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Structure and Stability of Rotating Fluid Disks Around Massive Objects. I. Newtonian Formulation

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Abstract. In this paper we have presented a very general class of solutions for rotating fluid disks around massive objects (neglecting the self gravitation of the disk) with density as a function of the radial coordinate only and pressure being nonzero. Having considered a number of cases with different density and velocity distributions, we have analysed the stability of such disks under both radial and axisymmetric perturbations. For a perfect gas disk with γ = 5/3 the disk is stable with frequency $(MG/r^3)^{1/2}$ for purely radial pulsation with expanding and contracting boundary. In the case of axisymmetric perturbation the critical γ_c for neutral stability is found to be much less than 4/3 indicating that such disks are mostly stable under such perturbations.

Key words: Fluid disks—stability

1. Introduction

Study of fluid disks around massive objects has been one of the most important aspects of theoretical astrophysics. Particularly in the case of binary systems where mass transfer is an important feature, one of the major sources of high energy radiation is supposed to be either the hot inner regions of the disk (in the case of continuum emission) or the impact of ruptured unstable disk in the case of burst emissions. A lucid summary with a detailed bibliography of the work on thin disks may be found in the article of Lightman, Shapiro and Rees (1978). Recently, Prasanna and Chakraborty (1981; now onwards referred to as Paper I) have studied the structure and stability of thin disks of charged fluid around compact objects wherein they found that such pressureless thin disks are stable under radial pulsations. It is well known

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that if the pressure is not negligible then the disk would no longer be thin and one has to consider the structure off the equatorial plane too. Fishbone and Moncrief (1976), and Kozlowski, Jaroszyński and Abramowicz (1978) have respectively considered for isentropic and barytropic fluid disks, solutions of Euler's equations with pressure and rotation. In this paper we consider a similar system (uncharged rotating fluid disks with nonzero pressure) without self gravitation and viscosity and obtain a class of steady state solution and further discuss the stability of such disks under axisymmetric perturbations. We presently restrict the analysis to Newtonian formulation only and shall consider the complete general relativistic discussion in a subsequent article.

We have developed earlier (Paper I) the general set of fluid equations for disks around central massive bodies (including charge density and conductivity); here we shall consider the Newtonian limit of these equations (with $\epsilon = 0$, $\sigma = 0$) and study the structure and stability.

2. Steady state solutions

The equations governing the dynamics of a perfect fluid rotating around a central gravitating source of mass M may be obtained from the general equations (Paper I, equations 2.15–2.17) and are given by

$$\rho\left[\frac{DV^r}{Dt}+\frac{MG}{r^2}-\frac{V^{\theta 2}+V^{\phi 2}}{r}\right]=-\frac{\partial p}{\partial r}, \tag{2.1}$$

$$\rho\left[\frac{DV^\theta}{Dt}+\frac{V^r\,V^\theta}{r}-\frac{\cot\theta\, V^{\phi 2}}{r}\right]=-\frac{1}{r}\frac{\partial p}{\partial\theta}, \tag{2.2}$$

$$\rho\left[\frac{DV^\phi}{Dt}+\frac{V^r\,V^\phi}{r}+\frac{\cot\theta\, V^\theta\, V^\phi}{r}\right]=-\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi} \tag{2.3}$$

And the continuity equation,

$$\rho\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\,V^r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,V^\theta)+\frac{1}{r\sin\theta}\frac{\partial V^\phi}{\partial\phi}\right]+\frac{D\rho}{Dt}=0, \tag{2.4}$$

wherein p, ρ, V^α denote the pressure, density and velocity as measured in the associated inertial frame and D/Dt is the rate of change operator following the fluid as given by

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+V^r\frac{\partial}{\partial r}+\frac{V^\theta}{r}\frac{\partial}{\partial\theta}+\frac{V^\phi}{r\sin\theta}\frac{\partial}{\partial\phi}. \tag{2.5}$$

For making the system determinate, we further assume the equation of state as expressed by the adiabatic law

$$\frac{D}{Dt}(p\,\rho^{-\gamma})=0, \tag{2.6}$$

γ being the adiabatic index C_p/C_v.

Presently we shall restrict ourselves to the case of pure rotational flow as expressed by $V_0^r = 0$ and $V_0^\theta = 0$ and $V_0^\phi = V_0$. Further as the disk is axisymmetric the equations are given by

$$\rho_0 \left[\frac{MG}{r^2} - \frac{V_0^2}{r}\right] = -\frac{\partial p_0}{\partial r} \tag{2.7}$$

