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CCD Star Images: On the Determination of Moffat's PSF Shape Parameters

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Abstract. Among the variety of empirical models of optical Point Spread Function used in the astronomical environment, only the Moffat's (1969) one is able to describe by means of two parameters (in the circular case) both the inner and the outer star image regions. In view of this very important feature, the problem of the simultaneous estimates of Moffat's PSF shape parameters, off-centring, and the background level in CCD star images has been investigated. The problem does not seem to be rigorously resolvable, but an approximate way to calculate all the parameters except off-centring is shown. It must be stressed that, the Moffat's PSF model being a softened power law belonging to the family of modified King and Hubble models, the present discussion can be of aid in many other research fields. Also, the integral equation enabling us to convolve a spherical source with Moffat's PSF is given and applied for comparison to Multi-Gaussian convolution.

Key words: point spread functions—CCD photometry

1. Introduction

It is well known that the observed intensity distribution in long-exposure star images can be described, in favourable background conditions, up to about 10 mag arcsec^{-2} below the central value by a variety of empirical smooth fitting functions. Among them, the more widely employed are the Moffat (1969) distribution, a central truncated Gaussian overlapping an exponential wing (King 1971), the Lorentzian (Franz 1973) and a sum of Gaussians (Brown 1974). The parameters of these distributions, under moderate or bad seeing conditions, are usually estimated with sufficient accuracy for most practical purposes by using standard fitting routines or graphical methods. But a serious problem arises with very good seeing conditions if the full resolution content of data is pursued. This is because neither the integration on the pixel surface nor the off-centring of the image with respect to the central pixel (*i.e.* that of local maximum intensity) can be neglected, when the seeing inner scale factor is of the order of the pixel size. In a previous paper (Bendinelli *et al.* 1987; hereafter Paper 1) all the parameters involved in the multi-Gaussian approximation of the intensity distribution in symmetrical CCD star images were simultaneously estimated. This was done by improving the centring algorithms of Van Altena & Auer (1975) and Chiu (1977) by useful analytical properties of Gaussians and using the Newton-Gauss regularized method to secure the convergence of the iterative process leading to the estimates of parameters. The aim of this work is to look for parameters using as star image model the Moffat's (1969) distribution, which contains only 2 parameters and shows a softened decreasing power law behaviour, so that it can describe better than a sum of Gaussians the PSF wings observed by King (1971), Kormendy (1973) and Capaccioli & de Vaucouleurs (1979). A reliable method to determine these PSF wings may greatly improve the analysis of many astrophysical observations such as, for example, imaging of distant extragalactic sources, stellar photometry in crowded fields and astrometry of double stars. Also, it should be noted that the bi-dimensional extension of Moffat's distribution to represent non-circular star images contains only one extra parameter. It seems therefore more useful, for understanding the effects of non-circular PSFs on extended sources, than a sum of bi-dimensional Gaussians, in which the number of parameters at least doubles and the useful analytical properties pointed out in Paper 1 are lost.

2. Estimation of model parameters

2.1 *Correct Formulation of the Problem*

Let us assume the normalized intensity distribution $F(r)$ in star images following Moffat(1969), *i.e.,*

$$F(r)=F_0[1+(r/\alpha)^2]^{-\beta}, \tag{1}$$

where $F_0 = (\beta - 1)(\pi\alpha^2)$. The integrated intensity in a pixel with size $2l$ centred at (x_n, y_m) is therefore given by

$$i(x_n, y_m)=i_0\int_{x_1}^{x_2}\int_{y_1}^{y_2}[1+(x'^2+y'^2)/\alpha^2]^{-\beta}\mathrm{d}x'\mathrm{d}y'+f(x_n, y_m), \tag{2}$$

where α and β are the seeing-dependent parameters, $i_0 = F_0 L_T$ ($L_T = \Sigma_n\Sigma_m\, i(x_n, y_m)$ being the integrated luminosity of the star), $f(x_n, y_m)$ the background point distribution and the integration limits x_1, x_2, y_1, y_2 read as

$$x_1=x_n-l-x_c \qquad y_1=y_m-l-y_c$$
$$x_2=x_n+l-x_c \qquad y_2=y_m+l-y_c$$

depending on the displacement x_c and y_c of the star image centre from the central pixel (x_0, y_0) with local maximum intensity in the frame. The searched parameters α, β, x_c, y_c, and f cannot be estimated by a Newton-like method from Eq. (2), since the double integral there is not expressible, unless β is an integer, by a differentiable function of the parameters themselves. Nevertheless, such a function can be found considering the mixed second order moments of the pixel intensity, *i.e.*,

