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Spacetime with Self-Gravitating Thick Disc

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Abstract. We compute the metric coefficients and study some properties of the spacetime comprising of a Schwarzschild hole distorted by a self gravitating thick disc.

Key words: black hole, accretion disc—self gravity

1. Introduction

Thick accretion discs are formed when the radiation emitted during accretion interacts with the accreting matter dynamically, resulting in puffing up of the disc. The description of such discs can be obtained in a number of papers (*e.g.* Abramowicz, Jaroszynski & Sikora 1978; Jaroszynski, Abramowicz & Paczyński 1980; Paczyński & Abramowicz 1982; Paczyński & Wiita 1980; Chakrabarti 1985a,b, henceforth referred to as Papers 1 & 2). In these works the disc structures are calculated assuming the self-gravity effect due to the disc is negligible. The characteristic density of the disc at which this effect becomes important is M^3_{hole}/r_c ($\simeq 0.05$ g cm $^{-3}$ for a hole of mass $10^8 M_\odot$, r_c is the radius of the centre of the disc). It is not clear at what maximum mass the disc is still stable against the local gravitational instabilities, but even when the ratio $M_{\text{disk}}/M_{\text{hole}} = m$ is as small as 0.01, the numerical works (*e.g.* Wilson 1981) with self-gravity effect has shown the outer edge of the disc to be sensitive to the disc mass. Some work on self-gravitating discs with pseudo-Newtonian geometry has been carried out by Abramowicz, Calvani, & Nobili (1983) and Abramowicz *et al.* (1984). In a somewhat different approach, Will (1974, 1975) has discussed the effect of a ring in Kerr geometry. His analysis was carried out by perturbation theory. However, for static, axially symmetric, spacetime the solution of the problem is exact. In this paper we determine the metric coefficients of the spacetime with a black hole surrounded by a self-gravitating disc and study some of the properties.

The approach of the present analysis will be the following: When some matter is distributed around a hole, the hole cannot be considered to be isolated. It becomes distorted, so to speak, due to the gravitating matter outside the horizon. Under the circumstances when the matter is static, and axially symmetrically distributed around a hole, it is possible to write down the metric of the region of the spacetime devoid of

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matter in terms of the so-called distorted hole metric (Mysak & Szekeres 1966; Geroch & Hartle 1982, henceforth referred as GH; Chandrasekhar 1983). Because of the complexity of the problem, we shall replace the disc by a ring of the same mass located at the centre of the disc. Since most of the mass of the disc *is* concentrated near the centre of the disc, this may not be a bad assumption. Essentially, we shall be studying the properties of the matter in the combined field of the black hole and the ring of matter instead of the field of the black hole alone.

Here we give expectations of what might happen when the self-gravity is turned on. One would naively think that the centripetal force upon an element of matter orbiting between the hole ($r = r_h$, r_h being the horizon size) and the ring ($r = r_c$) will be reduced because of the outward pull by the ring than otherwise. Thus the angular momentum needed by the element to stay in a Keplerian orbit would be less than what is needed if the ring were absent. Similarly, the required angular momentum will be more if the element orbits beyond the ring. This changes the nature of the pressure gradient force needed to support the disc for a given angular momentum distribution. Below we point out that one has to be somewhat careful in talking about cases with and without self-gravity side by side. This is essential because in the framework of general relativity, strictly speaking, it is meaningless to compare the two problems. For example, if one solves the problem without self-gravity turned on with a black hole mass M_{hole}, then when the ring of 'mass' M_{ring} is present, what mass of the hole should be used so that the comparison is a meaningful one? What is the definition of the mass of the hole anyway? The mass arises as a constant of integration which is identified as the mass of the hole because at infinity, the potential term 'looked' like that of a Newtonian body of mass M_{hole}. With the ring present, the potential at infinity will be that due to the sum of the hole mass and that of the disc. Thus, dealing with the properties of the hole between $r = r_h$ and r_{ring} one does not know how to separate the contributions of the mass of the disc, the mass of the hole and the binding energy between the hole and the disc. This problem appears because there is no 'Newtonian limit' of the region between the horizon and the ring. A similar problem arises in the definition of angular momentum. It can easily be shown that several different definitions of angular momentum (henceforth always assumed to be the specific quantity, namely, the angular momentum per unit mass) go over to the Newtonian value when the proper limit at a large distance is taken. For example, $-u_\phi$, $-u_\phi/u_t$, $u_\phi u_t$ all go over to ΩR^2 at infinity but between the hole and the ring only $-u_\phi$ can be considered as the conserved angular momentum. The situation is further aggravated by the question: suppose we have a well-defined quantity to compare. Then, one should quite reasonably compare them at the same spacetime point and not at the same coordinate point. For example, the potential at some $r = r_0$ before self-gravity of the disc is turned on, should not be compared with that after the self-gravity at $r = r_0$ where r is, say, the spherical polar coordinate used in both spacetimes. We shall put emphasis on these points clearly whenever such confusions arise.

