J. Astrophys. Astr. (1989) **10**, 347–365

Stability of Resonant Planetary Orbits in Binary Stars

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Received 1989 April 19; accepted 1989 July 8

Abstract. This paper contains a numerical study of the stability of resonant orbits in a planetary system consisting of two planets, moving under the gravitational attraction of a binary star. Its results are expected to provide us with useful information about real planetary systems and, at the same time, about periodic motions in the general four-body problem (G4) because the above system is a special case of G4 where two bodies have much larger masses than the masses of the other two (planets). The numerical results show that the main mechanism which generates instability is the destruction of the Jacobi integrals of the massless planets when their masses become nonzero and that resonances in the motion of planets do not imply, in general, instability. Considerable intervals of stable resonant orbits have been found. The above quantitative results are in agreement with the existing qualitative predictions.

Key words: planetary systems—resonance—stability

1. Introduction

In 1978 Hadjidemetriou and Michalodimitrakis proposed a new approach to the study of planetary systems based on the study of periodic orbits of the general *N*-body problem ($N \geqslant 3$). The periodic orbits are of the planetary type, *i.e.* one or two bodies (called Suns) have large masses and the remaining bodies (called planets) have small masses.

The above study provides us with useful information about the dynamics of real planetary systems because one can find which configurations are unstable and therefore not expected in planetary systems existing in nature. It has been found that resonance configurations are not necessarily unstable. On the other hand, the existence of several resonance configurations in our solar system, such as the Sun-Jupiter-Saturn at the 2:5 resonance, the Galilean satellites of Jupitter at the 4:2:1 resonance, the Hilda group of asteroids at the 3:2 resonance, supports the above conjecture.

A large part of the families of planetary-type periodic orbits computed so far for several planetary models corresponds to resonance motion (Hadjidemetriou & Michalodimitrakis 1978). The computation of their stability showed that there exist stable resonant orbits and the stability depends on several factors such as the relative masses of the planets, the relative dimensions of the orbits of the planets and the phase difference in the motion of the planets.

In a qualitative study of the stability of resonant orbits in planetary systems, Hadjidemetriou (1979, 1985) using the Krein (1950) — Gelfand & Lindskii (1955) theory investigated the "dangerous" resonances where instability may develop, the type of instability which can appear in each case and the mechanism by which such instabilities develop. In this study the qualitative difference between systems with two planets (or satellites) and systems with more than two planets (or satellites) is emphasized. When a third planet is added, the increase of the planetary masses from the zero value destroys the Jacobi integral of each planet and leaves only the Jacobi integral of the whole system. As a consequence the system becomes potentially unstable in the sense that there exists a Hamiltonian perturbation which generates instability. The destruction of the Jacobi integral of each planet is a new mechanism of instability generation in the case $N > 2$ (and one Sun) which has been verified by numerical computations in particular cases (Hadjidemetriou 1979; Hadjidemetriou & Michalodimitrakis 1981).

The case of a planetary system with two Suns is briefly mentioned by Hadjidemetriou (1979). If the system has one planet, Hamiltonian perturbations cannot, in general, create instability. Dvorac (1984) has studied numerically planetary orbits in double stars using the model of the elliptic restricted problem of three bodies.

If however the system has two or more planets, there exist a Hamiltonian perturbation which generates instability. Such a perturbation could be a change of the planetary masses and/or a change in phase of the unperturbed motion of the planets. As far as we know, no quantitative study of the above case exists. The purpose of this paper is to fill this gap by finding numerically families of resonant periodic orbits in a planetary system with a binary star and two planets and examining their stability. We expect that the results would provide us with useful information about the expected configurations of real planetary systems with a binary star. On the other hand, such a study can be viewed more generally as a contribution to the quantitative study of the general fourbody problem.

Let $m_i (i = 1, \ldots, 4)$ be the masses of the pointlike bodies P_i $(i = 1, \ldots, 4)$ where P_1 and P_2 represent the binary and P_3 and P_4 the planets. Let also GXY be an inertial frame of reference with its origin at the mass center G of the 4-body system. We choose a rotating frame of reference $0xy$ such that:

(a) Its origin coincides with the mass centre of P_1 and P_2

(b) Its x-axis always contains P_1 and P_2, with the positive direction from P_2 to P_1. This frame reduces, for $m_3 = m_4 = 0$, to the wellknown rotating frame of the restricted 3-body problem. The motion of $P_i (i = 1, \ldots, 4)$ takes place on the GXY plane which coincides with the $0xy$ plane.

With respect to the GXY frame the system has six degrees of freedom and we choose as generalized coordinates the Cartesian coordinates.

$$x_1 (>0), x_3, y_3, x_4, y_4$$

of P_1, P_3, P_4 with respect to the $0xy$ axes and the angle θ between the GX and the $0x$ axes (Fig. 1).

A well known (Hadjidemetriou 1975a, 1977) way to find periodic orbits of the general planar 4-body problem (hereafter G4) is to start from periodic orbits of the corresponding degenerate problem ($\dot{m}_3 = m_4 = 0$, hereafter DG4). By giving small, but nonzero, values to the planetary masses and taking as approximate initial conditions

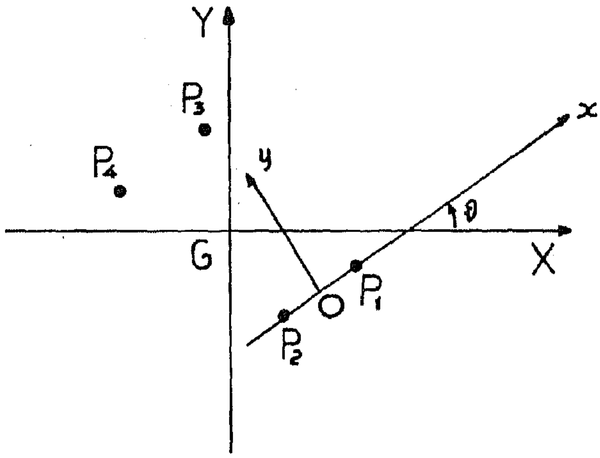

Figure 1. The inertial frame *GXY* and the rotating frame 0*xy*. *G* is the mass centre of the system and 0 is the mass centre of P_1 and P_2.

those of DG4 we find, by numerical continuation, periodic orbits (of the planetary type) of the G4.

To obtain periodic orbits of DG4 we start from the circular restricted 3-body problem with μ =½ (Copenhagen problem, hereafter CP) and put two massless points P_3, P_4 on suitable selected orbits of CP so that the motion of the system P_3 and P_4 is periodic. Here we consider the following cases:

(a) P_3 and P_4 revolve around the binary.

(b) P_3 revolves around P_1 and P_4 around P_2.

(c) P_3 and P_4 revolve around the same star (say P_1).

Case a: To obtain periodic orbits in this case we start from the well known families *m* and *l* of CP (Szebehely 1987) where the massless point revolves around both primaries P_1 and P_2. Putting two massless points P_3 and P_4 on the same orbit, of period *T*, but with a phase difference of 180° (see Fig. 2a) we obtain a periodic orbit of DG4 with period *T*. If we put P_3 and P_4 on different orbits with periods T_3 and T_4 (see Fig. 2b) such that T_4/T_3 = rational, we obtain a new periodic orbit of DG4. In this paper we study the case T_4/T_3 = ½. As a consequence of this we confine ourselves to the family *m* since there are no two orbits of *l* with T_4/T_3=½.

