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On a Nonlinear and Lorentz-Invariant Version of Newtonian Gravitation—II

J. V. Narlikar & T. Padmanabhan *Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005*

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Abstract. This paper gives a full nonlinear version of Newtonian gravity in which the gravitational energy acts as a source of the gravitational field. The generalized field equation for the scalar gravitational potential is solved for a spherically symmetric localized distribution of matter. It is shown that the perihelia of orbits of test particles in such a field precess steadily. The effect is, however, too small to account for the observed shift in the perihelion of planet Mercury. Further, the bending of light in this theory is zero. It is suggested that these inadequacies of the quasi-Newtonian framework call for more sophisticated approaches to gravity.

Key words: Newtonian gravitation, nonlinear—Lorentz-Invariant gravitation

1. Introduction

This paper forms a sequel to an earlier paper by Rawal & Narlikar (1982, hereafter referred to as Paper I) in which an attempt was made to combine the essentials of special relativity with those of Newtonian gravity to construct a scalar Lorentz invariant theory of gravity. Paper I was limited to discussing first order effects in which the mass-equivalent of gravitational energy acts as a source of the scalar gravitational potential ϕ. Here we generalize the framework to all orders in which the feedback of gravitational energy on ϕ in turn modifies the energy which further modifies ϕ and so on. We will then apply the field equation of the modified theory to study the gravitational effects of spherical distributions of matter. As in Paper I, the approach will be Lorentz invariant.

It will be shown that the above problem can be solved exactly and that the motion of a test particle in the field can be applied to study the orbits of planets around the Sun.

2. The field equations

As in Paper I we will begin by formulating the action. We first do so in an iterative fashion and later obtain the final answer in a closed form by a self-consistency argument. We choose units in which $c = 1$ and $\hbar = 1$. Thus

$$l_{\mathrm{P}} = \sqrt{4\pi G} \tag{2.1}$$

has the dimensions of length

The zero'th order action is written as

$$J^{(0)} = J^{(0)}_{\phi} + J^{(0)}_{\mathrm{Int}} + J_m. \tag{2.2}$$

Following the sign-convention of Paper I we have (with $\phi_i = \delta\phi/\delta x^i$),

$$J^{(0)}_{\phi} = \frac{1}{2l_{\mathrm{P}}^2}\int \phi^i \phi_i \, \mathrm{d}^4 x, \quad J^{(0)}_{\mathrm{Int}} = \int \phi \, T_{\mathrm{m}} \, \mathrm{d}^4 x, \quad J_{\mathrm{m}} = -\sum_{\mathrm{a}} \int m_{\mathrm{a}} \, \mathrm{d}s_{\mathrm{a}} \tag{2.3}$$

where m_{a} is the rest mass of typical particle 'a' and $\mathrm{d}s_{\mathrm{a}}$ the element of its proper time. T_m is the trace of the matter energy tensor. $\delta J^{(0)}/\delta\phi = 0$ gives us the zero'th order Lorentz invariant Poisson equation

$$\Box \, \phi = l_{\mathrm{P}}^2 T_{\mathrm{m}}. \tag{2.4}$$

To begin the iteration we must add to T_m. the trace of the energy tensor of the ϕ field. To calculate this energy tensor at any order of iteration we use the procedure outlined below (for a rationale, see, for example, Landau & Lifshitz 1975).

Given the action in flat spacetime as J, write it covariantly in a Riemannian spacetime with the flat spacetime metric η_{ik} replaced by g_{ik} (i, k = 0, 1, 2, 3; 0 timelike) and ordinary derivatives by covariant derivatives. Then consider the variation $g_{ik} \to g_{ik} + \delta g_{ik}$. suppose that

$$\delta J = -\tfrac{1}{2}\int T^{ik} \delta g_{ik} \sqrt{-g} \, \mathrm{d}^4 x. \tag{2.5}$$

Then T^{ik} is the required energy tensor. It can then be written down for flat spacetime.

