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A Note on the Self-Consistency of the EIH Equations of Motion

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Abstract. The self-consistency of the Einstein Infeld and Hoffman (EIH) equations of motion is critically examined in the limiting case of a three-body problem where two bodies are very close to each other and a third quite far removed from them.

Key words: self-consistency—post-Newtonian equations

1. Introduction

The field equations of General Theory of Relativity (GTR) as obtained by Einstien in 1915 do not yield exact analytical equations of motion for the many-body problem. Einstein, Infeld & Hoffmann (1938) obtained an approximate equation of motion for the many-body problem in GTR assuming the bodies to be point masses. Since the equations of motion thus obtained were one order higher with respect to a 'smallness parameter' ε^2 (Misner, Thorne & Wheeler 1970) than the Newtonian equations of motion they are called post-Newtonian equations. The equation of motion was first given explicitly by Eddington & Clark (1938). It was later reproduced in Misner, Thorne & Wheeler (1970) with minor modifications. In this paper we aim to examine the singularities of this EIH equation of motion when all bodies are assumed to be static, that is, the initial acceleration of a initially static body and its singularities are studied when other bodies are also assumed to be initially static. To do this we take an example of three bodies, two close to each other and a third slightly removed from them and we study the initial acceleration of the third body when the other two are put in various initial positions.

2. The EIH equations without velocity dependent terms

As given by Equation 39.64 of Misner, Thorne & Wheeler (1970, hereinafter MTW), the EIH equation of motion including the velocity dependent terms is given by (gravitational units, that is, G = 1 and c = 1 are always assumed in this work),

$$\frac{d^2\mathbf{x}_K}{dt^2} = \sum_{A\neq K} \mathbf{r}_{AK}\frac{M_A}{r_{AK}^3}\left[1-4\sum_{B\neq K}\frac{M_B}{r_{BK}} - \sum_{C\neq A}\frac{M_C}{r_{CA}}\left(1-\frac{\mathbf{r}_{AK}\cdot\mathbf{r}_{CA}}{2r_{CA}^2}\right)\right.$$
$$\left. + v_K^2 + 2v_A^2 - 4\mathbf{v}_A\cdot\mathbf{v}_K - \frac{3}{2}\left(\frac{\mathbf{v}_A\cdot\mathbf{r}_{AK}}{r_{AK}}\right)^2\right]$$

$$-\sum_{A\neq K}(\mathbf{v}_A-\mathbf{v}_K)\frac{M_A\mathbf{r}_{AK}\cdot(3\mathbf{v}_A-4\mathbf{v}_K)}{r_{AK}^3}+\frac{7}{2}\sum_{A\neq K}\sum_{C\neq A}\mathbf{r}_{CA}\frac{M_AM_C}{r_{AK}r_{CA}^3} \tag{1}$$

where all notations are exactly the same as given in MTW.

Since we are interested in the initial acceleration of a static body K when other bodies are also initially static we remove all velocity dependent terms from this equation which yields

$$\left.\frac{d^2\mathbf{x}_K}{dt^2}\right|_{t=0}=\sum_{A\neq K}\mathbf{r}_{AK}\frac{M_A}{r_{AK}^3}\left[1-4\sum_{B\neq K}\frac{M_B}{r_{BK}}-\sum_{C\neq A}\frac{M_C}{r_{CA}}\left(1-\frac{\mathbf{r}_{AK}\cdot\mathbf{r}_{CA}}{2r_{CA}^2}\right)\right]$$
$$+\frac{7}{2}\sum_{A\neq K}\sum_{C\neq A}\mathbf{r}_{CA}\frac{M_AM_C}{r_{AK}r_{CA}^3}. \tag{2}$$

Let us now have three bodies idealized as point masses and having the following initial rectangular Cartesian-like coordinates

$(0, 0, 0) \rightarrow$ body no. 1
$(\tilde{x}_1, \tilde{y}, 0) \rightarrow$ body no. 2
$(\tilde{x}_1, -\tilde{y}, 0) \rightarrow$ body no. 3

We wish to study the initial acceleration of the first body due to the gravitational interaction with bodies no. 2 and 3.