Other than the complexities mentioned above, the problem of finding the equipotentials for the discs is not at all different from what is done in Papers 1 and 2 as both the cases are stationary and axisymmetric. We remind the readers what is done in these papers: When the matter distribution is assumed to be barotropic, the surfaces of constant specific angular momentum l and the surfaces of the constant angular velocity Ω coincide provided matter is predominantly rotating. The solution of the Euler equation is obtained by using an angular momentum distribution as a power

Law of von-Zeipel parameter λ given by,

$$\lambda^2 = \frac{l}{\Omega}. \tag{1}$$

The behaviour of λ(Papers 1 and 2, also Abramowicz 1982, where it is denoted by *R*) is similar to axial distance *R* in Newtonian geometry and it is always possible to define this quantity in any axisymmetric spacetime. Thus, with $l = c\lambda^n$ (*c* and *n* being a pair of constants defining the distribution) one obtains the effective potential *W* defined to be $\mathrm{d}W = -\int \mathrm{d}p/(p+\varepsilon)$, (where, ρ and ε are the isotropic pressure and the energy density respectively and the equation of state is chosen to be barotropic) as given by,

$$\frac{e^W}{u_t(1-c^2\lambda^{2n-2})^\alpha} = \text{constant}. \tag{2}$$

Here α is given by,

$$\alpha = \frac{n}{2n-2}$$

and u_t is the specific binding energy and can be calculated from the normalization condition as,

$$u_t = \left[\frac{-(g_{t\phi}^2 - g_{tt}g_{\phi\phi})}{(g_{\phi\phi} + l g_{t\phi})(1-c^2\lambda^{2n-2})}\right]^{1/2} \tag{3}$$

The quantities $g_{\mu\nu}$ are the metric coefficients for the axisymmetric spacetime considered. Therefore, the same solution as given by equation (2) is valid in the case where the disc is massive. In Papers 1 and 2, λ was calculated using the Schwarzschild metric: $\lambda = \left(-\frac{g_{\phi\phi}}{g_{tt}}\right)^{1/2}$. Here, however, Λ (defined here in capital letter so as to distinguish from that without the self-gravity of the disc) has to be calculated with the new metric components. Thus, solving the problem with self-gravity boils down to determining the new metric coefficients.

2. The metric coefficients

The calculation of the metric coefficients is done with the assumption that the ring is static and is symmetrically placed in the equatorial plane around a Schwarzschild hole. It is easy to show that the rotational energy of the matter at the centre of the disc is less than 10 per cent of the rest mass energy. The rotational effect will introduce a small $g_{t\phi}$ component of the metric which is proportional to $m = M_{\text{disk}}/M_{\text{hole}}$ and the specific angular momentum of the ring. Thus, our analysis of replacing the rotating disc by a static ring is justified for $m < 1$.

In Weyl coordinates (ρ and z) the line element of a static axisymmetric spacetime takes the form (Synge 1964),

$$\mathrm{d}s^2 = \exp[2(\gamma-\psi)](\mathrm{d}\rho^2 + \mathrm{d}z^2) + \rho^2\exp(-2\psi)\mathrm{d}\phi^2 - \exp(2\psi)\mathrm{d}t^2 \tag{4}$$

where, ρ and z are the Weyl coordinates given by,

$$\rho = r\sin\theta\sqrt{\left(1-\frac{2}{r}\right)} \tag{5a}$$

and

$$z=(r-1)\cos\theta \tag{5b}$$

in the usual spherical coordinate system. The hole mass has been chosen to be unity as before, $\psi = \psi(\rho, z)$ and $\gamma = \gamma(\rho, z)$ are two scalar functions. In the empty spacetime, the Einstein equation $R_{\mu\nu} = 0$ takes the form,