Case b: To obtain periodic orbits we start from the families *f* and *g* of CP where the massless point revolves around one of the primaries with period *T*. We put P_3 on an orbit around P_1 and P_4 on the same (symmetrical) orbit around P_2. In this way we obtain a periodic orbit of DG4 with period *T*. This can be done in two different ways. Either we launch P_3 and P_4 in phase or with a phase difference of 180° as shown in Figs 2c–f. In this paper we also study the case where P_3 and P_4 revolve around different orbits with period ratio T_4/T_3 = ½ as shown in Figs 2g–j.

Case c: To obtain periodic orbits we put P_3 and P_4 on the same orbit of the families *f* and *g* or on different orbits around P_1 such that T_3/T_4 = ½ (see Fig 3).

From Figs 2 and 3 we note that we already have many families of periodic orbits for fixed values of the masses m_i (i = 1, . . . , 4). This explains the restrictions we posed in the beginning, for example the equality of the masses m_1 and m_2, the selection of the circular (instead of the elliptic) restricted problem as a starting (unperturbed) problem, the choice of the resonances T_4 = T_3 = 1:1 and 1:2 only, *etc.* In spite of these

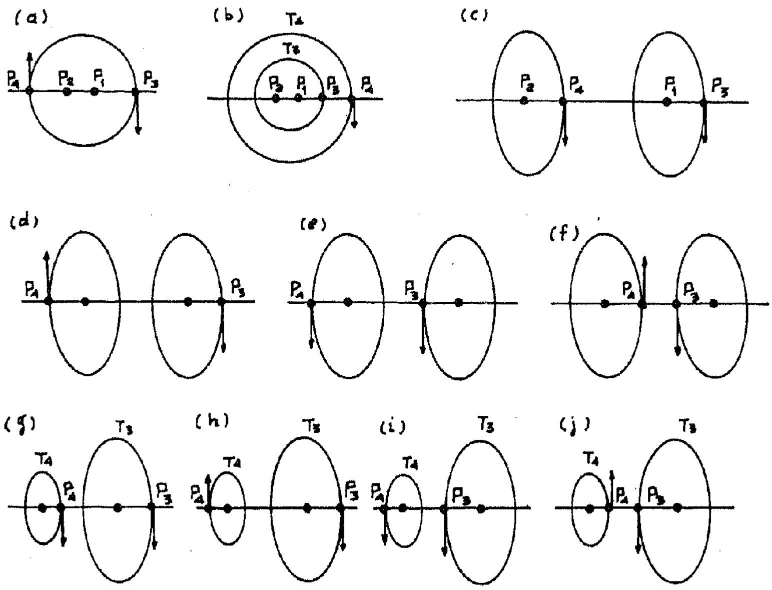

Figure 2. Periodic orbits in the cases where (i) P_3 and P_4 revolve around both stars, and (ii) P_3 revolves around P_1 while P_4 revolves around P_2 (shown schematically). The corresponding families of the Copenhagen problem are shown in Table 1.

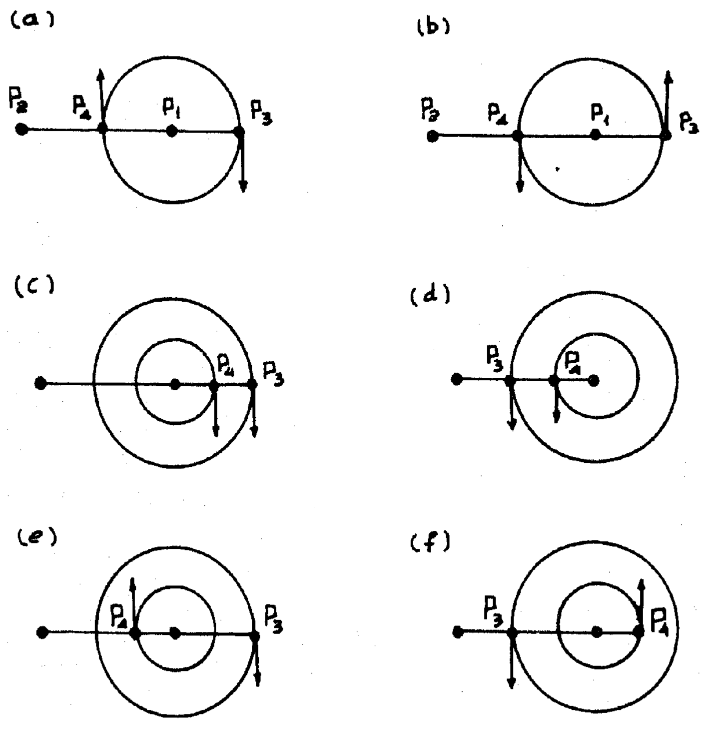

Figure 3. Periodic orbits in the case where P_3 and P_4 revolve around P_1 (shown schematically).

restrictions the volume of the results is still big enough, so we present here only some representative results to justify the overall conclusions.

2. Equations of motion in the rotating frame

We take

$$m_1 + m_2 + m_3 + m_4 = 1, \qquad K^2 = 1 \tag{1}$$

where K^2 is the gravitational constant.

By straightforward calculations it can be shown that the Lagrangian of the 4-body system expressed in terms of the generalized coordinates x_1, x_3, y_3, x_4, y_4, θ is

$$\begin{aligned} L = {} & \frac{1}{2}m_1\left(1+\frac{m_1}{m_2}\right)\dot{x}_1^2 + \frac{1}{2}m_3(1-m_3)(\dot{x}_3^2+\dot{y}_3^2) + \frac{1}{2}m_4(1-m_4)\cdot(\dot{x}_4^2+\dot{y}_4^2) \\ & - m_3m_4(\dot{x}_3\dot{x}_4+\dot{y}_3\dot{y}_4) + \frac{1}{2}\Big[m_1\left(1+\frac{m_1}{m_2}\right)x_1^2 + m_3(1-m_3)(x_3^2+y_3^2) \\ & + m_4(1-m_4)(x_4^2+y_4^2) - 2m_3m_4\cdot(x_3x_4+y_3y_4)\Big]\dot{\theta}^2 \\ & + [m_3(1-m_3)(\dot{y}_3x_3-\dot{x}_3y_3) + m_4(1-m_4)\cdot(\dot{y}_4x_4-\dot{x}_4y_4) \\ & + m_3m_4(\dot{x}_4y_3+\dot{x}_3y_4-\dot{y}_4x_3-\dot{y}_3x_4)]\dot{\theta} - V \end{aligned} \tag{2}$$

where

$$V = \sum_{i\neq j}\sum \frac{m_im_j}{r_{ij}} \qquad (i, j = 1, \ldots, 4)$$

and $r_{ij} = [(x_i - x_j)^2 + (y_i - y_j)^2]^{1/2}$ is the distance between P_i and P_j.