And

$$\rho_0 V_0^2 \cot\theta = \frac{\partial p_0}{\partial \theta}, \tag{2.8}$$

the remaining equations being identically satisfied. In the case when ρ_0 is independent of θ these equations may be solved exactly. Thus considering

$$\rho_0 = \rho_0(r), \tag{2.9}$$

equations (2.7) and (2.8) together give the equation

$$-\frac{1}{r}\frac{\partial}{\partial\theta}(\rho_0 V_0^2) + \cot\theta \frac{\partial}{\partial r}(\rho_0 V_0^2) = 0, \tag{2.10}$$

whose solution is given by

$$\rho_0 V_0^2 = A\, r^k \sin^k \theta, \tag{2.11}$$

A and k being constants. Substituting these in equations (2.7) and (2.8), we get

$$\frac{\partial p_0}{\partial r} = A\, r^{k-1} \sin^k\theta - \frac{MG\, \rho_0}{r^2} \tag{2.12}$$

and

$$\frac{\partial p_0}{\partial \theta} = A\, r^k \cot\theta \sin^k\theta, \tag{2.13}$$

whose solution may be obtained as

$$p_0 = \frac{A}{k} r^k \sin^k\theta - MG \int \frac{\rho_0\, dr}{r^2} + B \tag{2.14}$$

for $k \neq 0$. For the case $k = 0$, the pressure is given by

$$p_0 = A \ln (r \sin \theta) - MG \int \frac{\rho_0 \, dr}{r^2} + B. \tag{2.15}$$

Assuming

$$\rho_0 = \rho_c (r/m)^1 \tag{2.16}$$

wherein $m = MG/c^2$, ρ_c and l are constants, the expressions for p_0 are given by

$$p_0 = -\frac{MG\,\rho_c}{m^l} \frac{r^{l-1}}{l-1} + \frac{A}{k} r^k \sin^k \theta + B, \qquad l \neq 1, k \neq 0, \tag{2.17}$$

$$p_0 = -\frac{MG\,\rho_c}{m^l} \frac{r^{l-1}}{l-1} + A \ln (r \sin \theta) + B, \quad l \neq 1, k = 0, \tag{2.18}$$

$$p_0 = -\frac{MG\,\rho_c}{m} \ln r + \frac{A}{k} \sin^k \theta + B, \qquad l = 1, k \neq 0 \tag{2.19}$$

and

$$p_0 = A \ln (\sin \theta) + \left(A - \frac{MG\,\rho_c}{m}\right) \ln r + B, \quad l = 1, k = 0. \tag{2.20}$$

The special case when k = — 2 and l = 0, corresponds to that of Fishbone and Moncrief. The whole class of solutions obtained above are all physically plausible provided the pressure satisfies the condition $p_0 > O$ throughout the interior of the disk and $p_0 = 0$ over the boundary. The constants A and B may be obtained by using the relation $p_0 = 0$ at r_a and r_b, the inner and outer edges at the plane $\theta = \pi/2$ and from $p_0 > 0$ we can obtain the condition relating k and l. Evaluating the constants thus, we have the pressure given by the expressions

$$\frac{p_0}{c^2} = \frac{\rho_c}{(l-1)} \left\{ \frac{(b^{l-1} - a^{l-1}) R^k \sin^k \theta - R^{l-1} (b^k - a^k) + b^k a^{l-1} - a^k b^{l-1}}{(b^k - a^k)} \right\}, \quad l \neq 1, k \neq 0, \tag{2.21}$$

$$\frac{p_0}{c^2} = \frac{\rho_c}{(l-1)} \left\{ \frac{(b^{l-1} - a^{l-1}) \ln (R \sin \theta) - R^{l-1} (\ln b - \ln a) + a^{l-1} \ln b - b^{l-1} \ln a}{(\ln b - \ln a)} \right\}, \quad l \neq 1, k = 0 \tag{2.22}$$

And

$$\frac{p^0}{c^2} = \rho_c \left\{ \frac{R^k \sin^k \theta (\ln b - \ln a) - (b^k - a^k) \ln R + b^k \ln a - a^k \ln b}{(b^k - a^k)} \right\}, \quad l = 1, k \neq 0 \tag{2.23}$$

and $p_0 = 0$ for $l = 1$, $k = 0$. In the above, R, a and b are dimensionless quantities denoting r/m, r_a/m and r_b/m respectively. In order to get the boundary of the disk off the equatorial plane $\theta = \pi/2$, we solve these equations $p_0 = 0$ for sin θ and thus for every r we get θ and $(\pi - 0)$ corresponding to the edge of the disk in the meridional plane. Finally using the condition that $p_0 > 0$ throughout the interior of the disk we get the criterion connecting k and l as $k < l - 1$. If $k = l - 1$ then it follows immediately that $P_o = 0$, $\theta = \pi/2$ and

$$V_0^2 = \frac{MG}{r} \tag{2.24}$$

showing that the disk is a pressureless thin disk confined to the equatorial plane and having Keplerian motion, a well-known result. Thus the nonzero pressure would definitely require a structure off the equatorial plane. As l and k are then related through $k < l - 1$, taking $l - 1$ $k + n$, n being a positive real number, we can write the velocity function to be

$$V_0^2 = A\left(\frac{MG}{r^{1+n}}\right)\sin^k\theta, \tag{2.25}$$

where the constant A for the three different cases is given by

$$A = \frac{k}{l-1}\left(\frac{b^{l-1} - a^{l-1}}{b^k - a^k}\right), \qquad k \neq 0,\ l \neq 1,$$

$$A = \frac{1}{l-1}\left(\frac{b^{l-1} - a^{l-1}}{\ln b - \ln a}\right), \qquad k = 0,\ l \neq 1$$

and

$$A = k\left(\frac{\ln b - \ln a}{b^k - a^k}\right), \qquad k \neq 0,\ l = 1. \tag{2.26}$$

