J. Astrophys. Astr. (1988) **9,** 67–74

On the Scalar-Tensor Theory of Lau and Prokhovnik

S. D. Maharaj *Department of Mathematics and Applied Mathematics, University of Natal, King George V Avenue, Durban 4001, South Africa*

A. Beesham *Department of Mathematics and Applied Mathematics, University of Durban-Westville, Private Bag X54001, Durban 4000 South Africa*

Received 1987 August 19; accepted 1988 January 25

Abstract. Recently Lau & Prokhovnik (1986) have formulated a new scalar-tensor theory of gravitation which reconciles Dirac's large numbers hypothesis with Einstein's theory of general relativity. The present work points out an error in the time-dependent cosmological term and the scalar potential given by Lau and Prokhovnik. The correct forms for these quantities are derived. Further, a vacuum Robertson–Walker solution to the generalized field equations is obtained, under an anasatz that we propose, which illustrates that the theory is, in some sense, incomplete.

Key words: other theories of gravitation—Cosmology

1. Introduction

A conceptually simple way to extend Einstein's theory of general relativity is to suppose that the gravitational term G is time dependent. In the past there have been several attempts at constructing consistent theories with variable G (Dirac 1938; Brans & Dicke 1961; Hoyle & Narlikar 1964; Canuto *et al.* 1977). Possibly the most simple of the suggested theories is the theory based on the large numbers hypothesis (LNH) of Dirac. The LNH is essentially a statement of the simplicity of the mathematical relation between some very large numbers that occur in nature.

The theory based on the Dirac LNH is not compatible with a constant cosmological term λ (Lau 1985), and requires the use of 'two metrics' (Dirac 1975, 1979). To overcome these problems Lau (1985) proposed that, in addition to a time varying G, the cosmological term λ be nonzero and time dependent. With this conjecture the Dirac LNH and general relativity (with $G = G(t)$ and $\lambda = \lambda(t)$) are reconciled. Furthermore Lau's theory reduces to Einstein's theory of general relativity in a limiting case and yields a viable dust cosmological model. Motivated by the success of this approach Lau & Prokhovnik (1986) generalized Lau's theory by formulating a new scalar-tensor theory in terms of an action principle. A time dependent scalar potential $\psi=\psi(t)$ is introduced such that $\lambda = \lambda(\psi)$ and $G = G(\psi)$ are coupled.

In this paper we point out an error in the paper by Lau and Prokhovnik (LP). A differential equation is incorrectly given so that the resulting solutions for the cosmological term and the scalar potential have to be modified. In this paper we present the corrected solutions. Further we obtain a vacuum solution to the generalized field equations of LP for the flat Robertson–Walker Spacetimes. This is in contrast to the earlier theory of Lau (1985) in which vacuum solutions turn out to be identical to the corresponding general relativistic solutions with constant λ (Beesham 1987). Vacuum solutions are significant because in our present interpretation the vacuum itself has a dynamical effect, *i.e.* the vacuum has an energy and matter content.

We emphasize that we consider the LP theory to be a viable generalization of Einstein's general relativity. However our vacuum solution illustrates that the theory is incomplete in the sense that there are more variables than there are equations. We are therefore forced to assume an ansatz to solve the field equations. In the conclusion we indicate how the LP theory can be supplemented and made complete by an additional condition utilizing the technique of Beesham (1986).

A further motivation for the study of theories which incorporate variable cosmological terms is the following. Nowadays, the parameter Λ is believed to correspond to the vacuum energy density of the quantum field (Zeldovich 1968), and it is thought that Λ was large during the early stages of the evolution of the universe, having strongly influenced its dynamics (Kasper 1985; Villi 1985; DerSarkissian 1985). The mass of the Higgs boson is thought to be related to Λ and G(Dreitlein 1974). Many workers have suggested the possibility of Λ being a variable quantity (*e.g.* Bergmann 1968; Wagoner 1970), with Linde (1974) having proposed that Λ is a function of temperature and relating it to the process of broken symmetry. Further interest in Λ arises within the context of quantum gravity, supergravity theories, Kaluza-Klein theories, the inflationary universe scenario, particle physics and grand unified theories (see Singh & Singh 1983; Lorenz-Petzold 1984; Banerjee & Banerjee 1985). There has been renewed interest in cosmological models with variable Λ, and the problems of singularity, horizon, flatness and monopole can be solved in some such models (*e.g.* Ozer & Taha 1986, 1987). The problem of fine tuning can also be explained naturally in certain variable Λ theories (*e.g.* Canuto *et al.* 1977). We remark in this connection that inflation, which also purports to solve some of the problems of the standard model is beset with many difficulties (Rothman & Ellis 1985; Olivo-Melchiorri & Melchiorri 1985).