The Lagrange equations which correspond to the coordinates x_1, x_3, y_3, x_4, y_4, θ are respectively

$$\begin{aligned} & m_1\left(1+\frac{m_1}{m_2}\right)(\ddot{x}_1 - x_1\dot{\theta}^2) - m_1m_2\left(1+\frac{m_1}{m_2}\right)r_{12}^{-2} + m_1m_3(x_3-x_1)r_{13}^{-3} \\ & + m_1m_4(x_4-x_1)r_{14}^{-3} - m_1m_3\left(x_3+\frac{m_1}{m_2}x_1\right)r_{23}^{-3} - m_1m_4\left(x_4+\frac{m_1}{m_2}x_1\right)r_{24}^{-3} = 0, \end{aligned} \tag{3}$$

$$\begin{aligned} & m_3(1-m_3)\,(\ddot{x}_3 - \dot{y}_3\dot{\theta} - y_3\ddot{\theta} - (\dot{y}_3-y_4)\dot{\theta} - x_3\dot{\theta}^2] + m_3m_4[-\ddot{x}_4 + \dot{y}_4\dot{\theta} + y_4\ddot{\theta} + x_4\dot{\theta}^2] \\ & - m_1m_3(x_1-x_3)r_{13}^{-3} - m_2m_3\left(x_3+\frac{m_1}{m_2}x_1\right)r_{23}^{-3} + m_3m_4(x_4-x_3)r_{34}^{-3} = 0, \end{aligned} \tag{4}$$

$$\begin{aligned} & m_3(1-m_3)[\ddot{y}_3 + \dot{x}_3\dot{\theta} - (x_4-x_3)\ddot{\theta} - (\dot{x}_4-\dot{x}_3)\dot{\theta} - y_3\dot{\theta}^2] \\ & + m_3m_4(-\ddot{y}_4 - y_4\dot{\theta}^2 + x_4\dot{\theta}) - (m_1m_3r_{13}^{-3} - m_2m_3r_{23}^{-3} + m_3m_4r_{34}^{-3})y_3 = 0, \end{aligned} \tag{5}$$

$$\begin{aligned} & m_3(1-m_3)(\ddot{x}_4 - y_4\ddot{\theta} - x_4\dot{\theta}^2 + 2\dot{y}_4\dot{\theta}) - m_1m_4y_4r_{14}^{-3} \\ & + m_3m_4(-\ddot{x}_3 + y_3\ddot{\theta} + x_3\dot{\theta}^2 + 2\dot{y}_3\dot{\theta}) - m_2m_4x_4r_{24}^{-3} - m_3m_4r_{34}^{-3} = 0, \end{aligned} \tag{6}$$

$$\begin{aligned} & m_4(1-m_4)(\ddot{y}_4 + 2\dot{x}_4\dot{\theta} + x_4\ddot{\theta} - y_4\dot{\theta}^2) - m_1m_4y_4r_{14}^{-3} - m_3m_4(\ddot{y}_3 + 2\dot{x}_3\dot{\theta} + x_3\ddot{\theta} - y_3\dot{\theta}^2) \\ & - m_2m_4y_4r_{24}^{-3} - m_3m_4(y_4-y_3)r_{34}^{-3} = 0, \end{aligned} \tag{7}$$

$$\left[m_1\left(1+\frac{m_1}{m_2}\right)x_1^2+m_3(1-m_3)(x_3^2+y_3^2)+m_4(1-m_4)(x_4^2+y_4^2)\right.$$
$$\left.-2m_3m_4(x_3x_4+y_3y_4)\right]\dot\theta+m_3(1-m_3)(\dot y_3x_3-\dot x_3y_3)$$
$$+m_4(1-m_4)(\dot y_4x_4-\dot x_4y_4)+m_3m_4(\dot x_4y_3+\dot x_3y_4-\dot y_4x_3-\dot y_3x_4)=\text{const.}\equiv P_\theta. \quad (8)$$

Equation (8) is the angular momentum integral $P_\theta = \partial L/\partial\dot\theta$ of the system, which exists because θ is an ignorable coordinate ($\partial L/\partial\theta = 0$). With the aid of (8) we eliminate $\dot\theta$ and $\ddot\theta$ from Equations (3)–(7) and thus we find the equations of motion of P_i ($i = 1, \ldots, 4$) relative to the $0xy$ frame. This elimination introduces P_θ as a parameter in the equations of relative motion. In this way the motion of P_i relative to the $0xy$ frame can be studied independently of the rotation of $0xy$ as a problem with five degrees of freedom,

3. Families of symmetric periodic orbits in the rotating frame

The Equations of motion (3)–(8) remain invariant under the transformation

$$x_1 \to x_1,\ x_i \to x_i,\ y_i \to -y_i,\ \theta \to -\theta,\ t \to -t\ (i = 3,4).$$

Consequently if at $t = 0$ and $t = T/2$ the relations

$$y_3 = y_4 = 0,\ \dot x_1 = \dot x_3 = \dot x_4 = 0 \quad (9)$$

are verified, then the motion of the system is periodic with period T. This means that if P_3 and P_4 start moving perpendicularly to the Ox-axis while P_1 and P_2 are momentarily at rest and after a time $T/2$ they cross the Ox-axis perpendicularly while P_1 and P_2 are again momentarily at rest, the motion of the system is periodic and symmetric with respect to the Ox-axis. Therefore the initial conditions of such a symmetric periodic motion are

$$x_{10}, x_{30}, x_{40}, y_{30}, y_{40}.$$

Here we assume that the stars as well as the planets have equal masses, *i.e.*

$$m_1 = m_2, \qquad m_3 = m_4.$$

We remind that we also take $m_1 + m_2 + m_3 + m_4 = 1$.

From the above we conclude that the periodicity conditions (9) can be written in the form

$$y_i(x_{10}, x_{30}, x_{40}, \dot y_{30}, \dot y_{40}, m_3, T/2) = 0 \quad (i = 3, 4),$$
$$\dot x_j(x_{10}, x_{30}, x_{40}, \dot y_{30}, \dot y_{40}, m_3, T/2) = 0 \quad (j = 1, 3, 4).$$

These equations can be solved, in principle, for x_{j0}, y_{i0} to yield

$$x_{i0} = x_{io}(m_3, T),\ \dot y_{i0} = \dot y_{i0}(m_3, T).$$

Consequently the above symmetric periodic orbits belong to biparametric families with parameters m_3 and T.

In this paper we start with a small value of m_3 and we take the initial conditions of a periodic orbit of the restricted problem as approximate initial conditions of the periodic orbit corresponding to the chosen small value of m_3. We perform the numerical integration of the equations of motion for these approximate initial conditions until $y_3 = 0$ (we also take $\dot{\theta}_0 = 1$). We choose to keep x_{30} fixed and we compute the corrections δx_{10}, δx_{40}, $\delta \dot{y}_{30}$, $\delta \dot{y}_{40}$ to the initial conditions until the periodicity conditions are satisfied to a prescribed accuracy. Next we vary m_3 (or x_{30}) to obtain, in a similar way, new periodic orbits. In this way we obtain families of periodic orbits with fixed x_{30} (or m_3).

The integration of the equations of motion was performed by the time-series expansion method and an accuracy of 10 decimal places was retained throughout. For stability in computations the accuracy was increased to 14 or 16 decimal places.

Let M and *M'* be the biparametric families which result from the continuation of the orbits of the family *m* of CP. Family *M* consists of orbits like those shown in Fig. 2a while family *M'* of orbits like those shown in Fig. 2b. The symbols of the biparametric families of G4 studied here are shown in Table 1.

4. Linear stability of periodic orbits

Let

$$x_1(t), \quad x_3(t), \quad x_4(t), \quad y_3(t), \quad y_4(t),$$
$$\dot{x}_1(t), \quad \dot{x}_3(t), \quad \dot{x}_4(t), \quad \dot{y}_3(t), \quad \dot{y}_4(t), \tag{10}$$

be a solution of the equations of the relative motion which represents a periodic motion of period *T*. The corresponding variational equations are of the form

$$\ddot{\bar{\xi}} = A\bar{\xi} \tag{11}$$

where ξ_i ($i = 1, \ldots, 10$) are the wellknown variational variables which correspond to variations from the periodic orbit (10).

Table 1. Symbols for the biparametric families of G4.