3. Stability analysis

As was done earlier (Paper I) in order to discuss the stability of the configuration, we perturb the system and. consider the equations governing the axisymmetric perturbations and perform the normal mode analysis restricting the perturbations to linear terms only. The general procedure we use is as given by Chandrasekhar and Friedman (1972a, b). The complete set of equations governing the perturbations may be obtained from our earlier work (Paper I, equations 3.1–3.4), in the Newtonian limit as

$$\rho_0\left[\frac{\partial}{\partial t}\delta V^r - \frac{2}{r}V_0\,\delta V^\phi\right] + \delta\rho\left[\frac{MG}{r^2} - \frac{V_0^2}{r}\right] = -\frac{\partial}{\partial r}\delta p, \tag{3.1}$$

$$\rho_0 \left[\frac{\partial}{\partial t}\delta V^\theta - \frac{2}{r}\cot\theta\, V_0\, \delta V^\phi\right] + \delta\rho \frac{V_0^2}{r}\cot\theta = -\frac{1}{r}\frac{\partial}{\partial\theta}\delta p, \tag{3.2}$$

$$\frac{\partial}{\partial t}\delta V^\phi + \frac{1}{r}\left(\frac{\partial V_0}{\partial\theta} + V_0\cot\theta\right)\delta V^\theta + \left(\frac{\partial V_0}{\partial r} + \frac{V_0}{r}\right)\delta V^r = 0 \tag{3.3}$$

and

$$\rho_0\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\,\delta V^r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,\delta V^\theta)\right] + \frac{\partial}{\partial t}\delta\rho + \delta V^r\frac{\partial\rho_0}{\partial r} + \frac{\delta V^\theta}{r}\frac{\partial\rho_0}{\partial\theta} = 0, \tag{3.4}$$

whereas the condition of adiabaticity gives

$$\frac{\partial}{\partial t}(\rho_0^{-\gamma} - \gamma\, p_0\, \rho_0^{-\gamma-1}\,\delta\rho) + \delta V^r\frac{\partial}{\partial r}(p_0\,\rho_0^{-\gamma}) + \frac{\delta V^\theta}{r}\frac{\partial}{\partial\theta}(p_0\,\rho_0^{-\gamma}) = 0. \tag{3.5}$$

Assuming the time dependence of each of the perturbed variable to be of the form

$$e^{i\sigma t}, \tag{3.6}$$

we introduce the Lagrangian displacement ξ^i ($i = r,\theta$) through the relation

$$\frac{\partial\xi^i}{\partial t} = \delta V^i. \tag{3.7}$$

Denoting the perturbed variables to represent only their spatial part we obtain after some rearrangement of terms the following set of equations governing the perturbations

$$\delta V^\phi = -\frac{1}{r}\left(\frac{\partial V_0}{\partial\theta} + \cot\theta\, V_0\right)\xi^\theta - \left(\frac{\partial V_0}{\partial r} + \frac{V_0}{r}\right)\xi^r, \tag{3.8}$$

$$\delta\rho = -\rho_0\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\,\xi^r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,\xi^\theta)\right] - \xi^r\frac{\partial\rho_0}{\partial r} - \frac{\xi^\theta}{r}\frac{\partial\rho_0}{\partial\theta}, \tag{3.9}$$

$$\delta p = -\gamma\, p_0\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\,\xi^r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,\xi^\theta)\right] - \xi^r\frac{\partial p_0}{\partial r} - \frac{\xi^\theta}{r}\frac{\partial p_0}{\partial\theta}, \tag{3.10}$$

$$-\rho_0\,\sigma^2\,\xi^r = \frac{2\rho_0\, V_0}{r}\delta V^\phi - \left(\frac{MG}{r^2} - \frac{V_0^2}{r}\right)\delta\rho - \frac{\partial}{\partial r}\delta p \tag{3.11}$$

And

$$-\rho_0\,\sigma^2\,\xi^\theta = \frac{2\,\rho_0\, V_0}{r}\cot\theta\,\delta V^\phi + \frac{V_0^2}{r}\cot\theta\,\delta\rho - \frac{1}{r}\frac{\partial}{\partial\theta}\delta p. \tag{3.12}$$

Equations (3.8) to (3.10) are the initial value equations while (3.11) and (3.12) are the dynamical equations which should be solved as an eigenvalue equation to obtain σ^2. Equation (3.12), being the condition of adiabaticity, is identical with

$$\frac{\Delta p}{p} = \gamma \frac{\Delta \rho}{\rho},$$

wherein Δp and $\Delta \rho$ are the Lagrangian perturbations in p and ρ. At the edge of the disk, we need the boundary condition $\Delta p = 0$, which is satisfied by restricting ξ^i and their derivatives to remain finite everywhere