$$\psi_{,\rho\rho}+\frac{1}{\rho}\psi_{;\rho}+\psi_{,zz}=0 \tag{6a}$$

$$\gamma_{,\rho}=\rho[(\psi_{,\rho})^2-(\psi_{,z})^2], \tag{6b}$$

and

$$\gamma_{,z}=2\rho\psi_{,\rho}\psi_{,z} \tag{6c}$$

where the comma in front of ρ and z denotes the differentiation with respect to them. Notice from equation (6a) that ψ satisfies the 'flat' metric Laplacian. In particular, this means that the sum of two solutions, namely, ψ_h (potential due to the hole alone) and ψ_r (potential due to the ring alone) is another solution. In our case, ψ_h will be given by,

$$\psi_h=\tfrac{1}{2}\log\left(1-\frac{2}{r}\right) \tag{7}$$

corresponding to the Schwarzschild metric. The potential ψ_r will be due to the ring and $\psi_t = \psi_h + \psi_r$ will be the total potential due to the hole ring system, γ is calculated by the requirement that it should vanish on the axis of symmetry and then performing the integration,

$$\gamma=\int_{\text{ABC}}\rho\left[\left[\left(\frac{\partial\psi^2}{\partial\rho}\right)^2-\left(\frac{\partial\psi}{\partial z}\right)^2\right]\mathrm{d}\rho+2\frac{\partial\psi}{\partial\rho}\frac{\partial\psi}{\partial z}\mathrm{d}z\right], \tag{8}$$

where, ABC denotes any path lying entirely in the region devoid of matter, beginning on the axis of symmetry and ending on the field point (see Fig. 1). The solution for ψ_r and γ are obtained (Weyl 1917; Bach & Weyl 1922) in terms of the complete elliptic functions and are given by,

$$\psi_r=-\frac{\frac{2}{\pi}mK(\kappa)}{[(\rho+b)^2+z^2]^{1/2}}. \tag{9a}$$

When the hole is absent, so $\psi_h = 0$ and Equations (5) become $p = r\sin\theta$ and $Z = r\cos\theta$, the Bach-Weyl solution for γ is

$$\gamma=\frac{m^2\kappa^4}{4\pi^2 b\rho}[-K^2+4(1-\kappa^2)K\dot{K}+4\kappa^2(1-\kappa^2)\dot{K}^2]$$
$$+\frac{m^2\kappa^4}{4\pi^2 b^2}[-K^2+4(1-\kappa^2)K\dot{K}-4\kappa^2(1-\kappa^2)(2-\kappa^2)\dot{K}^2], \tag{9b}$$

where, $K = \mathrm{d}K/\mathrm{d}\kappa^2$ and

$$\kappa\equiv\left[\frac{4b\rho}{(\rho+b)^2+z^2}\right]^{1/2}. \tag{9c}$$

The ring is placed at $\rho = b$ and m is the mass of the ring.

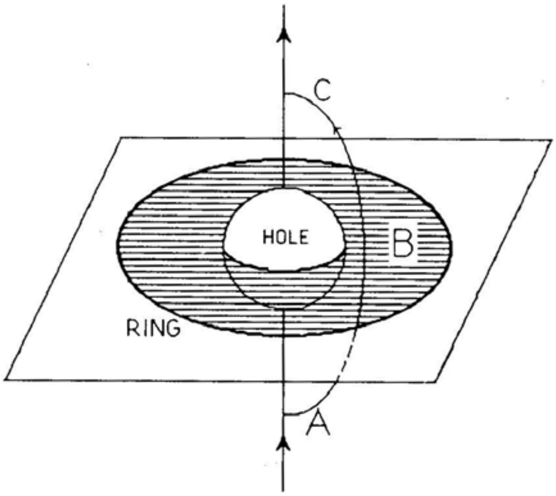


Figure 1. The geometry of hole and thick disc is modeled as a Schwarzschild hole surrounded by a ring on the equatorial plane. Integration in equation (8) is done along the curve ABC which lies entirely in the empty space between the hole and the ring intersecting the equatorial plane in the shaded region.