Family of G4	Corresponding family of CP	Orbits are shown schematically in figure
M	m	2a
L	l	2a
F_1	f	2c
F_2	f	2d
G_1	g	2e
G_2	g	2f
F'	f	3a
G'	g	3b
M'	m	2b
F'_1	f	2g
F'_2	f	2h
G'_1	g	2i
G'_2	g	2j
F'''_1	f	3c
F'''_2	f	3e
G'''_1	g	3d
G'''_2	g	3f

Let $\Delta(t)$ be the monodromy matrix of (11). If all the eigen-values λ_i (i = 1, . . . , 10) of $\Delta(T)$ lie on the unit circle the periodic orbit is stable. The existence of the Jacobi integral for the system implies, in general, that two of the eigen-values are equal to unity. Therefore the characteristic equation of $\Delta(T)$ can be written in the form

$$(\lambda^2-1)(\lambda^2+k_1\lambda+1)(\lambda^2+k_2\lambda+1)(\lambda^2+k_3\lambda+1)(\lambda^2+k_4\lambda+1)=0.$$

By following a procedure analogous to that of the general 3-body problem (Brouke 1969; Hadjidemetriou 1975b) it can be shown (Hadjidemetriou 1988) that a periodic orbit is stable if the following conditions hold

$$D_1>0, \quad D_2>0, \quad |k_i|<2 \; (i=1, \ldots, 4)$$

where

$$D_1=k_1^2-4(k_2-2), \quad D_2=k_2^2-4(k_4-2).$$

The stability parameters k_t are expressed in terms of the elements of Δ (T). By integrating numerically the variational equations we compute the elements of $\Delta(T)$ and the corresponding values of D_1, D_2, k_i. To check the above results and to obtain a better picture of the stability properties of the periodic orbits we also make a direct computation of the eigenvalues of $\Delta(T)$.

As we know, by putting two massless points P_3 and P_4 on suitably selected periodic orbits of the CP we obtain periodic orbits of the DG4. If the orbits of CP are stable one might conclude that the resulting periodic orbits of DG4 and the periodic orbits of G4, obtained by continuation of DG4 with respect to m_3 and m_4, are also stable. However it is known that planetary systems with two suns and more than one planet are in general unstable in the sense that there exist Hamiltonian perturbations which destabilize the system (Hadjidemetriou 1979, 1985, Personal communication).

For example, in the case of a planetary system with two massless planets, the monodromy matrix $\Delta(T)$ has a 4-fold unit eigenvalue due to the existence of two Jacobi integrals (one for P_3 and one for P_4). If we perturb the system by giving small values to the planetary masses the motions of the planets cease to be independent. As a result, only one Jacobi integral remains, that of the whole system. So the 4-fold unit eigenvalue is transformed to a 2-fold unit eigenvalue and the other two eigenvalues are free to move outside the unit circle generating instability in this way.

The destruction of the Jacobi integrals is an important destabilization mechanism. In general, a perturbation destabilizes the system by moving some eigenvalues of $\Delta(T)$ out of the unit circle. As it is known (Yakubovich & Strarzinskii 1975) if a perturbation causes a meeting of eigenvalues of the same kind on the unit circle they cannot move off the circle but if the eigenvalues are of different kinds they may move off the circle. When the eigenvalues which move off the circle are complex the corresponding type of instability is known as complex instability. In terms of the stability parameters, complex instability exists if D_1 and/or D_2 are negative.

5. Numerical results

We computed families of periodic orbits for the following values of the planetary masses

$$M_3 = 10^{-6}, 10^{-5}, 10^{-4}, 10^{-3} \qquad (m_4 = m_3, m_1 = m_2).$$

In some cases we also computed the families $m_3 = 10^{-2}, 10^{-1}$. In what follows we shall describe the properties of these families. We begin with the description of those families where P_3 and P_4 are placed on orbits of CP with the same period.

Family M:

The initial conditions for P_4 are

$$x_{40} = -x_{30}, \quad \dot{y}_{40} = -\dot{y}_{30}.$$

The computed orbits extend from x_{30} =0.65 up to x_{30} = 8 and the planetary masses from $m_3 = 10^{-6}$ up to $m_3 = 10^{-1}$. Fig. 4 shows the projection of the characteristic curve of the family $m_3 = 10^{-6}$ on the $x_{30}\ \dot{y}_{30}$ plane of the initial conditions space ($x_{10}\ x_{30}\ \dot{y}_{30}$ ($x_{10} \simeq 0.5$) and the corresponding evolution of the period T along the family. The corresponding curves for the other families ($m_3 = 10^{-5}$ up to $m_3 = 10^{-1}$) are omitted (in order to save space). Table 2 shows some representative members of the above families.

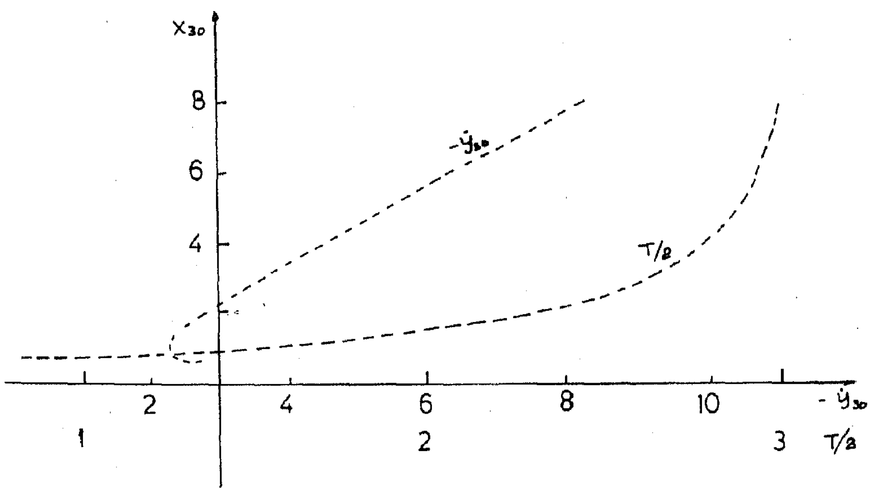

Figure 4. Family *M*: projection of the characteristic curve of the monoparametric family $m_3 = 10^{-6}$ and the evolution of the period T along the family.

Table 2. Initial conditions of some orbits of the families *M* and *l*.

Family	m_3	x_{10}	x_{30}	$\dot{y}_{30}$	T
M	10^{-6}	0.500000	8	−8.354116	6.017
M	10^{-3}	0.499668	8	−8.353805	6.017
M	10^{-1}	0.464935	4	−4.456826	5.674
L	10^{-6}	0.500000	2.2	−1.528960	9.191
L	10^{-3}	0.499867	2.2	−1.529673	9.212
L	40×10^{-3}	0.497034	2.2	−1.561171	10.609

As it is known (Henon 1965) the orbits of the family m of CP are stable for $x_{30} > 0.617$. Despite this, all computed orbits of M are found to be unstable. Fig. 5 shows, schematically, a typical position of the eigenvalues $\lambda_1, \ldots, \lambda_{10}$ of $\Delta(T)$ for the family M. In the degenerate case ($m_3 = m_4 = 0$) we have a 4-fold unit eigenvalue, say $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$, because of the existence of two Jacobi integrals, one for P_3 and one for P_4. We also have a double complex eigenvalue on the unit circle, say $\lambda_5 = \lambda_6$ (and the corresponding conjugates $\lambda_7 = \lambda_8$), because P_3 and P_4 are moving independently and symmetrically on the same stable orbit of the family m.

For $m_3 \neq 0$ only one Jacobi integral remains (for the whole system) and correspondingly a double unit eigenvalue, say $\lambda_1 = \lambda_2$. The other two eigenvalues λ_3, λ_4 are displaced off the unit circle as shown in Fig. 5 causing the appearance of instability. At the same time λ_5 and λ_6 separate from each other since the motions of P_3 and P_4 are no longer independent, but they still remain on the unit circle. For the same reason the pair λ_9, λ_{10}, which corresponds to the motion of P_1 remains also on the unit circle.