$$M(x_n)=|x_n-x_0|\sum_m|y_m-y_0|\,i(x_n, y_m)$$

$$M(y_m)=|y_m-y_0|\sum_n|x_n-x_0|\,i(x_n, y_m), \tag{3}$$

where the background is assumed constant on account of simplicity. This assumption can be released, if necessary, by introducing two or more background parameters, whose estimate, in any case, is less difficult than shape and centring parameters, being the starting Eq. (2) more linear in background parameters (see also Paper 1). The calculated distribution $M(x_n)$ sufficiently approximates the value of the integral

$$M(x_n)=i_0\int_{x_1}^{x_2}\int_{-\infty}^{+\infty} x'y'[1+(x'^2+y'^2)/\alpha^2]^{-\beta}\,\mathrm{d}x'\mathrm{d}y'+|x_n-x_0|\sum_m |y_m-y_0|f, \tag{4}$$

in which the double integration can be easily performed, giving the equation

$$M(x_n)=i_0\alpha^4[2(\beta-1)(\beta-2)]^{-1}\{[1+(x_n-l-x_c)^2/\alpha^2]^{2-\beta}-[1+(x_n+l-x_c)^2/\alpha^2]^{2-\beta}\}+|x_n-x_0|\sum_m |y_m-y_0|f, \tag{5}$$

and a corresponding one holds for $M(y_m)$, containing y_m, y_c and y_0 instead of x_n, x_c and x_0. The analytical expressions of $M(x_n)$ and $M(y_m)$ enable us to estimate, in principle, the model parameters by the Newton-Gauss regularized method, because they are differentiable with respect to the searched parameters. Further, Eq. (5) can be immediately extended to a bidimensional elliptical Moffat's PSF model, if, an uncommon case, the image axes and the CCD frame axes are parallel, or if the data consist of PDS scans of a photographic plate previously oriented. In practice, unfortunately, the parameters are strongly linked in Eq. (5), violating the identifiability conditions (i.e., the derivatives of the fitting function with respect to the parameters are not linearly independent, see for more details Beck & Arnold 1977), so that parameters themselves cannot be simultaneously and uniquely estimated.

2.2 *An Approximate Solution*

Let us consider the sequence of integrated luminosities $L(x_n)$ over the squares A_n with side $2(2n - 1)l$, centred on the. star image (neglecting its off-centring), i.e.,

$$L(x_n)=\int_{A_n} i(x,y)\mathrm{d}A+A_n f. \tag{6}$$

It can be easily verified that the difference between $L(x_n)$ and $L(r_n)$, the integrated luminosity over a circle of the same area as the square, is some per cent at the centre and rapidly reduces outwardly, so that $L(x_n)$ can be considered as an approximate value of $L(r_n)$. But, in terms of the normalized Moffat's distribution, $L(r_n)$ reads as

$$L(r_n)=L_T[1-(1+r^2/\alpha^2)^{1-\beta}]+A_n f, \tag{7}$$

which is an analytical expression enabling us to estimate the parameters by the Newton-Gauss regularized method. The link between α and β violates again the identifiability condition, but the convergence of the iterative process, with the present reduced number of parameters, is due to use of Penrose (1956) inversion which, as pointed out in Paper 1, works also with nearly singular matrices. The fitting procedure closely follows that extensively described in Paper 1 to derive from CCD frames the multi-Gaussian approximation of the PSF. In effect, the same code has been used, adding only few statements to extract from CCD original data the new 'input data' (see Eq. 7) and to specify the new PSF model and its derivatives with respect to parameters. A block diagram description of the code is reported in Bendinelli, Parmeggiani & Zavatti (1984). As far as the off-centring is concerned, it can be derived, before the search of shape parameters, from the moments of the marginal pixel intensity distributions (see Chiu 1977).