Although, these solutions are in closed form, and for some extreme limiting cases they can be written down in terms of elementary functions, for our purposes we would like to re-derive ψ_r by using a new 'spherical' polar coordinate system Π and Θ, such that, $z = \Pi\cos\Theta$ and $\rho = \Pi\sin\Theta$. Thus, in terms of the usual polar coordinates r and θ,

$$\Pi^2 = r^2 - 2r + \cos^2\theta \tag{10a}$$

and

$$\Theta = \tan^{-1}\frac{\rho}{z} = \frac{\sqrt{r(r-2)}\tan\theta}{(r-1)}. \tag{10b}$$

If now the ring be kept on the equatorial plane $\left(\theta = \frac{\pi}{2} = \Theta\right)$ at $r = r_c$, then in terms of the new coordinate system it is located at $\Pi_c = \sqrt{r_c(r_c - 2)} = b$. The solution of the Laplace equation for the ring of matter is given by,

$$\psi_{r<} = -m \sum_{l(\text{even})} \frac{\Pi^l}{\Pi_c^{\,l+1}} P_l(\cos\Theta) P_l(0) \tag{11a}$$

for $\Pi < \Pi_c$, and

$$\psi_{r>} = -m \sum_{l\,(\text{even})} \frac{\Pi_c^l}{\Pi^{l+1}} P_l(\cos\Theta) P_l(0) \tag{11b}$$

for $\Pi > \Pi_c$. The P_l are the Legendre polynomials of order l. Odd l polynomials give zero contribution so the sum is over even l only. Here m denotes the ratio of the disc mass in the unit of the original hole mass. Upon expansion, Equations (1 1a,b) become,

$$\psi_{r<} = -m\left[\frac{1}{\Pi_c} - \frac{\Pi^2}{4\Pi_c^3}(3D-1) + \frac{3\Pi^4}{64\Pi_c^5}(35D^2 - 30D + 3) - \ldots\right] \tag{12a}$$

and

$$\psi_{r>} = -m\left[\frac{1}{\Pi} - \frac{\Pi_c^2}{4\Pi^3}(3D-1) + \frac{3\Pi_c^4}{64\Pi^5}(35D^2 - 30D + 3) - \dots\right] \tag{12b}$$

Where

$$D = \cos^2\Theta = \frac{(r-1)^2\cos^2\theta}{r^2 - 2r + \cos^2\theta}.$$

At this point it is important to remember that Equation (6a) is linear, so that even if the series expansion for ψ_r is terminated after some finite terms, ψ_r still remains the solution of that equation. But, unless all the terms are kept in the series, ψ_r will not represent the potential due to a ring but some other matter distribution close to it. Fig. 2 shows the surface of constant ψ_r with terms up to $l = 6$. (Ignore the kinks in the contours, as they are due to the choice of a coarse grid.) Near the ring it resembles the potential due to a torus, but far away from the hole the potential is almost spherically symmetric as expected. With this potential the new metric coefficients g_{tt} and $g_{\phi\phi}$ can be calculated from,

$$g_{tt} = \left(1 - \frac{2}{r}\right)e^{2\psi_r} \tag{13a}$$

and

$$g_{\phi\phi} = -r^2\sin^2\theta e^{-2\psi_r}. \tag{13b}$$

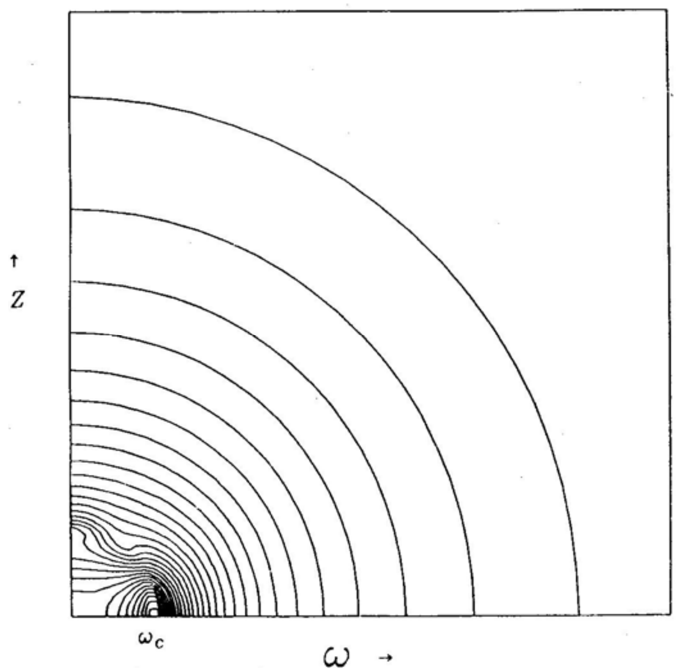


Figure 2. The equipotential surfaces due to the ring in Schwarzschild geometry with terms up to $l = 6$ (see equations 1 la and 11b). Ignore the kinks in the contours, as they are due to choice of a coarse grid. Near the ring it resembles the potential due to a torus, and far away the potential is almost spherically symmetric and is due to a total mass of $M_{\text{hole}} + M_{\text{ring}}$ as expected.