Following the procedure of Chandrasekhar and Friedman (1972 a, b), we multiply the dynamical equations (3.11) and (3.12) by $\bar{\xi}^r$ and $\bar{\xi}^\theta$ respectively, add them and integrate with respect to r and θ over the entire region of the disk,, Here $\bar{\xi}^r$ and $\bar{\xi}^\theta$ are the ‘ trial functions ’ which satisfy the same boundary conditions as the true eigen functions ξ^r and ξ^θ, but otherwise completely arbitrary. By performing several integrations by parts and using the steady state relations, we can then bring the resultant equation to symmetrical form in barred and unbarred displacements as

$$\begin{aligned}
\sigma^2 \int\int \rho_0\, r^2 \sin\theta\, (\bar{\xi}^r \xi^r + \bar{\xi}^\theta \xi^\theta)\, dr\, d\theta
&= \int\int 2\, \rho_0\, r\, V_0 \sin\theta \left\{ \left(\frac{\partial V_0}{\partial r} + \frac{V_0}{r}\right) \bar{\xi}^r \xi^r + \frac{\cot\theta}{r} \left(\frac{\partial V_0}{\partial \theta} + \cot\theta\, V_0\right) \bar{\xi}^\theta \xi^\theta \right. \\
&\quad \left. + \frac{V_0}{r} \cot\theta\, (\bar{\xi}^\theta \xi^r + \bar{\xi}^r \xi^\theta) \right\} dr\, d\theta - MG \int\int \frac{\partial \rho_0}{\partial r} \sin\theta\, \bar{\xi}^r \xi^r\, dr\, d\theta \\
&\quad + \int\int V_0^2\, r \sin\theta \left\{ \bar{\xi}^r \xi^r \frac{\partial \rho_0}{\partial r} + \frac{\bar{\xi}^\theta \xi^\theta}{r} \cot\theta \frac{\partial \rho_0}{\partial \theta} \right\} dr\, d\theta \\
&\quad - \int\int \left\{ r^4 \sin\theta\, \bar{\xi}^r \xi^r \frac{\partial}{\partial r}\left(\frac{1}{r^2} \frac{\partial p_0}{\partial r}\right) + \sin^2\theta\, \bar{\xi}^\theta \xi^\theta \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta} \frac{\partial p_0}{\partial \theta}\right) \right\} dr\, d\theta \\
&\quad - \int\int \left\{ p_0 \frac{\partial}{\partial r}(r\, \bar{\xi}^r) \frac{\partial}{\partial \theta}(\sin\theta\, \xi^\theta) + p_0 \frac{\partial}{\partial \theta}(\sin\theta\, \bar{\xi}^\theta) \frac{\partial}{\partial r}(r\, \xi^r) \right. \\
&\quad - p_0 \frac{\partial}{\partial \theta}(r\, \bar{\xi}^r) \frac{\partial}{\partial r}(\sin\theta\, \xi^\theta) - p_0 \frac{\partial}{\partial r}(\sin\theta\, \bar{\xi}^\theta) \frac{\partial}{\partial \theta}(r\, \xi^r) \\
&\quad \left. - (\bar{\xi}^\theta \xi^r + \bar{\xi}^r \xi^\theta) \sin\theta \frac{\partial p_0}{\partial \theta} \right\} dr\, d\theta \\
&\quad + \gamma \int\int \left\{ \frac{p_0 \sin\theta}{r^2} \frac{\partial}{\partial r}(r^2\, \bar{\xi}^r) \frac{\partial}{\partial r}(r^2\, \xi^r) + \frac{p_0}{r} \left[\frac{\partial}{\partial r}(r^2\, \bar{\xi}^r) \frac{\partial}{\partial \theta}(\sin\theta\, \xi^\theta) \right. \right. \\
&\quad \left. \left. + \frac{\partial}{\partial \theta}(\sin\theta\, \bar{\xi}^\theta) \frac{\partial}{\partial r}(r^2\, \xi^r) \right] + \frac{p_0}{\sin\theta} \frac{\partial}{\partial \theta}(\sin\theta\, \bar{\xi}^\theta) \frac{\partial}{\partial \theta}(\sin\theta\, \xi^\theta) \right\} dr\, d\theta.
\end{aligned} \tag{3.13}$$

As shown by Chandrasekhar and Friedman, the symmetrical form of σ^2 equation implies a variational principle; for identifying $\bar{\xi}^r$ with ξ^i one can write the following equation for σ^2.