For the ϕ-field using $J^{(0)}_{\phi}$ in place of J above we get

$$T^{ik}_{\phi} = \frac{1}{l_{\mathrm{P}}^2}(\phi^i\phi^k - \tfrac{1}{2}\eta^{ik}\phi^l\phi_l). \tag{2.6}$$

Therefore to T_m we must add

$$T_{\phi} = -\frac{1}{l_{\mathrm{P}}^2}\phi^l\phi_l. \tag{2.7}$$

However, addition of T_ϕ to T_m. in the interaction term J^0_{int} further modifies T^{ik}_{ϕ} by our prescription (2.5). This is where the iteration begins. So we write the complete action as

$$J = J_{\phi} + J_{\mathrm{Int}} + J_{\mathrm{m}} \tag{2.8}$$

where J_{m} is unchanged but

$$J_{\phi} \equiv \int L \mathrm{d}^4 x \equiv \int \sum_{n=0}^{\infty} L^{(n)} \mathrm{d}^4 x. \tag{2.9}$$

Here the term $L^{(n)}$ arises from the interaction term:

$$\int \phi T^{(n-1)}_{\phi} \mathrm{d}^4 x \equiv \int L^{(n)} \mathrm{d}^4 x \tag{2.10}$$

where $T^{(n-1)}$ is obtained from $L^{(n-1)}$ by the prescription (2.5)

The following ansatz gives us the iterative solution. Write

$$L^{(n)} = \tfrac{1}{2} a_n \phi^n \phi^i \phi_i, \quad a_0 = \frac{1}{l_{\mathrm{P}}^2}. \tag{2.11}$$

This gives us from (2.5)

$$T^{(n)}_{\phi} = -a_n \phi^n \phi^i \phi_i. \tag{2.12}$$

But from (2.10) we get

$$-a_{n-1} = \tfrac{1}{2} a_n; \quad i.e., \; a_n = \frac{(-2)^n}{l_{\mathrm{P}}^2}. \tag{2.13}$$

Therefore

$$L = \frac{1}{2l_P^2}\sum_{n=0}^{\infty}(-2\phi)^n\phi_i\phi^i$$
$$= \frac{1}{2l_P^2}(1+2\phi)^{-1}\phi_i\phi^i. \tag{2.14}$$

Thus we get

$$J_\phi = \frac{1}{2l_P^2}\int(1+2\phi)^{-1}\phi_i\phi^i\,d^4x. \tag{2.15}$$

The situation is actually more complicated than we have so far anticipated; for there is another iteration involved! Consider $J^{(0)}_{\rm int}$. It can be expressed in the form

$$J^{(0)}_{\rm Int} = \int\phi T_{\rm m}\,d^4x = \sum_{\rm a}\int\phi m_{\rm a}\,ds_{\rm a}. \tag{2.16}$$

When we apply (2.5) to the above action, it yields additional contribution to T_ϕ :

$$\delta\sum_{\rm a}\int\phi m_{\rm a}\,ds_{\rm a} = \tfrac{1}{2}\int\phi T^{ik}_{\rm m}\delta g_{ik}\sqrt{-g}\,d^4x \tag{2.17}$$

which gives the additional contribution as $-\phi T_{\rm m}$. This generates further terms $+\phi^2 T_{\rm m}$,$-\phi^3 T_{\rm m}$ *etc.* in the same manner as obtained earlier for J_ϕ. Therefore we get on summation

$$J_{\rm Int} = \sum_{\rm a}\int\frac{\phi}{1+\phi}m_{\rm a}\,ds_{\rm a}. \tag{2.18}$$

Putting all three terms of (2.8) together we get

$$J = \frac{1}{2l_P^2}\int(1+2\phi)^{-1}\phi_i\phi^i\,d^4x - \sum_{\rm a}\int(1+\phi)^{-1}m_{\rm a}\,ds_{\rm a} \tag{2.19}$$

as the complete nonlinear action.

It is possible to derive this expression by a short-cut route using the consistency argument. Let J be written in the form

$$J = \frac{1}{2l_P^2}\int f(\phi)\phi^i\phi_i\,d^4x + \sum_{\rm a}\int m_{\rm a}g(\phi)\,ds_{\rm a} - \sum_{\rm a}\int m_{\rm a}\,ds_{\rm a} \tag{2.20}$$

where the three terms are respectively J_ϕ, $J_{\rm int}$ and $J_{\rm m}$. Then from (2.5) we get

$$T_\phi = -\frac{f(\phi)}{l_P^2}\phi^l\phi_l - g(\phi)T_{\rm m}. \tag{2.21}$$

Now rewrite (2.20) in the form

$$J = \frac{1}{2l_P^2}\int\phi_i\phi^i\,d^4x - \int(T_{\rm m}+T_\phi)\phi\,d^4x - \sum_{\rm a}\int m_{\rm a}\,ds_{\rm a}. \tag{2.22}$$

A variation of ϕ in this action can lead to a Poisson-type equation if T_ϕ in the second term is kept unchanged like $T_{\rm m}$. With this interpretation (2.20) may be equated to (2.22)

$$f(\phi) = (1+2\phi)^{-1}, \quad g(\phi) = \phi(1+\phi)^{-1} \tag{2.23}$$

and (2.19) is obtained.

