-\frac{rh\nu_s}{\beta}\right)$$

$$= \frac{A^s \exp\left(-h\nu^s/\beta\right)}{1 - \exp\left(-h\nu^s/\beta\right)}$$

Because of the above stated value of A^s, it is also true that

$$E = \sum_s \frac{8\pi h\nu^{s^3} d\nu^s}{c^3} V \frac{\exp\left(-h\nu^s/\beta\right)}{1 - \exp\left(-h\nu^s/\beta\right)}$$

Using the preceding results, one finds also that

$$S = k \left\{\frac{E}{\beta} - \sum_s A^s \ln\left[1 - \exp\left(\frac{h\nu^s}{\beta}\right)\right]\right\};$$

from this it follows that $\beta = kT$, because of the condition $\partial S/\partial E = 1/T$. Substituting kT for β in the above equation for E, one obtains

$$E = \sum_s \frac{8\pi h\nu^{s^3}}{c^3} V \left[\exp\left(\frac{h\nu^s}{kT}\right) - 1\right]^{-1} d\nu^s.$$

which is equivalent to Planck's formula.