3. Results and discussion

The approximate method described in the above last sub-section was tested by fitting the *V* band star images in the field of M 31 obtained by S. G. Djorgovski at the Kitt Peak 4-m telescope with a TI 800 × 800 CCD (pixel size of 0.298 arcsec). The intensity distributions in a star image obtained by the present Moffat's model and by the 3-Gaussians model of Paper 1 are shown in Fig. 1 and compared there with the brightness profile derived in a model-independent way using the QPHOT code (Djorgovski 1987, personal communication). The model parameters are reported in Table 1, and the joint residuals (calculated — model brightness) are shown in Fig. 2. It is evident from Fig. 1 that the brightness profiles computed by both models agree

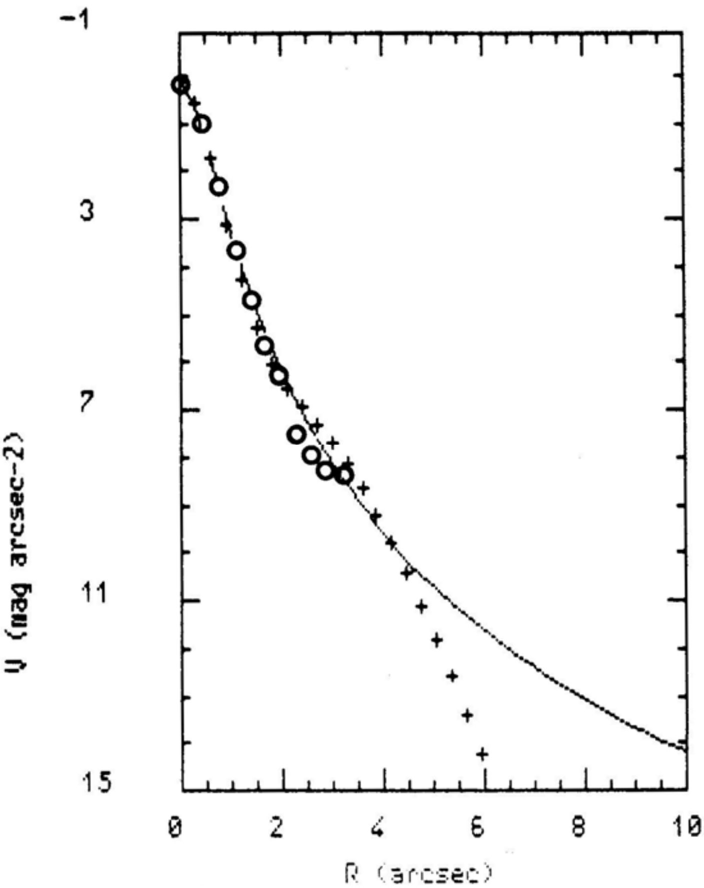

Figure 1. Comparison between the observed *V* brightness distribution in star images (circles) and the calculated ones by Moffat (dotted line) and multi-Gaussian (crosses) PSF models.

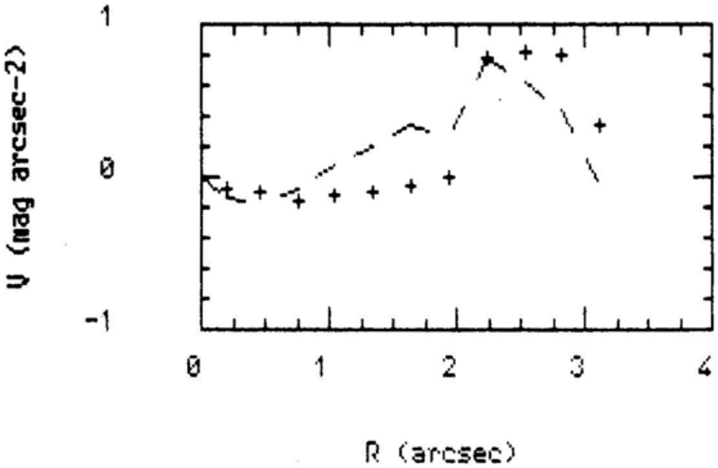

Figure 2. Residual (see text) of Moffat (dotted line) and multi-Gaussian (crosses) PSF models.

Table 1. Parameters of Moffat's and multi-Gaussian PSF models.