$$\sigma^2 \int\int \rho_0\, r^2 \sin\theta\, ({\xi^r}^2 + {\xi^\theta}^2)\, dr\, d\theta$$

$$= \int\int 2\rho_0 V_0\, r \sin\theta \left\{ \left(\frac{\partial V_0}{\partial r} + \frac{V_0}{r}\right) {\xi^r}^2 + \frac{\cot\theta}{r} \left(\frac{\partial V_0}{\partial \theta} + \cot\theta\, V_0\right) {\xi^\theta}^2 \right.$$

$$\left. + \frac{2V_0}{r} \cot\theta\, \xi^r \xi^\theta \right\} dr\, d\theta - MG \int\int \frac{\partial \rho_0}{\partial r} \sin\theta\, {\xi^r}^2\, dr\, d\theta$$

$$- \int\int \left\{ r^4 \sin\theta\, {\xi^r}^2 \frac{\partial}{\partial r}\left(\frac{1}{r^2} \frac{\partial p_0}{\partial r}\right) + \sin^2\theta\, {\xi^\theta}^2 \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta} \frac{\partial p_0}{\partial \theta}\right) \right\} dr\, d\theta$$

$$+ \int\int V_0^2 r \sin\theta \left\{ {\xi^r}^2 \frac{\partial \rho_0}{\partial r} + \frac{{\xi^\theta}^2}{r} \cot\theta \frac{\partial \rho_0}{\partial \theta} \right\} dr\, d\theta$$

$$- \int\int \left\{ 2p_0 \left[\frac{\partial}{\partial r}(r\, \xi^r) \frac{\partial}{\partial \theta}(\sin\theta\, \xi^\theta) - \frac{\partial}{\partial \theta}(r\, \xi^r) \frac{\partial}{\partial r}(\sin\theta\, \xi^\theta) \right] \right.$$

$$\left. - 2 \frac{\partial p_0}{\partial \theta} \sin\theta\, \xi^r \xi^\theta \right\} dr\, d\theta + \gamma \int\int \left[\frac{p_0}{r^2} \sin\theta \left\{ \frac{\partial}{\partial r}(r^2\, \xi^r) \right\}^2 \right.$$

$$\left. + \frac{2p_0}{r} \frac{\partial}{\partial r}(r^2\, \xi^r) \frac{\partial}{\partial \theta}(\sin\theta\, \xi^\theta) + \frac{p_0}{\sin\theta} \left\{ \frac{\partial}{\partial \theta}(\sin\theta\, \xi^\theta) \right\}^2 \right] dr\, d\theta. \tag{3.14}$$

Now if one evaluates equation (3.14) by two trial displacements ξ^i and $\xi^i + \delta\xi^i$ such that the resultant variation in σ^2 is $\delta\sigma^2$, and trace back the calculations that lead to equation (3.13) starting from equations (3.8) – (3.12), one gets,

$$\delta\sigma^2 \int\int \rho_0\, r^2 \sin\theta\, ({\xi^r}^2 + {\xi^\theta}^2)\, dr\, d\theta$$

$$= \int\int -2\, \delta\xi^r\, r^2 \sin\theta \left[\rho_0 \sigma^2 \xi^r + \frac{2\rho_0 V_0}{r} \delta V^\phi - \left(\frac{MG}{r^2} - \frac{V_0^2}{r}\right) \delta\rho - \frac{\partial}{\partial r} \delta p \right] dr\, d\theta$$

$$+ \int\int -2\, \delta\xi^\theta\, r^2 \sin\theta \left[\rho_0 \sigma^2 \xi^\theta + \frac{2\rho_0 V_0}{r} \cot\theta\, \delta V^\phi \right.$$

$$\left. + \frac{V_0^2}{r} \cot\theta\, \delta\rho - \frac{1}{r} \frac{\partial}{\partial \theta} \delta p \right] dr\, d\theta. \tag{3.15}$$

It is clear from equation (3.15) that demanding $\delta\sigma^2 = 0$ amounts to solving the original eigenvalue equations (3.11) and (3.12) along with the initial value equations

(3.8) – (3.10). Rewriting equation (3.14) in terms of dimensionless quantities R, and V_0, we get

$$\frac{m^2\sigma^2}{c^2}\int\int \rho_0 R^2 \sin\theta\,(\xi^{r^2}+\xi^{\theta^2})\,dRd\theta$$

$$=\int\int\Big[2\rho_0 \mathbf{V}_0 R\sin\theta\left(\frac{\partial \mathbf{V}_0}{\partial R}+\frac{\mathbf{V}_0}{R}\right)\xi^{r^2}-\frac{\partial\rho_0}{\partial R}\sin\theta\,\xi^{r^2}$$

$$-R^4\sin\theta\frac{\xi^{r^2}}{c^2}\frac{\partial}{\partial R}\left(\frac{1}{R^2}\frac{\partial p_0}{\partial R}\right)+\mathbf{V}_0^2 R\sin\theta\,\xi^{r^2}\frac{\partial\rho_0}{\partial R}\Big]dR\,d\theta$$

$$+\int\int\Big[2\rho_0\,\mathbf{V}_0\cos\theta\left(\frac{\partial\mathbf{V}_0}{\partial\theta}+\cos\theta\,\mathbf{V}_0\right)\xi^{\theta^2}-\sin^2\theta\frac{\xi^{\theta^2}}{c^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial p_0}{\partial\theta}\right)$$

$$+\xi^{\theta^2}\mathbf{V}_0^2\cos\theta\frac{\partial\rho_0}{\partial\theta}\Big]dR\,d\theta+\int\int\Big[4\rho_0\,\mathbf{V}_0^2\cos\theta\,\xi^r\,\xi\theta$$