Hence the von Zeipel parameter Λ is given by,

$$\Lambda = \frac{r\sin\theta}{\sqrt{1-\frac{2}{r}}}\, e^{-2\psi_r} \tag{14}$$

One word of caution is in order. A natural, geometrical definition of 'radius' and a definition intimately connected with angular momentum is (radius) $\equiv \varpi$= (circumference)/2π.In the metric without the ring $\varpi = r\sin\theta$, while in the new metric $\varpi = \sqrt{-g_{\phi\phi}} = r\sin\theta\, e^{-\psi_r}$. Hence any property at $R = r\sin\theta$ in the old metric should be compared with that property at $Re^{-\psi_r}$ and not at R.

3. Properties of hole-ring spacetime

We now study very briefly the properties of the hole-ring spacetime. Notice first that both the $\psi_{r<}$ and $\psi_{r>}$ are regular in their respective domains. On the axis with $r<r_c$,

$$\psi_{r<}|_{\text{axis}} = -m\left[\frac{1}{\Pi_c} - \frac{\Pi^2}{2\Pi_c^3} + \frac{3\Pi^4}{8\Pi_c^5} - \dots\right] \tag{15}$$

and on the equatorial plane,

$$\psi_{r<}|_{\text{eq}} = -m\left[\frac{1}{\Pi_c} + \frac{\Pi^2}{4\Pi_c^3} + \frac{9\Pi^4}{64\Pi_c^5} + \dots\right] \tag{16}$$

Now the equation of the horizon is given by,

$$g_{tt} = 0 = \left(1-\frac{2}{r}\right)e^{2\psi_r}. \tag{17}$$

On the equator, the horizon is located at ,$\varpi_{EH} = 2e^{-\psi_{r<}|_{EH}}$, with ($\Pi^2 = 0$),

$$\psi_{r<}|_{EH} = -\frac{m}{\Pi_c},$$

whereas on the axis, the horizon is at $\varpi_{AH} = 2e^{-\psi_{r<}|_{AH}}$, with ($\Pi^2 = 1$),

$$\psi_{r<}|_{AH} = -m\left(\frac{1}{\Pi_c} - \frac{1}{4\Pi_c^3} + \frac{9}{64\Pi_c^5} - \dots\right).$$

Clearly, as a result of the ring, the horizon is of oblate shape and bulged in the equatorial plane.

The Keplerian angular momentum can be derived from the geodesic equation,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = 0. \tag{18}$$

Here $\Gamma^\mu_{\alpha\beta}$ is the Christoffel symbol and τ is the proper time measured along the geodesic. The first term vanishes since $u^\phi = d\phi/d\tau$ and $u^t = dt/d\tau$, which arc the only nonvanishing components of the velocity vector, are along Killing directions. Expanding the second term in terms of the components we obtain.

$$\Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = (\Gamma_{rtt}dt^2 + 2\Gamma_{rt\phi}dt\,d\phi + \Gamma_{r\phi\phi}d\phi^2)d\tau^{-2} = 0. \tag{19}$$

Substituting the Christoffel symbol we find the Keplerian frequency **Ω_k** to be,

$$\Omega_k \equiv \frac{d\phi}{dt} = \sqrt{-\frac{g_{tt,r}}{g_{\phi\phi,r}}} = \Omega_0 e^{2\psi_r}, \qquad (20)$$

where,

$$\Omega_0 = \left[\frac{1 + r(r-2)\dfrac{\partial\psi_r}{\partial r}}{r^3 - r^4\dfrac{\partial\psi_r}{\partial r}}\right]^{1/2}$$

Hence the Keplerian specific angular momentum is given by,

$$l_k = \Omega_0 e^{-2\psi_r}\frac{r^2}{1-\dfrac{2}{r}}. \qquad (21)$$

The distribution is clearly discontinuous at $\Pi = \Pi_c$ because of the discontinuity of $\partial\psi_r/\partial_r$.