3-*Gaussian*			
	$a_1=0.523$	$a_2=0.375$	$a_3=0.101$
	$\sigma_1=0.302$	$\sigma_2=0.545$	$\sigma_3=1.483$
Moffat			
	$\alpha=0.6135$	$\beta=2.3124$	

Note: a_i is the partition coefficient of the ith Gaussian.

reasonably with the observed data. The magnitude of residuals is not surprising if one keeps in mind that in reality the star images overlap outer regions of M31, and not an uniform background. The models diverge out of about 4.5 arcsec (i.e. roughly speaking $3\sigma_3$ and 7α), but it should be stressed that the integrated luminosity of Moffat model from 7α to infinity (see Eq. 7) is less than one hundredth of the total, so that the true brightness distribution in extreme wings is irrelevant for any reasonable application (see, for instance, Appendix A). Further, the radial range in which the Moffat curve is overlapped by a sum of N Gaussians depends only on N, as shown for example in Bendinelli et al. (1984) and in Paper 1. In conclusion, it seems that parameters of both the multi-Gaussian and Moffat models can be calculated taking into account the finite pixel size, with practically the same accuracy. The choice of the model evidently depends on the particular research field. For instance in astrometry the multi-Gaussian is surely preferable, giving also the off-centring, while in extra-galactic astronomy the other one should be used (see Schweizer 1981, and Djorgovski 1984 for the effects of PSF wings on the appearance of distant sources). Finally, one must stress the impossibility of deriving the parameters of King's and Lorentzian PSF models taking into account the pixel integration. In the first case owing to the discontinuity at the connection point between the inner Gaussian and the outer exponential, in the second since parameters are linked in a too complicated way.

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Appendix A

Convolution of a spherical source with Moffat's PSF

In a series of papers (see Bendinelli *et al.* 1986, and references therein) it has been shown that convolution-deconvolution of a spherical source with the PSF, approximated by a sum of weighted Gaussians, can be performed by the monodimensional integral equation

$$f(r)=\sum_{i=1}^{N}(a_i/\sigma_i^2)\exp(-r^2/2\sigma_i^2)\int_0^\infty \exp(-\rho^2/2\sigma_i^2)I_0(r\rho/\sigma_i^2)\phi(\rho)\,\mathrm{d}\rho \qquad \text{(A1)}$$

relating the true brightness distribution $\phi(r)$ and the observed one $f(r)$. Let us assume

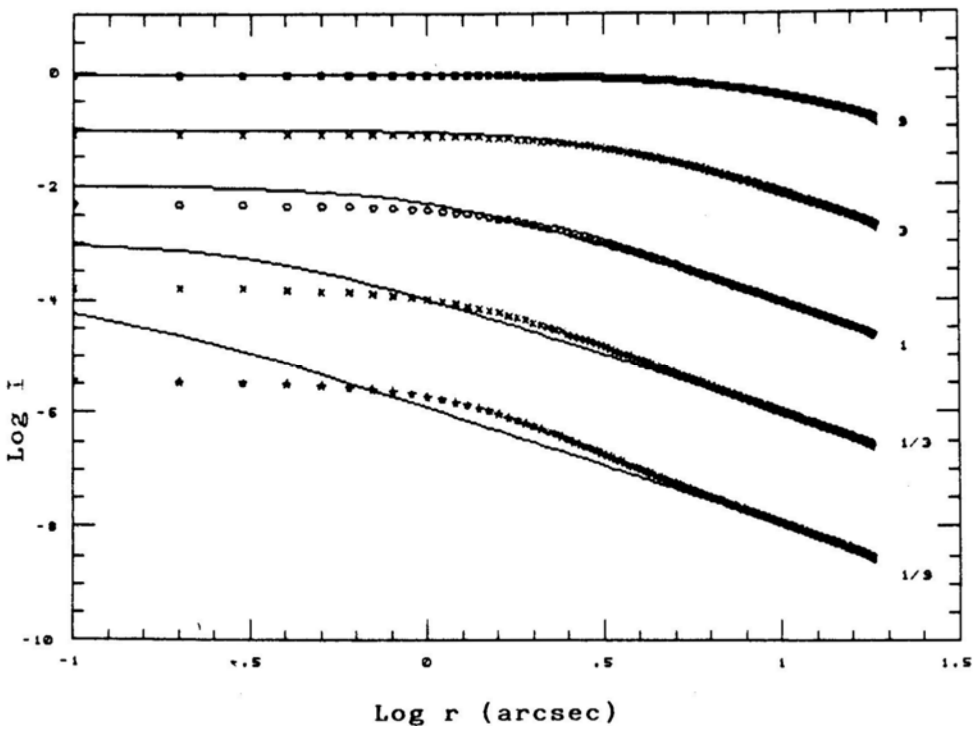

Figure A1. Set of empirical King's models with $\log(r_t/r_c)$ = 2.25 and r_c indicated at the end of each profile (continuous lines), vertically shifted, and their · convolution with Moffat's PSF of Table 1 (symbols).