It appears that for the computed orbits of M the destruction of one of the Jacobi integrals of the degenerate problem is the only mechanism which generates instability.

The evolution of the polar angle of the eigenvalues λ_5 and λ_9 as we proceed along the family $m_3 = 10^{-6}$ from $x_{30} = 8$ to $x_{30} = 0.65$ is shown in Fig. 6. For large x_{30} all eigenvalues lie in a small angular interval near + 1. As we proceed to smaller x_{30}, λ_5, λ_6 ($\simeq \lambda_5$), λ_9 move counter-clockwise on the unit circle. On the other hand, λ_5 and λ_6 reach a maximum angle 155° at $x_{30} = 3.2$ and subsequently move clockwise. For the whole family $m_3 = 10^{-6}$, λ_5 and λ_6 practically coincide and λ_3, λ_4 remain very close to the unit circle. This is also true for all values of m_3 if x_{30} is relatively large. However, for relatively small x_{30} as m_3 increases λ_5 and λ_6 become well separated and λ_3, λ_4 move further away from the unit circle.

Family L:

The initial conditions for P_4 are $x_{40} = -x_{30}$, $\dot{y}_{40} = -\dot{y}_{30}$. The computed orbits extend from $x_{30} = 2.2$ up to $x_{30} = 8$ and the planetary masses from $m_3 = 10^{-6}$ up to $m_3 = 10^{-1}$. Fig. 7 shows the projection of the characteristic curve of the family $m_3 = 10^{-6}$ on the $x_{30}\ y_{30}$ plane and the evolution of the period T along the family. The

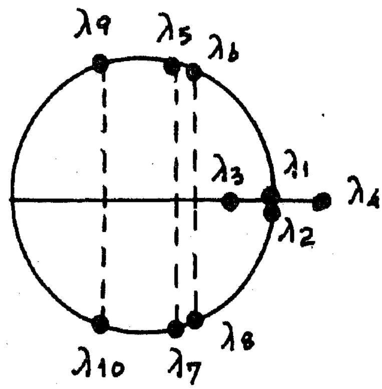

Figure 5. Family M: a typical (schematic) distribution of the eigenvalues λ_i ($i = 1, \ldots, 10$) of the monodromy matrix on the unit circle.

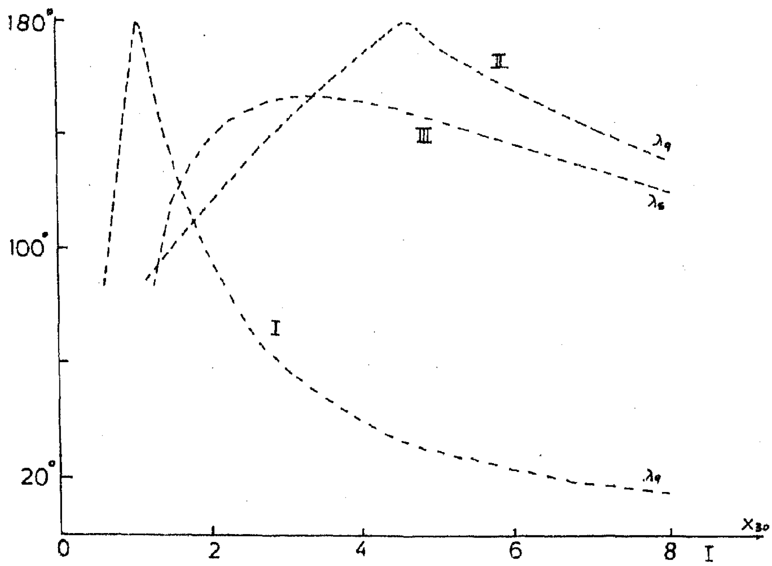

Figure 6. Family *M*: the evolution of the eigenvalues λ_5 and λ_9 along the family $m_3 = 10^{-6}$.

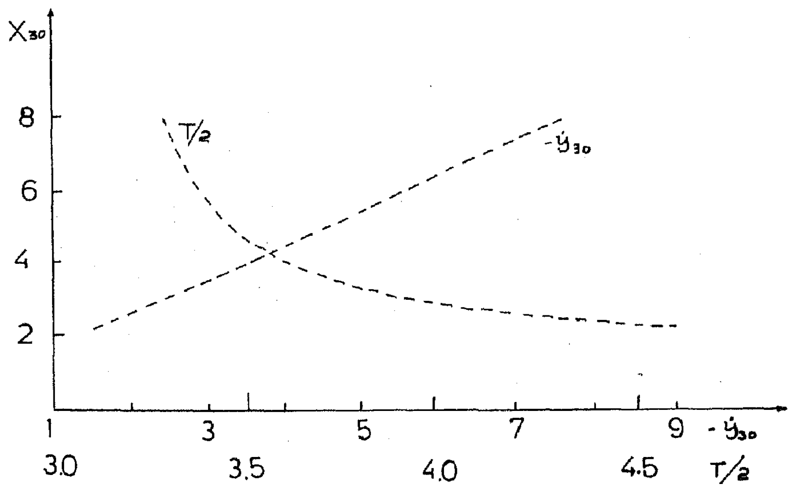

Figure 7. Family *L*: projection of the characteristic curve of the monoparametric family $m_3 = 10^{-6}$ and the evolution of the period *T* along the family.

corresponding curves for the other families ($m_3 = 10^{-5}$ upto $m_3 = 10^{-1}$) are omitted. Table 2 shows some representative members of *L*.

As it is known (Henon 1965), the orbits of the family *l* of CP are stable for $x_{30} > 1.9079$. Despite this, all computed orbits of *L* are found to be unstable and the behaviour of the eigenvalues of $\Delta(T)$ is qualitatively similar to that of the family *M*. However, for relatively small x_{30} complex instability appears for relatively large values of m_3.

Families F_1 and F_2:

The initial conditions for P_4 are $x_{40} = -x_{3f}$, $y_{40} = y_{3f}$ in the family F_1 and $x_{40} = -x_{30}$, $\dot{y}_{40} = -\dot{y}_{30}$ in the family F_2, where x_{3f} and $\dot{y}_{3f}$ are the "final" (for $t = T/2$) conditions of P_3.

The computed orbits extend from $x_{30} = 0.57$ up to $x_{30} = 1.7$ and the planetary masses from $m_3 = 10^6$ to $m_3 = 10^{-3}$. Fig.8 shows the projections of the characteristic curve of the family $m_3 = 10^{-6}$ of F_1 on the $x_{30}\,\dot{y}_{30}$, $x_{30}\,\dot{y}_{40}$,$x_{30}\,x_{40}$ planesof the initial condition space $x_{10}\,x_{30}\,x_{40}\,\dot{y}_{30}\,\dot{y}_{40}$ and the evolution of the period T along the family. The characteristic of the family ($m_3 = 10^{-6}$ of F_2 is a curve in the initial condition space $x_{10}\,x_{30}\,y_{30}$ and its projection on the $x_{30}\,y_{30}$ plane, as well as its T-curve, practically coincide with the corresponding curves of the family $m_3 = 10^{-6}$ of F_1. The corresponding curves for the other families ($m_3 = 10^{-5}$ upto $m_3 = 10^{-3}$ of F_1 and F_2) are omitted. Table 3 shows the initial conditions of some representative orbits of F_1 and F_2.