In this paper we adopt the convention that the signature of the metric tensor is –2 and the speed of light is taken to be unity. For notational convenience dots will denote differentiation with respect to time, commas denote partial derivatives and semicolons denote covariant derivatives.

2. Field equations

Lau & Prokhovnik (1986) obtained the generalized field equations

$$R_{ij}-\tfrac{1}{2}Rg_{ij}+\Lambda g_{ij}=-8\pi G T_{ij}-\psi_{,i}\psi_{,j} \tag{1}$$

using a variational principle formulation. R_{ij} *is* the Ricci tensor, $R = R_i^i$ is the Ricci scalar, T_{ij} is the energy-momentum tensor and g_{ij} is the metric tensor. Λ becomes the new cosmological term and is related to λ by

$$\Lambda=\lambda(t)-\tfrac{1}{2}g^{00}\dot{\psi}^2.$$

In general Λ is also dependent on the spatial coordinates because of the term containing g^{00}. But for spatially homogeneous spacetimes g^{00} can be taken to be unity, in which case $\Lambda = \Lambda(t)$. The scalar potential ψ is strictly time dependent and couples Λ and G:

$$\psi = \psi(t),\ \Lambda = \Lambda(\psi),\ G = G(\psi).$$

Coupled to Equation (1) is the field equation for ψ,

$$\dot{\psi}\,\Box\psi + \dot{\Lambda} + \tfrac{1}{2}\dot{g}^{00}\dot{\psi}^2 + g^{00}\dot{\psi}\ddot{\psi} + 8\pi\dot{G}L_{\mathrm{m}} = 0, \tag{2}$$

where L_m is the matter Lagrangian density including all non-gravitational fields and

$$\Box\psi = g^{ij}\psi_{;i;j}.$$

All of the field Equations (1) and (2) are not independent because of the Bianchi identities. On taking the divergence of Equation (1) we obtain

$$g_i^j\Lambda_{,j} = -8\pi\dot{G}T_{io} - (\psi_{,i}\psi^{,j})_{;j}, \tag{3}$$

thereby relating Λ and G.

In the special case of vacuum ($L_{\mathrm{m}} = T_{ij} = 0$) we can immediately predict the behaviour of the gravitational term G. For vacuum we observe that G is absent from the field Equations (1) and (2). Hence G can be arbitrarily chosen.

3. Dirac LNH

The spatially homogeneous and isotropic Robertson-Walker specetimes with flat spatial sections are characterized by the metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - l^2(t)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2). \tag{4}$$

The energy-momentum tensor

$$T^{ij} = \rho u^i u^j$$

represents a pressure-free (‘dust’) perfect fluid where u^i is the comoving fluid four-velocity and ρ is the energy density of the fluid. For compatibility with the Dirac LNH we must have (Dirac 1938, 1979; Lau 1985)

$$l^2(t) = \beta t^{2/3}, \tag{5}$$

$$G(t) = \beta_1\frac{1}{t}, \tag{6}$$

$$(L_{\mathrm{m}} =)\rho(t) = \beta_2\frac{1}{t}, \tag{7}$$

where β, β_1, β_2 are constants.