Moffat's PSF approximation; then convolution is expressed by the double integral equation

$$f(r) = [2(\beta - 1)/\pi\alpha^2] \int_0^\infty \rho F(\rho) \int_0^\pi [1 + (\rho^2 + r^2 - 2\rho r \cos\theta)/\alpha^2]^{-\beta} \, d\rho d\theta \qquad \text{(A2)}$$

which requires, to be computed or inverted up to the radial distance where the left-hand side term becomes negligible, about a factor of four in time more than Equation (Al). To prove the substantial equivalence of Equations (Al) and (A2), a set of empirical King's model have been chosen. They are characterized by the same, concentration index c = 2.25, with core radius r_c varying from 9 to 0.111 arcsec, in order to simulate the distance effect or smaller and smaller intrinsic sizes. Convolutions of models with the Moffat PSF specified in Table 1 are shown in Fig. Al, and differences between convolutions of any model with both Moffat and multi-Gaussian PSFs in Fig. A2. It seems that from these figures some main conclusions can be drawn:

1. Dealing with large sources (r_c > 1 arcsec) both PSF approximations give practically the same results.
2. For small sources (r_c < 1 arcsec) local differences of the order of 1 mag arcsec^{-2} between PSF approximations may cause about 0.1 mag arcsec^{-2} in convolved profiles, but in this case the problem is which is the better approximation.

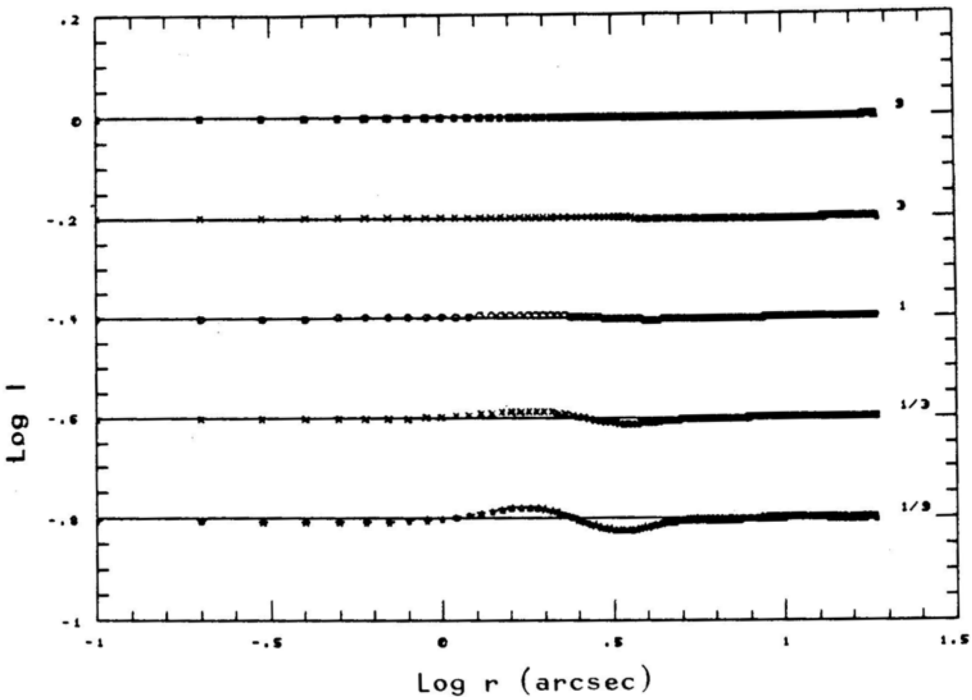

Figure A2. differences between convolutions of King's models with respectively Moffat's and multi-Gaussian PSF approximations of Table 1.

3. Convolution effects become negligible for both PSF approximations at compareable distances from the centre, roughly speaking at $3\sigma_3$ or 7α, so that we must be reasonably confident that the outer profile of distant sources is not an artefact of convolution with seeing.

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