Equation (2) reduces to

$$\left.\frac{d^2\mathbf{x}_1}{dt^2}\right|_{t=0}=\mathbf{r}_{21}\frac{M_2}{r_{21}^3}+\mathbf{r}_{31}\frac{M_3}{r_{31}^3}-4\mathbf{r}_{21}\frac{M_2}{r_{21}^3}\left(\frac{M_2}{r_{21}}+\frac{M_3}{r_{31}}\right)-4\mathbf{r}_{31}\frac{M_3}{r_{31}^3}\left(\frac{M_2}{r_{21}}+\frac{M_3}{r_{31}}\right)$$
$$-\mathbf{r}_{21}\frac{M_2}{r_{21}^3}\left(\frac{M_3}{r_{32}}-\frac{M_3}{r_{32}}\frac{\mathbf{r}_{21}\cdot\mathbf{r}_{32}}{2r_{32}^2}+\frac{M_1}{r_{12}}-\frac{M_1}{r_{12}}\frac{\mathbf{r}_{21}\cdot\mathbf{r}_{12}}{2r_{12}^2}\right)$$
$$-\mathbf{r}_{31}\frac{M_3}{r_{31}^3}\left(\frac{M_2}{r_{23}}-\frac{M_2}{r_{23}}\frac{\mathbf{r}_{31}\cdot\mathbf{r}_{23}}{2r_{23}^2}+\frac{M_1}{r_{13}}-\frac{M_1}{r_{13}}\frac{\mathbf{r}_{31}\cdot\mathbf{r}_{13}}{2r_{13}^2}\right)$$
$$+\frac{7}{2}\mathbf{r}_{32}\frac{M_2M_3}{r_{21}r_{32}^3}+\frac{7}{2}\mathbf{r}_{12}\frac{M_2M_1}{r_{21}r_{12}^3}+\frac{7}{2}\mathbf{r}_{23}\frac{M_3M_2}{r_{31}r_{23}^3}+\frac{7}{2}\mathbf{r}_{13}\frac{M_3M_1}{r_{31}r_{13}^3}, \tag{3}$$

where we have taken $K= 1$, summation over A and B to include bodies nos 2 and 3 respectively since we have $A\neq K$ *and* $B\neq K$, and summation over C to exclude body no. 2 when $A = 2$ or to exclude body no. 3 when $A = 3$.

If we take $\hat{x}$ and $\hat{y}$ to be unit vectors along x- and y- axes respectively, we have

$$\mathbf{r}_{21}=-\mathbf{r}_{12}=\tilde{x}_1\hat{x}+\tilde{y}_1\hat{y},$$
$$\mathbf{r}_{31}=-\mathbf{r}_{13}=\tilde{x}_1\hat{x}-\tilde{y}_1\hat{y},$$
$$\mathbf{r}_{23}=-\mathbf{r}_{32}=2\tilde{y}_1\hat{y}.$$

Let us assume all masses to be unity $M_1=M_2=M_3 = 1$, and therefore, Equation (3) reduces to

$$\left.\frac{d^2\mathbf{x}_1}{dt^2}\right|_{t=0}=\frac{2\tilde{x}_1\hat{x}}{(\tilde{x}_1^2+\tilde{y}_1^2)^{1.5}}-\frac{26\tilde{x}_1\hat{x}}{(\tilde{x}_1^2+\tilde{y}_1^2)^2}-\frac{5}{4}\frac{\tilde{x}_1\hat{x}}{\tilde{y}_1(\tilde{x}_1^2+\tilde{y}_1^2)^{1.5}}. \tag{4}$$

3. Results and discussions

The acceleration of body no. 1 is broken up into three components corresponding directly to the RHS of Equation (4) as

$$\text{Accn (1)=Newtonian term—post Newtonian term no. 1} \\ \text{—post Newtonian term no. 2.} \quad (5)$$

The results are plotted in Fig. 1, where each of the above terms are plotted with $\tilde{x}_1$ = 100 gravitational units and $\tilde{y}_1$ varying from 5 to 0. At $\tilde{y}_1$=0 a singularity occurs in the post Newtonian term no. 2 which appears to be abnormal due to the following reason.