The above equations have been written down in spherical polar coordinates. They become much simpler and more transparent if the following tetrads arc chosen instead, namely,

$$\lambda_t^{(t)} = \sqrt{\left(1-\frac{2}{r}\right)}\exp(\psi), \qquad (22a)$$

$$\lambda_\phi^{(\phi)} = (r\sin\theta)\exp(-\psi), \qquad (22b)$$

$$\lambda_\rho^{(\rho)} = \exp(\gamma - \psi), \qquad (22c)$$

$$\lambda_z^{(z)} = \exp(\gamma - \psi). \qquad (22d)$$

It is useful to write down the equations in the form in which a comparison can be made with the result calculated without self-gravity (WSG) terms. First we note that (with self-gravity included) the use of a new radial distance $\omega = re^{-\psi_r}$ and the use of the unit of $M' \rightarrow M_{hole}\, e^{-\psi_r}$, simplify the equations very much. (This 'reduced' mass of the hole goes over to $Me^{-\psi_{r(horizon)}}$ as prescribed by Geroch & Hartle 1982.) In this unit, we derive,

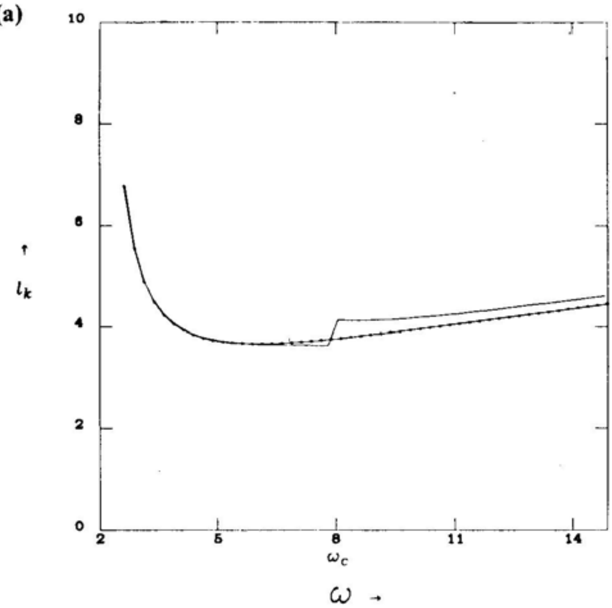
$$\Lambda(\omega) = \frac{\omega\sin\theta}{\sqrt{1-\dfrac{2}{\omega}}}, \qquad (3)$$

$$\Omega_k(\omega) = \left[1 + \omega(\omega-1)\frac{\partial\psi_r}{\partial\omega}\right]^{1/2}\frac{1}{\omega^{3/2}}, \qquad (3)$$

$$l_k(\omega) = \Omega_k\Lambda^2 = \left[1 + \omega(\omega-1)\frac{\partial\psi_r}{\partial\omega}\right]^{1/2}\frac{\omega^{3/2}}{(\omega-2)}. \qquad (3)$$

(The Equations 23b–c are defined in the plane of the ring.) The plot of Keplerian specific angular momentum versus ω is shown in Fig. 3(a–c) for various values of m, the ratio of the disc mass to the hole mass. Angular momentum has been measured in the unit of M' since the hole size is changed. The dotted line corresponds to the

(a)



(b)

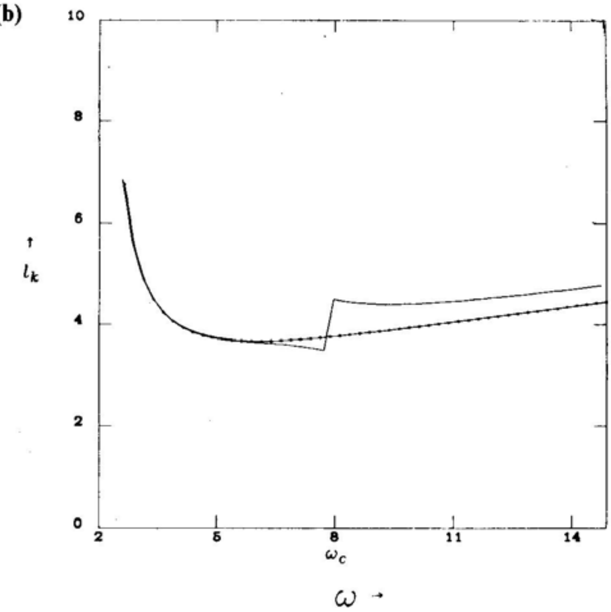


Figure 3. The Keplerian angular momentum distribution is compared without (dotted curve, plotted against $\omega = R$) and with (undotted curve, plotted against $\omega = Re^{-\psi_r}$) the self-gravity effect for various values of $m = M_{hole} / M_{ring}$. The dotted curve is given by Equation (23c); the undotted curve is the same equation with $\psi_r = 0$. In (a), $m = 0.05$, in (b) $m = 0.1$, and in (c) $m = 0.2$. Because of the ring, angular momentum needed to stay in a circular orbit is less or greater than what is necessary when the ring is absent, depending upon whether the particle is between the hole and the ring or between the ring and infinity.