This argument illustrates the fact that although T_ϕ may be looked upon as a source of ϕ in much the same way as T_m, it hides the nonlinearity inherent in the gravitational interaction. This nonlinearity is seen in the correct field equation obtained from $\delta J = 0$ by a free variation of ϕ:

$$(1+2\phi)^{-1} \square \phi - (1+2\phi)^{-2}\phi_i\phi^i = l_P^2(1+\phi)^{-2}T_m. \tag{2.24}$$

This generalized Poisson equation is simplified by the transformation

$$\Omega^2 = 1 + 2\phi \tag{2.25}$$

to the form

$$\square \Omega = \frac{4l_P^2\Omega}{(1+\Omega^2)^2}T_m. \tag{2.26}$$

3. Spherically symmetric potential

To solve (2.26) in the empty spacetime outside a spherical distribution is easy. Since $T_m = 0$ we get the solution as

$$\Omega = A + \frac{B}{r} \tag{3.1}$$

where A, B are arbitrary constants.

The potential f is then given by

$$\phi = \frac{1}{2}(A^2 - 1) + \frac{AB}{r} + \frac{B^2}{2r^2}. \tag{3.2}$$

If we assume that the matter is localized and at large r there is no 'cosmological' contribution to ϕ then a comparison with the Newtonian theory in the 'weak field approximation at large r' gives

$$A = 1, \quad AB = \frac{l_P^2}{4\pi}M. \tag{3.3}$$

Restoring G, c, $\hbar$ to the cgs units we therefore get

$$\phi = \frac{GM}{r}\left\{1 + \frac{GM}{2c^2 r}\right\}. \tag{3.4}$$

A word of caution is needed here. As we shall shortly show, the constants A and B are not so trivial in the present nonlinear theory as they are in the linear Newtonian theory. This is seen by considering the equations of motion of a test particle 'a'.

The variational principle $\delta J/\delta s_a = 0$ gives the equations of motion of 'a'. Writing r as the position vector of 'a', we get the 'energy integral' as

$$\dot{\mathbf{r}}^2 = 1 - \frac{k^2}{(1+\phi)^2}, \quad k = \text{constant}. \tag{3.5}$$

Here $\dot{\mathbf{r}}$ denotes the velocity of the test particle in the rest frame of the source distribution. The appearance of the $(1 + \phi)^2$ term in the denominator underscores the need for fixing A and B unambiguously.

Consider for example the effect of A and B on the precession of the perihelion of a planet around the Sun. A straightforward calculation gives the answer for the rate of precession of the perihelion as

$$\omega = \omega_E \cdot \frac{1}{6} \cdot \frac{1 - 5A^2}{A^2(1 + A^2)} \tag{3.6}$$

where ω_E is the Einstein value for the precession rate in general relativity. In (3.6) we have already fixed B by the requirement $AB = GM$, M being the mass of the Sun.

Thus for $A = 1$ and the potential (3.4) we get a retardation of perihelion. This result corrects the earlier erroneous conclusion of Paper I based on only the first term of the iterative process. However, we also see that if asymptotic conditions at infinity require $A \neq 1$ then a different value of the perihelion precession is found.

R. Nityananda & A. Samuel (1984, private communication) have pointed out that there is no bending of light in a coupling of the type ϕT_m since for photons (or for electromagnetic fields in general) $T_m = 0$. An analysis of particle trajectories with non-zero restmass but with $|\dot{\mathbf{r}}| \simeq c$ at infinity also shows that as the relativistic parameter

$$\gamma \equiv \left(1 - \frac{|\dot{\mathbf{r}}|^2}{c^2}\right)^{-1/2} \tag{3.7}$$

tends to infinity the bending angle drops off as γ^{-2}.

4. Conclusion

The preceding sections describe a logically complete and mathematically consistent theory of gravity. However, this theory cannot claim to have anything to do with reality because it fails to explain correctly (i) the bending of light and (ii) the precession of planetary orbits. Nevertheless the above exercise has the advantage in that it demonstrates the need for a more sophisticated theory, as a generalization of the Newtonian law of gravitation. In view of the fact that the next possible generalization of the Newtonian concept of matter density is to the second rank tensor T_{ik}, we expect the theory to be a tensorial one. Whether such a theory, satisfying the present observational tests can be constructed within a flat spacetime, is currently under investigation. If such a theory also fails we have a strong reason why general relativity, a tensorial theory in curved spacetime is needed to describe gravity.

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