As it is known (Henon 1965) the orbits of the family f of CP are stable for $0.5 < x_{30} < 1.4410$ with the exception of a small interval of instability ($1.0875 < x_{30} < 1.20103$). Despite this, all computed orbits of F_1 are found unstable and in some cases complex instability appears. On the contrary, a small interval of stable orbits of F_2 has been found near $x_{30} = 1.35$.

Families G_1 and G_2:

The initial conditions for P_4 are $x_{40} = -x_{3f}$, $\dot{y}_{40} = -\dot{y}_{3f}$ in the family G_1 and $x_{40} = -x_{30}$, $\dot{y}_{40} = -\dot{y}_{30}$ in the family G_2 . The computed orbits extend from $x_{30} = 0.22$ up to

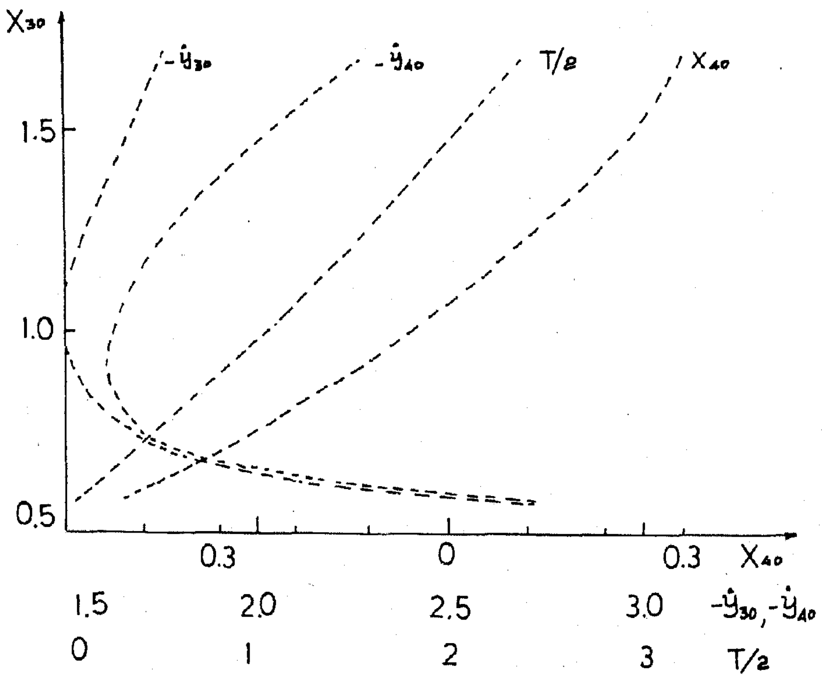

Figure 8. Family F_1: projection of the characteristic curve of the monoparametric family $m_3 = 10^{-6}$ and the evolution of the period T along the family.

Table 3. Initial condition's of some orbits of the families F_1, F_2, G_1, G_2.

Family	m_3	x_{10}	x_{30}	x_{40}	$\dot{y}_{30}$	$\dot{y}_{40}$	T
F_1	10^{-6}	0.500000	1.7	0.303049	−1.762887	−2.311786	4.823
F_1	10^{-2}	0.502328	1.3	0.139603	−1.587210	−1.826639	3.361
F_2	10^{-6}	0.500000	1.4	−1.4	−1.624452	1.624452	3.728
F_2	10^{-3}	0.500957	1.3	−1.3	−1.583467	1.583467	3.363
G_1	10^{-6}	0.500000	0.220	−0.741483	−1.085009	−1.290306	1.621
G_1	10^{-3}	0.499906	0.220	−0.741460	−1.085485	−1.289896	1.622
G_2	10^{-6}	0.500002	0.220	−0.220	−1.085002	1.085002	1.621
G_2	10^{-6}	0.500005	0.465	−0.465	−3.744243	3.744243	0.059

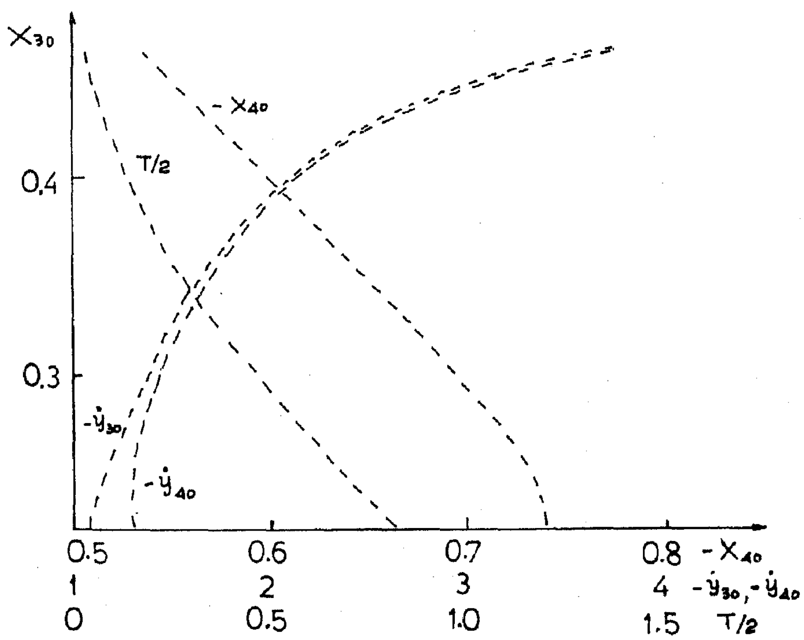

Figure 9. Family G_1: projection of the characteristic curve of the monoparametric family $m_3 = 10^{-6}$ and the evolution of the period T along the family. The curves for $\dot{y}_{30}$ and $T/2$ of the family G_2 almost coincide with the corresponding curves of the family G_1.

$x_{30} = 0.465$ and the planetary masses from $m_3 = 10^{-6}$ up to $m_3 = 10^{-3}$. Fig. 9 shows some projections of the characteristic curve of the family $m_3 = 10^{-6}$ of G_1 and G_2. Table 3 shows the initial conditions of some representative orbits of G_1 and G_2.

As it is known (Henon 1965) the orbits of the family g of CP are stable for $0.5 > x_{30} > 0.1813$ (with $\dot{y}_{30} < 0$). The computation of the stability parameters showed that for a large part of the families G_1 and G_2 the values of some $|K_i|$ are so close to the critical value 2, that we cannot make a definite statement about their stability (within the best accuracy of our computations).

Families F' and G':

The computed orbits extend from x_{30} = 0.6 up to x_{30} = 1.7 for the family F' and from x_{30} =0.22 up to x_{30} = 0.465 for the family G'. The planetary masses extend from $m_3 = 10^{-6}$ up to $m_3 = 10^{-3}$. All orbits of these families are found to be unstable.

In what follows we shall describe the properties of the families in which the ratio T_4/T_3 of the periods of P_4 and P_3 respectively in the degenerate case is equal to 1/2.

Family M':

The computed orbits extend from x_{30}= 1.4,.$x_{40} \simeq$ 0.7 up to x_{30} = 8, $x_{40} \simeq$ 1 and the planetary masses from $m_3 = 10^{-6}$ upto $m_3 = 10^{-3}$. Fig. 10 shows some projections of the characteristic curve of the family $m_3 = 10^{-6}$ and Table 4 shows the initial conditions of some orbits of the family M'. All computed orbits of M' are found to be stable. However the maximum of $|K_i|$ is very close to the critical value 2.

Families F'_1 and F'_2:

The computed orbits extend from x_{30} = 0.6125 up to x_{30} = 1.7 and the planetary masses from $m_3 = 10^{-6}$ up to $m_3 = 10^{-3}$. Figs 11 and 12 show, respectively, some projections of the characteristic curve of the family $m_3 = 10^{-6}$ of F_1 and F_2 Table 4 shows some representative orbits of the families F'_1 and F'_2.