Using the metric (4) it is easy to show that the field Equation (2) reduces to (see Appendix)

$$2\dot{\psi}\ddot{\psi} + \dot{\Lambda} + 3\frac{\dot{l}}{l}\dot{\psi}^2 + 8\pi\dot{G}\rho = 0. \tag{8}$$

The term $3\dot{l}\dot{\psi}^2/l$ in equation (8) is missing from the corresponding equation obtained by LP (p. 344). Consequently the forms for ψ and Λ will be modified on incorporation of the missing term. Substituting Equations (5)–(7) in Equation (8) we obtain

$$(\dot{\psi}^2)^{\dot{}} + \dot{\Lambda} + \frac{1}{t}\dot{\psi}^2 = 8\pi\beta_1\beta_2\frac{1}{t^3}. \tag{9}$$

The (0, 0) component of Equation (1) is given by

$$3\frac{\dot{l}^2}{l^2} - \Lambda = \dot{\psi}^2 + 8\pi G\rho. \tag{10}$$

Substituting Equations (5)–(7) in Equation (10) we obtain

$$\frac{1}{3t^2} - \Lambda - \dot{\psi}^2 = 8\pi\beta_1\beta_2\frac{1}{t^2}. \tag{11}$$

On differentiating Equation (11) with respect to time we have

$$-\frac{2}{3t^3} - \dot{\Lambda} - (\dot{\psi}^2)^{\dot{}} = -16\pi\beta_1\beta_2\frac{1}{t^3}. \tag{12}$$

Adding Equations (9) and (12) we obtain

$$\dot{\psi}^2 = (\tfrac{2}{3} - 8\pi\beta_1\beta_2)\frac{1}{t^2}. \tag{13}$$

Equation (13) has the general solution

$$\psi = (\tfrac{2}{3} - 8\pi\beta_1\beta_2)^{1/2}\ln t + A, \tag{14}$$

where A is constant. Substituting Equation (13) in Equation (11) yields

$$\Lambda = -\frac{1}{3t^2}. \tag{15}$$

Note that the corresponding expression for Λ found by LP contains an additional term, $-\dot{\psi}^2$. Thus we have obtained expressions for ψ and Λ, namely Equations (14) and (15), thereby correcting the solutions found by Lau and Prokhovnik. Since $\Lambda \sim t^{-2}$ the qualitative features of the solution obtained by LP remain unchanged. In particular the magnitude of Λ will decrease as the age of the universe increases.

4. Vacuum solutions

The existence of vacuum solutions to the field Equations (1) and (2) is of particular interest. With $T_{ij} = 0$ the field equation (1) can be written as

$$R_{ij} - \frac{1}{2}Rg_{ij} = -(\Lambda g_{ij} + \psi_{,i}\psi_{,j}). \tag{16}$$

The term $-(\Lambda g_{ij} + \psi_{,i}\psi_{,j})$ may be interpreted as the energy-momentum tensor of the vacuum in the presence of the gravitational field. This interpretation is possible because of the theory of modern quantum electrodynamics. An empty space is not considered as an inert region anymore, instead it continually gives rise to quantum fluctuations. Thus the vacuum itself has energy and momentum.

The theory Equations (16) for the metric (4) yield

$$3\frac{\dot{l}^2}{l^2}-\Lambda=\dot{\psi}^2, \tag{17}$$

$$2\frac{\ddot{l}}{l}+\frac{\dot{l}^2}{l^2}-\Lambda=0, \tag{18}$$

and Equation (8) becomes

$$2\dot{\psi}\ddot{\psi}+\dot{\Lambda}+3\frac{\dot{l}}{l}\dot{\psi}^2=0. \tag{19}$$

The divergence relation (3) also gives the result (19). The Equations (17)–(19) are a system of three equations for the three unknowns l, Λ and ψ. Therefore it seems possible to specify the general solution of this system. However this is not the case since Equations (17)–(19) are not independent. A linear combination of Equations (17) and (18) gives

$$2\frac{\ddot{l}}{l}-2\frac{\dot{l}^2}{l^2}=-\dot{\psi}^2. \tag{20}$$

On differentiating Equation (17) with respect to time we have

$$\dot{\Lambda}+2\dot{\psi}\ddot{\psi}=6\frac{\dot{l}}{l}\left(\frac{\ddot{l}}{l}-\frac{\dot{l}^2}{l^2}\right). \tag{21}$$

Substituting Equation (20) into Equation (21) we obtain exactly Equation (19), thereby demonstrating that only two of the Equations (17)–(19) are independent. This means that we have to specify one of l, Λ or ψ. Then Equations (17)–(19) will yield the remaining two variables.