$$-\frac{2p_0}{c^2}\frac{\partial}{\partial R}(R\,\xi^r)\frac{\partial}{\partial\theta}(\sin\theta\,\xi^\theta)+\frac{2p_0}{c^2}\frac{\partial}{\partial\theta}(R\,\xi^r)\frac{\partial}{\partial R}(\sin\theta\,\xi^\theta)$$

$$+\frac{2}{c^2}\frac{\partial p_0}{\partial\theta}\sin\theta\,\xi^r\,\xi^\theta\Big]dR\,d\theta+\gamma\int\int\Big[\frac{p_0}{R^2c^2}\sin\theta\left\{\frac{\partial}{\partial R}(R^2\,\xi^r)\right\}^2+\frac{p_0}{c^2\sin\theta}$$

$$\left\{\frac{\partial}{\partial\theta}(\sin\theta\,\xi^\theta)\right\}^2+\frac{2p_0}{Rc^2}\frac{\partial}{\partial R}(R^2\,\xi^r)\frac{\partial}{\partial\theta}(\sin\theta\,\xi^\theta)\Big]dR\,d\theta,\qquad \mathbf{V}_0=\frac{V_0}{c}. \tag{3.16}$$

In order to evaluate σ^2, we choose two kinds of trial functions (i) with fixed boundary, *i.e.* the Lagrangian displacements ξ^i vanish at the boundary and (ii) expanding or contracting boundary.

For Case (i), corresponding to the three different types of solutions (2.21) – (2.23), we choose a function 'q'

$$q=\sin^k\theta-\frac{k}{AR^k}\left(\frac{R^{l-1}}{l-1}-B\right),\qquad l\neq 1, k\neq 0, \tag{3.17}$$

$$q=\sin\theta-\exp\left\{-\frac{R^{l-1}}{A\,(l-1)}-\frac{B}{A}-\ln R\right\},\qquad l\neq 1, k=0, \tag{3.18}$$

and

$$q=\sin^k\theta-\frac{k}{AR^k}(\ln R-B),\qquad l=1, k\neq 0, \tag{3.19}$$

which vanishes at the boundary and set $\xi^i = q$. With this choice of ξ^r and ξ^θ in equation (3.16) we evaluate the critical value of adiabatic index γ_{c_1} for $\sigma^2 = 0$, which would give the neutral stability. In order to increase the accuracy, we take

$$\xi^r = q + \alpha q^2, \quad \xi^\theta = q + \beta q^2, \tag{3.20}$$

wherein α and β are adjustable parameters determined by extremising the expression for σ^2. With such a choice of ξ^r and ξ^θ, one could obtain better values for γ_{c_1} for the onset of instability

In Case (ii) (a nonstationary boundary), we first consider the case of radial perturbation with $\xi^\theta = 0$. The equations governing such radial perturbation are obtained from the original set of equations (3.8) – (3.12) as follows

$$\delta V^\phi = -\left(\frac{\partial V_0}{\partial r} + \frac{V_0}{r}\right) \xi^r, \tag{3.21}$$

$$\delta \rho = -\frac{\rho_0}{r^2}\frac{\partial}{\partial r}(r^2 \xi^r) - \xi^r \frac{\partial \rho_0}{\partial r}, \tag{3.22}$$

$$\delta p = -\frac{\gamma p_0}{r^2}\frac{\partial}{\partial r}(r^2 \xi^r) - \xi^r \frac{\partial p_0}{\partial r} \quad , \tag{3.33}$$

$$-\rho_0 \sigma^2 \xi^r = \frac{2\rho_0 V_0}{r}\delta V^\phi - \left(\frac{MG}{r^2} - \frac{V_0^2}{r}\right)\delta\rho - \frac{\partial}{\partial r}\delta p \tag{3.24}$$

and

$$\frac{2\rho_0 V_0}{r} \cot\theta\, \delta V^\phi + \frac{V_0^2}{r}\cot\theta\, \delta\rho - \frac{1}{r}\frac{\partial}{\partial\theta}\delta p = 0. \tag{3.25}$$

Using initial value equations (3.21) – (3.23) in (3.25) and assuming ξ^r to be a function of r only we obtain the differential equation

$$(1-\gamma)\frac{d}{dr}(r^2 \xi^r) + 2r\,\xi^r = 0 \tag{3.26}$$

for ξ^r, whose solution is given by

$$\xi^r = \eta\, r^{\frac{4-2\gamma}{\gamma-1}} \tag{3.27}$$

where η is constant of integration. Using this solution for ξ^r in (3.24) we get

$$\rho_0 \sigma^2 \xi^r = \eta \frac{\rho_c MG}{m^l}\left\{\frac{2\gamma(3\gamma-5)}{(\gamma-1)^2} B + A \sin^k\theta \frac{(3\gamma-5)(k\gamma-k+2\gamma)}{k(\gamma-1)^2} r^k\right.$$