The problem of two bodies having masses M_1 = 1 and M_2 = 2 with initial co-ordinates (0,0,0) and ($\tilde{x}_1$,0,0) respectively gives the expression for the initial acceleration of the first body as (both bodies assumed to be initially static)

$$\left.\frac{d^2\mathbf{x}_1}{dt^2}\right|_{t=0} = \frac{2\hat{x}}{\tilde{x}_1^2} - \frac{26\hat{x}}{\tilde{x}_1^3}, \quad (6)$$

following Equation (2). However, in the problem of the three bodies situated at the points (0,0,0), ($\tilde{x}_1$, $\tilde{y}_1$, 0)and ($\tilde{x}_1$, $-\tilde{y}_1$, 0) each with unit mass we expect as $\tilde{y}_1$ tends to zero (the masses remaining static of course) the initial acceleration of the first body to tend to that in the limit of the two body problem given by Equation (6). Or in other words, the Equation (4) should tend to Equation (6) in the limit $\tilde{y}_1 \rightarrow 0$ This is because to the first body the presence of two different distant bodies which are close to each other should appear more and more as a single body when their mutual distance of separation that is the distance between second and third bodies is continually decreased. But the RHS of Equation (4) *does not* tend to that of Equation (6) when $\tilde{y}_1 \rightarrow 0$. The Newtonian term does as is expected. The post Newtonian term no. 1, that is, the second term on the RHS of (4) reduces to the only post Newtonian term of (6). But the post Newtonian term no. 2, that is, the third term on the RHS of (4) diverges as $\tilde{y}_1 \rightarrow 0$ whereas there is no such term present on the RHS of(6). So the two limits are not mathematically agreeing with each other.

It is desirable that no fundamental force of nature shows such an anomaly in the asymptotic limits. In reality $\tilde{y}_1$=0 is of course ruled out since no body can be collapsed to exactly a point mass with infinite density. But in the present case we have already assumed the particles to be point-like. However for the sake of argument one can say that the black hole radius of r = 2 gravitational units can be considered a practical limit which a body of mass = 1 can be contracted to. So in Fig. 1 we show it to be a limit of practical possibility to which the second and third bodies can be brought close to each other which is $\tilde{y}_1$=2. In this case, the post Newtonian term remains still finite and is not negligible even at $\tilde{y}_1$>6 or so which shows that the problem persists in the practical cases such as stably revolving close pair of neutron stars.

The post-Newtonian approximation as introduced by Damour (1989) introduces five parameters in his section 6.9. These are $\alpha=L/R$ the 'geometric coupling' parameter, β_e , β_i, γ_e and γ_i as defined therein. As noted by him in section 6.10 of the same article that the last four of these parameters have to be much smaller than unity for the post-Newtonian approximations to be valid, that is, in other words, the post-Newtonian approximations are made under these assumptions. But in a previous

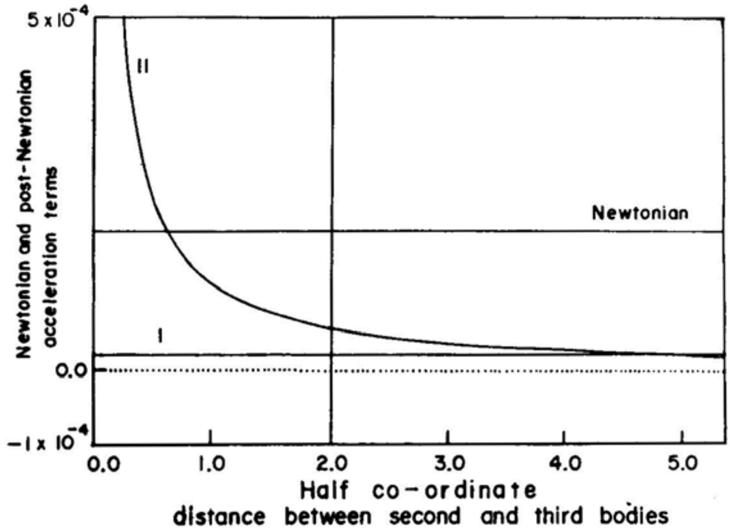

Figure 1. Relative strengths of the Newtonian and the post-Newtonian accelerations produced by a close pair of particles on another particle at far off distance as a function of separation between the pair. The Newtonian term remains fairly constant but one of the non-Newtonian terms coming from EIH equation is increasing indefinitely as the separation of the doublet is reduced. The non-Newtonian term exceeds the Newtonian value at a separation of about one unit of gravitational distance.