For $m_3 = 10^{-6}$ a small interval of stable orbits of F' has been found around x_{30} = 1.35. However for larger values of m_3 this interval disappears and all orbits of F' become unstable. For example, for $m_3 = 10^{-3}$, the orbitas which were stable for $m_3 = 10^{-6}$ now show complex instability.

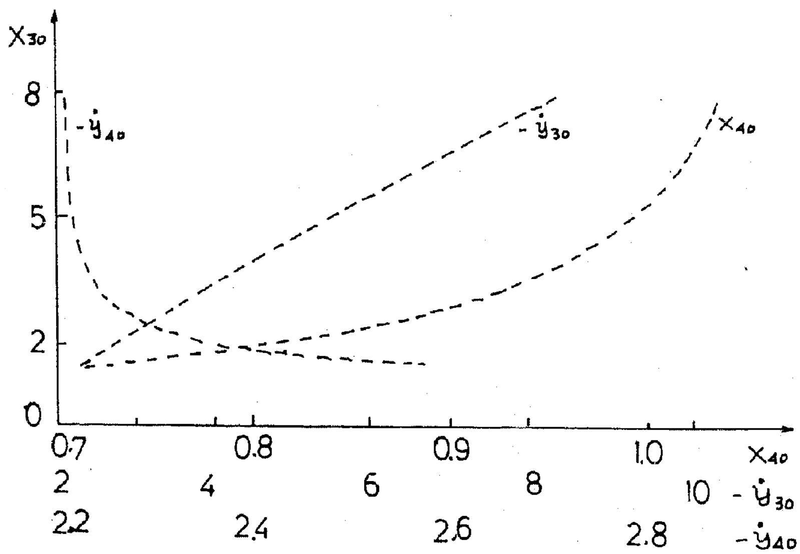

Figure 10. Family *M'*: projections of the characteristic curve of the monoparametric family $m = 10^{-6}$.

Table 4. Initial conditions of some orbits of the families M', F'_1, F'_2, G'_1, G'_2.

Family	m_3	x_{10}	x_{30}	x_{40}	$\dot{y}_{30}$	$\dot{y}_{40}$	T
M'	10^{-6}	0.500000	8.0	1.031868	−8.354117	−2.208424	6.017
M'	10^{-6}	0.500000	1.4	0.718565	−2.319518	−2.578500	3.825
M'	10^{-3}	0.499755	8.0	1.031882	−8.355317	−2.207578	6.021
M'	10^{-3}	0.499813	2.2	0.838199	−2.895288	−2.313923	4.789
F'_1	10^{-6}	0.500000	1.5	−0.062966	−1.669827	−1.620857	4.091
F'_1	10^{-3}	0.499892	1.4	−0.090596	−1.626204	−1.609773	3.730
F'_2	10^{-6}	0.500001	1.5	−0.982632	−1.669826	1.506907	4.091
F'_2	10^{-6}	0.500002	0.9	−0.727055	−1.523649	1.712516	1.663
G'_1	10^{-6}	0.500000	0.220	−0.682046	−1.085009	−1.499871	1.621
G'_1	10^{-3}	0.499820	0.220	−0.681820	−1.085771	−1.500103	1.621
G'_2	10^{-6}	0.500000	0.220	−0.312883	−1.085002	1.450458	1.621
G'_2	10^{-6}	0.502150	0.220	−0.313062	−1.078186	1.442771	1.642

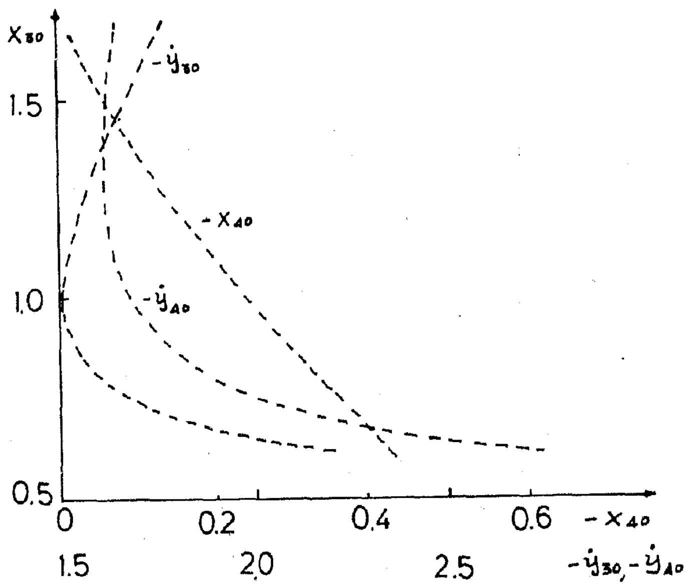

Figure 11. Family F'_1 : projections of the characteristics curve of the monoparametric family $m_3 = 10^{-6}$.

For $m_3 = 10^{-6}$ the orbit of F'_2 with $x_{30} = 1.7$ is strongly unstable. As we proceed to smaller x_{30} the instability weakens. For $x_{30} = 1.4$, 1.3 the orbits become stable. For smaller x_{30} the orbits are unstable up to $x_{30} \simeq 1$. For $x_{30} < 1$ all computed orbits are stable but the maximum of $|K_i|$ is very close to the critical value 2.

For larger values of m_3 the stability is not destroyed as in the case of F'_1. On the contrary the above-mentioned stability properties of the family $m_3 = 10^{-6}$ are preserved.

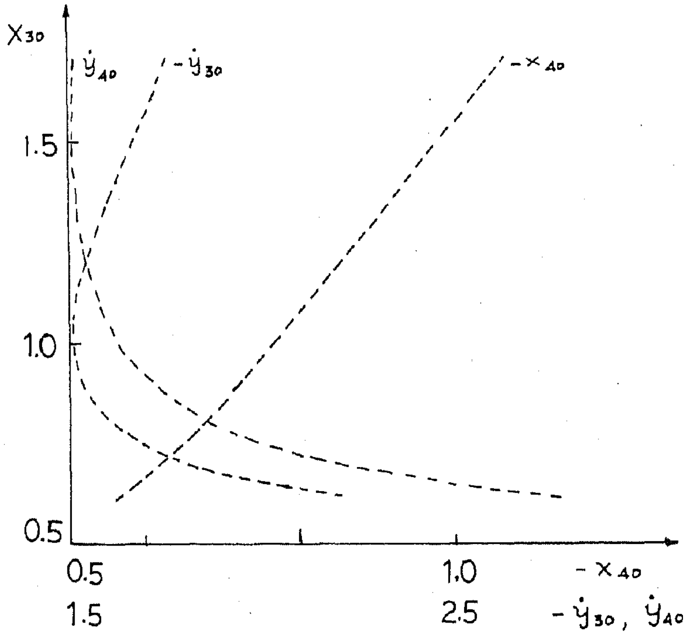

Figure 12. Family F'_2: projections of the characteritics curve of the monoparametric family $m_3 = 10^{-6}$.

Families G'_1 and G'_2:

The computed orbits extend from $x_{30} = 0.22$ up to $x_{30} = 0.435$ and the planetary masses from $m_3 = 10^{-6}$ up to $m_3 = 10^{-3}$.Figs 13 and 14 show some projections of the characteristic curve of the family $m_3 = 10^{-6}$ of G'_1 and G'_2 respectively. Table 4 shows the initial conditions of some orbits of G'_1 and G'_2.