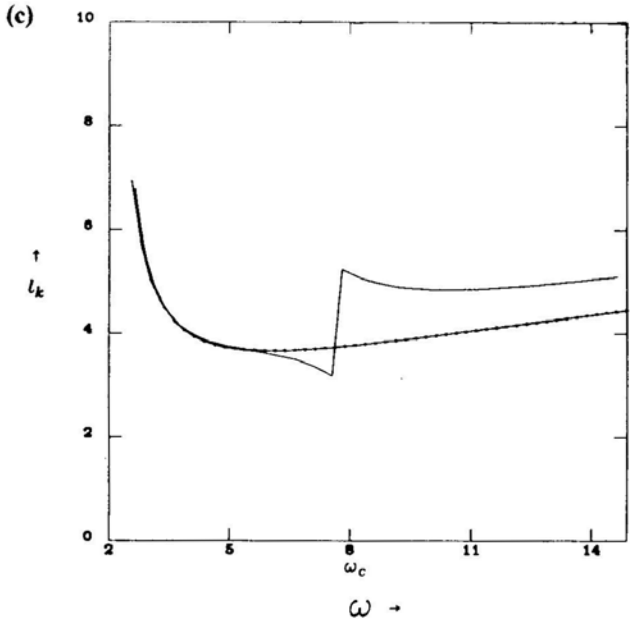


Figure 3. Continued

Keplerian angular momentum in WSG case. The unit of both ω and l_k is such that M' = 1. Very near the hole, angular momentum remains almost the same but near the ring $l_k(\omega)$ decreases for $r<r_c$. Away from the ring it increases in the same way as we discussed earlier. Notice the discontinuity of the angular momentum at the ring. For a real disc with nonsingular matter distribution, the angular momentum increases smoothly from the lower value to the higher value passing through the original Keplerian curve at r_c.

4. Discs and inner jets in the hole-ring spacetime

In Papers 1 and 2, we have shown the surfaces of constant potentials for the discs and the inner jets. By using the von-Zeipel parameter Λ and the new metric coefficients as discussed above the effective potential W and the enthalpy density h of Papers 1 and 2 can be calculated in the case where the self-gravity is important. They are given by,

$$e^{(W-\psi_r)} = C_1\sqrt{(1-2/\omega)}(1-c_\phi^2\Lambda^{2n-2})^{\frac{1}{2n-2}} \tag{24a}$$

and

$$he^{\psi_r} = \frac{C_2}{\sqrt{(1-2/\omega)}(1-c_\phi^2\Lambda^{2n-2})^{\frac{1}{2n-2}}}, \tag{24b}$$

where C_1 and C_2 are the constants to be evaluated by using proper boundary conditions. The presence of ψ_r in both the equations and ω and Λ instead of r and λ respectively, make all the difference between the two cases. One has to remember that both ω and Λ were measured in the units of the variable 'hole mass' $M' = M_{\rm hole}e^{-\psi_r}$. This makes the comparison at the same spacetime point meaningful. The decrement of the potential makes the disc more strongly bound. In fact, it has been recently shown (Goodman & Narayan 1988) that moderate self-gravity *damps* out the non-axisymmetric instabilities of the disc.

One can calculate the variation of the total energy content of the disc when some small amount of matter (few baryons) is transferred from the ring to the hole. The potential of the ring at the horizon is obtained by puting $z = \pm M_{\text{hole}}$ (Notice that in Weyl coordinate the black hole is represented as a line mass of length $2M_{\text{hole}}$ placed along the axis.),

$$\psi_{\text{h}} = \frac{m}{(1+b^2)^2}. \tag{25}$$

Now, if n baryons of total rest mass ε accretes axisymmetrically to the hole, the variation of the total energy content of the ring is calculated from the first law of thermodynamics (see for example, GH),

$$\delta Q = \delta M_{\text{hole}} - \frac{\kappa \delta A}{8\pi}. \tag{26}$$

As seen from infinity, $\delta M_{\text{hole}} = 0$. The change in area δA is given by,

$$\delta A = 2\varepsilon A \left[1 + \frac{1}{(1+b^2)^{1/2}} + \frac{m}{(1+b^2)^{3/2}} \right]. \tag{27}$$