$$+\left[\frac{4-2\gamma}{\gamma-1}-\frac{2\gamma}{(\gamma-1)(l-1)}\left(\frac{3\gamma-5}{\gamma-1}\right)\right]r^{l-1}\Bigg\}\, r^{\frac{6-4\gamma}{\gamma-1}},$$

$$l\neq 1,\ k\neq 0, \qquad (3.28)$$

$$\rho_0\,\sigma^2\,\xi^r=\eta\frac{\rho_c\,MG}{m^l}\left\{\frac{2\gamma\,(3\gamma-5)}{(\gamma-1)^2}B+2A\ln\,(r\sin\,\theta)\frac{(3\gamma-5)}{(\gamma-1)^2}+A\frac{(3\gamma-5)}{\gamma-1}\right.$$

$$+\left[\frac{4-2\gamma}{\gamma-1}-\frac{2\gamma}{(\gamma-1)(l-1)}\left(\frac{3\gamma-5}{\gamma-1}\right)\right]r^{l-1}\Bigg\}\, r^{\frac{6-4\gamma}{\gamma-1}},$$

$$l\neq 1,\ k=0, \qquad (3.29)$$

$$\rho_0\,\sigma^2\,\xi^r=\eta\frac{\rho_c\,MG}{m}\left\{\frac{2\gamma\,(3\gamma-5)}{(\gamma-1)^2}B+A\sin^k\theta\,\frac{(3\gamma-5)\,(k\gamma-k+2\gamma)}{k\,(\gamma-1)^2}r^k\right.$$

$$+\left(\frac{4-2\gamma}{\gamma-1}\right)-(\ln\,r)\frac{2\gamma}{\gamma-1}\left(\frac{3\gamma-5}{\gamma-1}\right)\Bigg\}\, r^{\frac{6-4\gamma}{\gamma-1}},\quad l=1,\ k\neq 0. \qquad (3.30)$$

For the special case of ordinary gas with γ = 5/3 the above equations for σ^2 reduce to a very simple form

$$\sigma^2=\frac{MG}{r^3} \qquad (3.31)$$

showing that the disks are stable with 'local' frequency being proportional to $r^{-3/2}$ irrespective of the other parameters like *l*, *k*, *a* and *b*. Incidentally this value of γ = 5/3 makes the function ξ^r to be ηr which is exactly the form as used by Bisnovatyi-Kogan and Blinnikov (1972) for analysing the stability of thin gas-disks against expansion and contraction. It is interesting to note that the same frequency is obtained as above for the radial oscillation of a pressureless disk confined to $\theta = \pi/2$ plane, which is in Keplerian motion with. $V_0 = (MG1r)^{1/2}$, as may be seen from equations (3.21) – (3.24) with δp =0.

To consider the stability with a non-stationary boundary (Case and with axi-symmetric perturbations, we choose

$$\xi^r=R+\alpha\,q,\quad \xi^\theta=R+\beta\,q, \qquad (3.32)$$

evaluate σ^2 from equation (3.16) and calculate γ_{c_2} by setting σ^2 = 0 after extremising it by adjusting α and β as in Case (i).

4. Discussion and conclusions

As the general solution obtained above refers to a class of solutions with parameters *l* and *k* referring to different density and velocity distributions, we have considered a number of cases for various values of l and k for different cases of disk radii. The thickness of the disk varies from one case to another depending upon the density and velocity distributions. The general structure of the disks is presented through figures and tables. Figs 1 and 2 show the upper half of the meridional section (r, θ) plane for two typical cases with (i) $a = 4$, $b = 20$, $l = 1$, $k = -1$ and (ii) $a = 4$, $b = 100$, $l = 0$, $k = -2$. For these two cases the corresponding profiles of pressure, density and velocity are plotted in Figs 3 and 4.

Study of the disk structure for a number of cases revealed that the maximum thickness h_m (= $R \cos \theta_m$, θ_m being the minimum value of θ for a given disk) of the disk as well as the shape change with the velocity (n) and density (l) profiles of the disk.

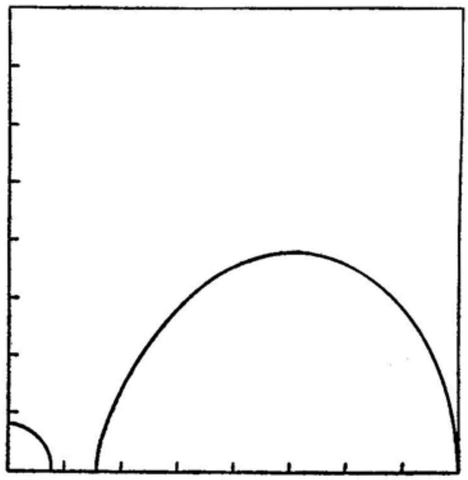

Figure 1. Upper half of the meridional section of the disk for $a = 4$, $b = 20$, $l = 1$, $k = -1$. The circular part at the left is the section of the central massive object with $R = 2$.