For $m_3 = 10^{-6}$ the orbits of G'_1 in the interval $0.22 < x_{30} < 0.34$ are stable, but the maximum of $|K_i|$ is very close to the critical value 2. For $x_{30} > 0.34$ the orbits become mildly unstable. For larger values of m_3 the stability is not destroyed. For example, for $m_3 = 10^{-3}$ the orbits of G_1 in the interval $0.22 < x_{30} < 0.385$ are stable with the exception of a small instability interval near $x30 = 0.24$.

For $m_3 = 10^{-6}$ the orbits of G'_2 are mildly unstable except for a small interval of stability $0.365 < x_{30} < 0.385$. For larger m_3 all computed orbits of G'_2 are found to be unstable.

Families F'''_1, F'''_2, G'''_1, G'''_2:

The computed orbits extend from $x_{30} = 0.6125$ up to $x_{30} = 1.7$ for F'''_1, F'''_2 and from $x_{30} = 0.22$ up to $x_{30} = 0.435$ for G'''_1, G'''_2: .The planetary masses extend from $m_3 = 10^{-6}$ up to $m_3 = 10^{-3}$.Table 5 shows the initial conditions of some representative orbits of these families for $m_3 = 10^{-6}$. All computed orbits are found to be mildly unstable.

All previous results are based on the assumption that the planets have equal masses ($m_3 = m_4$). The question naturally arises whether these results depend, in an essential

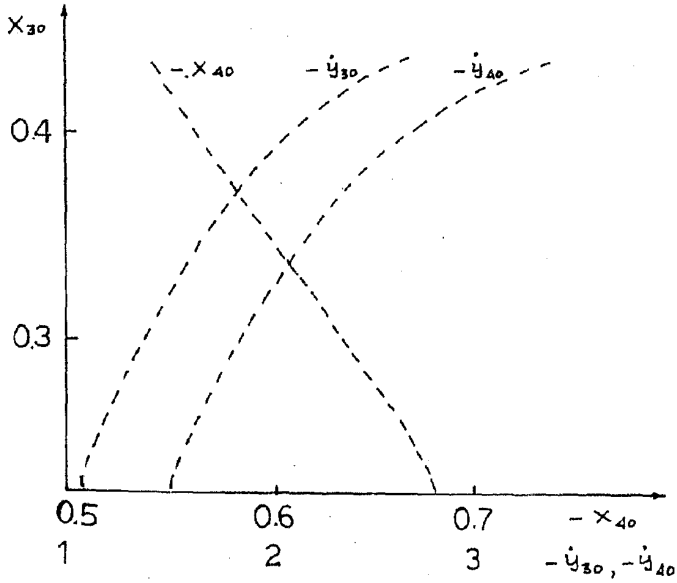

Figure 13. Family G'_1: projections of the characteristics curve of the monoparametric family $m_3 = 10^{-6}$.

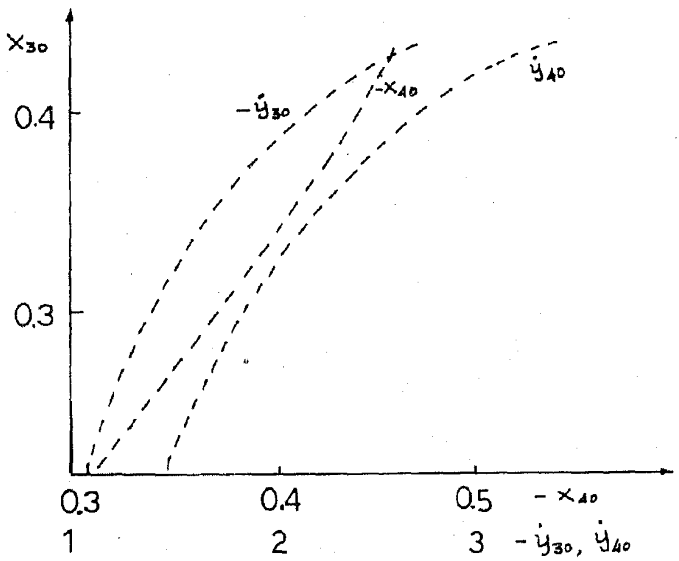

Figure 14. Family G'_2: projections of the characteristics curve of the monoparametric family $m_3 = 10^{-6}$.

way, on this assumption. So we selected some representative orbits of the previous families and varied m_3 (with $m_4 = 10^{-6}$) to obtain nearby periodic orbit with $m_3 \neq m_4$. The computations showed that the inequality of m_3 and m_4 does not change qualitatively the picture obtained for $m_3 = m_4$.

Table 5. Initial conditions of some orbits of the familie $F_1''', F_2''', G_1''', G_2'''$:**for** $m_3 = 10^{-6}$

Family	x_{10}	x_{30}	x_{40}	$\dot{y}_{30}$	$\dot{y}_{40}$	T
F_1'''	0.500001	1.2	0.862173	−1.545072	−1.542005	2.967
F_1'''	0.500003	0.65	0.591922	−1.975860	−2.423967	0.476
F_2'''	0.500000	1.2	0.158605	−1.545067	1.616129	2.967
F_2'''	0.500000	0.65	0.408360	−1.975850	2.431136	0.476
G_1'''	0.500002	0.22	0.312378	−1.085010	−1.450416	1.621
G_1'''	0.500004	0.435	0.458735	−2.708268	−3.439108	0.150
G_2'''	0.500000	0.22	0.682036	−1.084933	1.499868	1.621
G_2'''	0.500000	0.435	0.541243	−2.708247	3.441017	0.150

6. Discussion and conclusions

The results reported in this paper provide us with useful quantitative information about the stability properties of planetary systems with a binary star and two planets and, at the same time, about periodic motions in the general fourbody problem.

From qualitative studies it is known that such planetary systems are potentially unstable in the sense that there exists a Hamiltonian perturbation of the degenerate system which generates instability through the destruction of the Jacobi integral of each planet separately. It is also known that commensurabilities do not play a very important role in the stability of the system and in particular they do not always imply instability.

We computed many families of periodic orbits starting from the (degenerate) orbits of the Copenhagen problem. We confined ourselves to the resonances $T_4 : T_3 = 1{:}1$ and 1:2 but we selected a wide range of values for the masses of the planets.

Our results are in agreement with the above-mentioned qualitative results. For example, the detailed numerical study of the evolution of the eigenvalues λ_i ($i = 1, \ldots 10$) of the monodromy matrix for the orbits of the families *M* and *L* showed that the destruction of one of the Jacobi integrals of the degenerate problem seems to be the only mechanism which generates instability. For the family *L*, in particular, another kind of instability (complex instability) appears for large values of the planetary masses due to the fact that complex eigenvalues of different kinds collide and subsequently move off the unit circle. The differences, as far as stability is concerned, between the resonances 1:1 and 1:2 are clearly shown in the case of the families *M* and *M'*. Both families result from the family *m* of the Copenhagen problem. The family *M* corresponds to the resonance 1:1 and it is unstable while *M'* corresponds to the resonance 1:2 and it is stable.

The main conclusions of this work are:

1.The destruction of the Jacobi integrals of the degenerate problem is the main mechanism which generates instabilities in the cases we studied.

2.The families corresponding to the resonance 1:1 are in general unstable while those corresponding to the resonance 1:2 have considerable stable parts.

3.The families in which both planets revolve around the same star are unstable for both resonances 1:1 and 1:2.

4. The instability is, in general, mild in the sense that the maximum $|K_i|$ of the stability parameters is relatively close to the critical value 2. Complex instability is rare and appears for relatively large values of the planetary masses.

5. The stability is not necessarily destroyed by the increase of the planetary masses (families M', F'_1, G'_1).

6. The above results do not change qualitatively when the planets have unequal masses.

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