In Section 3 we specified the scale factor $l(t)$ and obtained Λ and ψ. In this section we specify a form for ψ and attempt to obtain l and Λ. The simplest form of ψ would be linear. However, this is not a physical choice because then Λ grows larger as the age of the universe increases.

Guided by the form specified for ψ for compatibility with the Dirac LNH (*cf.* Equation 14) we choose

$$\psi=A\ln t+B, \tag{22}$$

where A and B are constants. Using Equation (22) we can write Equation (20) as

$$2\frac{\dot{l}}{l}=A^2\left(\frac{1}{t}\right)^{\cdot},$$

which has the general solution

$$2\frac{\dot{l}}{l}=A^2\frac{1}{t}+C \tag{23}$$

where C is a constant. Equation (23) can also be integrated and we obtain the solution

$$l^2(t)=\exp(A^2\ln t+Ct+D), \tag{24}$$

where D is a constant. Finally substituting Equation (24) in Equation (18) we get

$$\Lambda = -A^2\frac{1}{t^2} + \frac{3}{4}\left(A^2\frac{1}{t} + C\right)^2. \tag{25}$$

Equations (22), (24) and (25) constitute a vacuum solution to the generalized field equations of LP.

5. Conclusion

LP have presented a very interesting scalar-tensor theory which generalizes the earlier theory proposed by Lau (1985). However, our vacuum solution illustrates a somewhat unpleasant feature of the theory, *viz.,* that, in a certain sense, it is incomplete. There are more unknowns than equations and thus, to obtain specific solutions, one has to assume some ansatz. The situation here is very similar to that in the scale-covariant theory of Canuto *et al.* (1977). LP give no indication of how to overcome the difficulty present in their theory. This feature of their theory contrasts sharply with general relativity, which, in the above sense, is complete.

We indicate how it may be possible to achieve completeness by applying the ideas of Beesham (1986). For definiteness, we consider the Robertson-Walker models with energy density ρ and pressure p. From the field Equations (1), the Robertson-Walker metric (4) and the perfect fluid form of the energy-momentum tensor, we derive the following equation

$$\dot{\rho} + \rho\frac{\dot{G}}{G} + 3(\rho + p)\frac{\dot{l}}{l} + \frac{1}{8\pi G}\dot{\Lambda} + \frac{1}{8\pi G}\left[(\dot{\psi}^2)^{\cdot} + 3\frac{\dot{l}}{l}\dot{\psi}^2\right] = 0.$$

This reduces to the usual equation of conservation of mass-energy

$$\dot{\rho} + 3(\rho + p)\frac{\dot{l}}{l} = 0,$$

if we take

$$\rho\dot{G} = -\dot{\Lambda}/8\pi - \frac{1}{8\pi}\left[(\dot{\psi}^2)^{\cdot} + 3\frac{\dot{l}}{l}\dot{\psi}^2\right].$$

With this ansatz, the condition $G \sim 1/t$ (Dirac 1938), and an equation of state relating ρ to p, the solution to the field equations is fully determined.

Appendix

In this appendix we obtain the field Equation (8), governing the behaviour of ψ, and explicitly identify the missing term in the corresponding equation given by LP. The field equation for ψ is just Equation (2):

$$\dot{\psi}\,\Box\psi + \dot{\Lambda} + \tfrac{1}{2}\dot{g}^{00}\dot{\psi}^2 + g^{00}\dot{\psi}\ddot{\psi} + 8\pi\dot{G}L_{\mathrm{m}} = 0 \tag{A1}$$

where $\psi = \psi(t)$. The matter Langrangian density L_m is equal to the energy density ρ since we are considering a 'dust' distribution of matter and g^{00} is unity by the metric

(4). Hence Equation (Al) reduces to

$$\dot{\psi}\,\Box\psi + \dot{\Lambda} + \dot{\psi}\ddot{\psi} + 8\pi\rho\dot{G} = 0. \tag{A2}$$