The surface gravity κ is calculated from $\kappa = e^{2\psi_{\text{T}}}/4$. The change in the energy content δQ is therefore,

$$\delta Q = -\varepsilon \left[1 + \frac{1}{(1+b^2)^{1/2}} + \frac{m}{(1+b^2)^{3/2}} \right]. \tag{28}$$

The first term is just the change in the baryon number in the ring. The second term is due to the reduction of the potential energy of the ring when the baryons separated. The third term is also the loss of the potential energy due to the baryon loss as the distance between the ring and the hole increases by a small distance. Since there was no rotation of the hole or the ring, no other energy terms appear in the expression for δQ.

If the self-gravity of the surrounding matter were important, any small perturbation (caused by passing compact object in highly eccentric orbit, say) on the black hole would cause the black hole to oscillate up and down. Classically, the frequency of vertical oscillation is given by,

$$\nu = (m/b^3)^{1/2}. \tag{29}$$

The estimate of the luminosity of the gravity wave generated is calculated from (see, for example, Misner, Thorne & Wheeler 1973),

$$L_{\text{GW}} = \frac{1}{5} \frac{G}{c^5} \langle \dddot{I} \rangle^2 \tag{30}$$

which, in our present case is given by,

$$L_{\text{GW}} \approx \frac{G}{c^5} (M_{\text{hole}} a^2)^2 \nu^6 \text{ erg s}^{-1}. \tag{31}$$

This is usually insignificant unless the ring is also of comparable mass and both the ring and hole take part in large amplitude oscillation. Assume a hole of mass 10 $M_{\odot}$

being surrounded by a ring of radius b = 10 and of equal mass. The classical angular frequency of the oscillation is ν = 600 Hz and the timescale of oscillation is ~ 10^{-2} s, whereas, the light crossing time in the hole is $t_h = 10^{-4}$ s. Hence the estimation made by classical approach may be in the right range. This system, being very compact, will have strong radiation reaction. The timescale in which such oscillation damps out will be on the order $\tau_r = \frac{c^5 4\pi^2}{\nu^2 G E_k}$, where $E_k = \frac{1}{2} M_{hole}\, a^2 \nu^2$ is the kinetic energy of the hole. This is about 5000 seconds for the parameters considered! For the observational purposes, one will like to calculate the perturbation to the flat metric at earth by this oscillation. The estimation is made by expanding g_{tt} in series so that the effective 'Newtonian' potential Φ, at a large distance r becomes:

$$\Phi = -\frac{G(M_{hole}+m)}{r} - \frac{Gmb^2}{4r^3}(3\cos^2\Theta - 1)(3\sin^2\alpha - 1) - \ldots \tag{32}$$

where, $\alpha = \alpha_0 \cos(\nu t)$ is the instantaneous angular amplitude of the oscillation ($\alpha_0 = a/b$). The quadrupole moment is obtained from the second term of the expansion and metric perturbation h becomes,

$$h = \frac{2\ddot{I}}{r} = \frac{3\alpha_0^2 \nu^2 G m b^2}{c^4 r} \tag{33}$$

For the parameters mentioned above and with α_0 = 0.1, and the source located at the centre of our own galaxy, we obtain $h = 2.5 \times 10^{-17}$ which is comparable to other suggested sources and should be detectable. Realistically, however, the mass of the stable disc may be much smaller compared to the mass of the hole which reduces the effect very much.

The self-gravity may have other important effects. For example, the lowering of Keplerian distribution of the angular momentum near the centre of the disc implies that the binding energy release will also be higher. This will cause the temperature to be higher. Since matter is more tightly bound near the centre, the residence time may be higher unless some instability shows up. Both of these effects will enhance the rate of thermonuclear reactions at the centre of the disc (Chakrabarti 1986, Chakrabarti, Jin, & Arnett 1987, Chakrabarti 1988). Fig. 1 of Chakrabarti (1988) shows how the temperature at the centre of the disc goes up in the self-gravity regime. The increase in temperature may even produce neutron tori postulated recently by Hogan & Applegate (1987). Clearly, many questions remain unanswered. We hope to be able to explore them in future.

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