section (vide Equation (78) of the same reference) Damour makes a 'minimal assumption' $\alpha \ll 1$ which in our opinion is never an assumption under which post-Newtonian approximations (PNA) are valid. In fact in his chapter 9 Weinberg (1972) introduces $G\bar{M}/\bar{r}$ and $\bar{v}^2$ as the 'small parameters' of PNA (for $c = 1$), which is tantamount to stating that β_e, β_i, γ_e and γ_i are all $\ll 1$. Although it is true that in the real world all heavenly bodies are separated by much greater distances than their physical dimensions, it is possible to describe the mutual gravitational interaction of two hypothetically very close bodies and their combined interaction on a distant third body to a high degree of accuracy by Newtonian gravity and hence to a higher degree of accuracy by terms including upto PNA. This is to say that two earths or two suns or even two much denser and 'strongly' self-gravitating objects can be brought close enough so as to almost touch each other's surfaces and their gravitational interaction (mutual or otherwise) can be described perfectly well by Newtonian gravity and hence it must be possible to do it even better by including the terms of PNA. We have shown in this note the anomalous behaviour of one of the PNA terms in a special case where two close spherical bodies assumed to be point masses do not appear more and more as a single body to a distant observer unlike that in Newtonian gravity when the mutual distance between the two close bodies is reduced indefinitely. This in our opinion might cause problems with Weinberg's definition of quadrupole moment (Weinberg 1972). Damour (1989) further in his section 6.13 introduces an idea of 'effective' mass $m = m_0 + \gamma_i m_1 + \ldots$ (see page 168 of the article) for very strongly self-gravitating bodies to make PNA valid close to and inside the bodies. However by the phrase 'strongly self-gravitating' is meant bodies whose radii are very close to their black hole horizon which in the above discussion we have shown is not the only case where the second PNA term is anomalous in its behaviour. Thus the problem persists.

Another problem about such equations of motion is that an assumption has been made that all the bodies which are in actual cases nearly spherical can be represented as point masses with their entire mass concentrated at their centres. In Newtonian gravity such an assumption was later justified as a theorem by integration (summation of field due to infinitesimal mass points vectorially) over the whole body. To our knowledge such a theorem does not exist for the terms including up to the post-Newtonian approximation of GTR. This is still an open question. The last point to be noted is the fact that the velocity dependent terms in the PNA equations of motion can be neglected in studying the special case of our problem simply because velocities of all particles depend on the equations of motion and the initial conditions. Newtonian gravity describes the motion of bodies momentarily static in an inertial frame perfectly well and so should PNA since it does not invalidate any of the assumptions as stated earlier in deriving the PNA equations. Even in the case of all heavenly bodies having velocities such that $\bar{v}^2 \sim G\bar{M}/\bar{r}$ our conclusions remain valid.

However it should be noted that the basic assumptions behind the derivation of the EIH equation of motion is that the bodies should obey the post-Newtonian approximations, namely the mutual distances between any two bodies in gravitational units must be $\gg M$ where M can be considered as the mass of the heaviest object in the system. In that case, all the singularities are avoided. So the formulation remains althrough consistent, provided we never allow any two bodies to come very close to each other violating the basic assumption of the PPN which has been used in deriving the EIH equations of motion.

In conclusion, in this note we point out the fact that the Einstein, Infeld and Hoffman equations of motion contain some singularities that appear to violate the limiting nature of the gravitational forces, which can be avoided only if we strictly adhere to obeying the basic assumptions used to derive the EIH equations of motion during the course of dynamical evolution of a finite number of point-like particles.

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