We now evaluate the quantity $*\psi$:

$$\begin{aligned}\Box\psi &= \psi_{;i;j}g^{ij}\\ &= (\psi_{,i})_{;j}g^{ij}\\ &= (\psi_{,i,j} - \psi_{,k}\Gamma^k_{ij})g^{ij},\end{aligned} \tag{A3}$$

where Γ^k_{ij} are the connection coefficients. (Note that $\psi_{;i} = \psi_{,i}$ because ψ is a scalar). Since $\psi = \psi(t)$ equation (A3) becomes

$$\begin{aligned}\Box\psi &= \ddot{\psi} - \dot{\psi}\Gamma^0_{ij}g^{ij}\\ &= \ddot{\psi} - \tfrac{1}{2}g^{ok}(g_{jk,i} + g_{ki,j} - g_{ij,k})g^{ij}\\ &= \ddot{\psi} + \tfrac{1}{2}\dot{\psi}g_{ij,0}g^{ij}\\ &= \ddot{\psi} + 3\frac{\dot{l}}{l}\dot{\psi}\end{aligned} \tag{A4}$$

The appearance of the term $3(l/l)\psi$ is directly related to the nonvanishing of the connection coefficients Γ^0_{ij}. The term involving Γ^0_{ij} arises because covariant differentiation is not commutative. Substituting Equation (A4) in (A2) we obtain

$$2\dot{\psi}\ddot{\psi} + \dot{\Lambda} + 3\frac{\dot{l}}{l}\dot{\psi}^2 + 8\pi\dot{G}\rho = 0 \tag{A5}$$

and we have derived Equation (8). The term $3(\dot{l}/l)\dot{\psi}$ in Equation (A5) has been omitted from the corresponding field equation in the paper by LP.

Acknowledgements

The authors are grateful to the referees for useful suggestions which led to an improvement in the manuscript.

References

Banerjee, A., Banerjee, N. 1985, *Astrophys. Sp. Sci.,* **114**, 165.
Beesham, A. 1986, *Int. J. Theor. Phys.,* **25**, 1295.
Beesham, A. 1987, *Austr. J. Phys.,* **40**, 23.
Bergmann, P. G, 1968, *Int. J. Theor. Phys.,* **1**, 25.
Brans, C., Dicke, R. H. 1961, *Phys. Rev.,* **124**, 925.
Canuto, V., Adams P. J., Hsieh, S. H., Tsiang, E. 1977, *Phys. Rev.,* **D 16**, 1643.
DerSarkissian, M. 1985, *Nuovo Cim.,* **88B**, 29.
Dirac, P. A. M. 1938, *Proc. R. Soc. London,* **A 165**, 199.
Dirac, P. A. M. 1975, *The General Theory of Relativity,* Wiley, New York
Dirac, P. A. M. 1979, *Proc. R. Soc. London,* **A 365**, 19.
Dreitlein, J. 1974, *Phys. Rev. Lett.,* **33**, 1243
Hoyle, F., Narlikar, J. V. 1964, *Proc. R. Soc. London,* **A282**, 191.
Kasper, V. 1985, *Gen. Relativ. Gravitation,* **17**, 725.

Lau, Y. K. 1985, *Austr. J. Phys.,* **38**, 547.
Lau, Y. K., Prokhovnik, S. J. 1986, *Austr. J. Phys.,* **39**, 339.
Linde, A. D. 1974, *JETP Lett.,* **19**, 183.
Lorenz-Petzold, D. 1984, *Astrophys. Sp. Sci.,* **100**, 461.
Olivo-Melchiorri, B., Melchiorri, F. 1985, *Riv. del Nuovo Cim.,* **8**, 1.
Ozer, M., Taha, M. O. 1986, *Phys. Lett.* B, **171**, 363.
Ozer, M., Taha, M. O. 1987, *Nucl. Phys.,* **B287**, 776.
Rothman, T., Ellis, G. F. R. 1985, *Metqflation,* Univ. Cape Town preprint 85/18.
Singh, T., Singh, Tarkeshwar, 1983 *Astrophys. Sp. Sci.,* **97**, 127.
Villi, C. 1985, *Nuovo Cim.,* **85A**, 175.
Wagoner, R. V. 1970, *Phys. Rev.,* **Dl** , 3209.
Zel'dovich, Ya. B. 1968, *Sov. Phys.,* USPEKHI, **11**, 381.