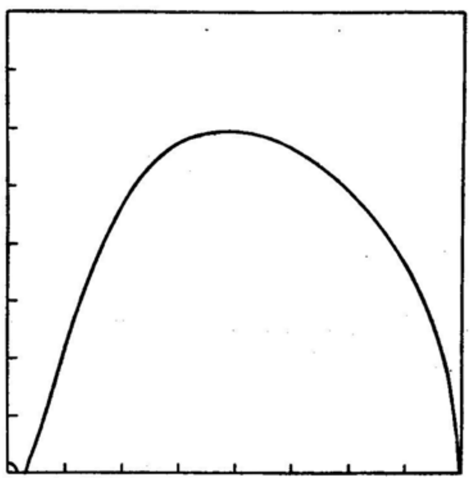

Figure 2. Upper half of the meridional section of the disk for $a = 4$, $b = 100$, $l = 0$, $k = -2$. The circular part at the left is the section of the central massive object with $R = 2$.

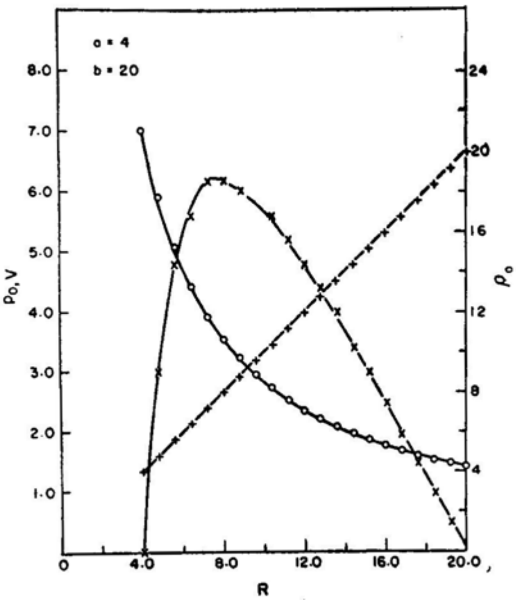

Figure 3. Profiles of pressure, density and velocity at $\theta = \pi/2$ plane for the disk described in Fig. 1; (+ : density, 0 velocity $V_0 \times 10$, × : pressure $P_0 \times 20$).

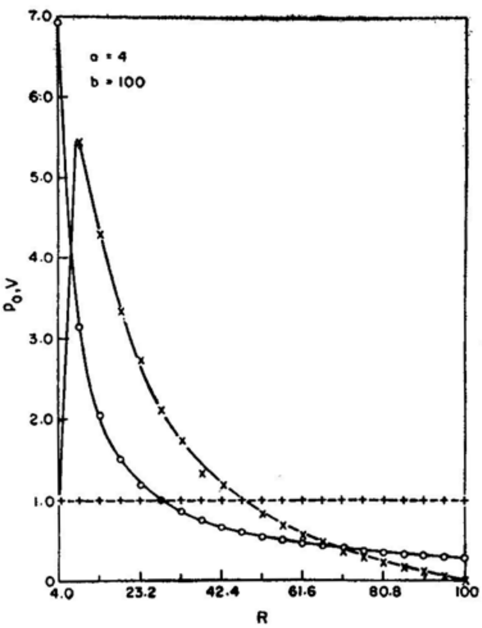

Figure 4. Profiles of pressure, density and velocity at $\theta = \pi/2$ plane for the disk described in Fig. 2; (+ : density, o: velocity $V_0 \times 10$, × : pressure P_0).

For a disk with $a = 4$, $b = 20$, Figs 5 to 8 reveal the nature of such changes. As may be seen from the Figs 5 and 6 the maximum thickness increases as n increases, which is in conformity with the known result that disks with larger angular momentum are thinner than the ones with lesser angular momentum. Also, as regards the shape

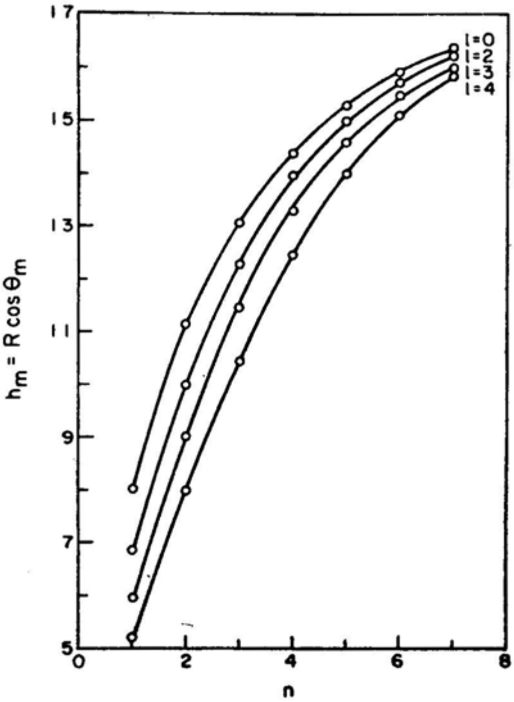

Figure 5. Maximum height $h_m = R \cos \theta_m$ as a function of n for different values of l for the disk with $a = 4$, $b = 20$.

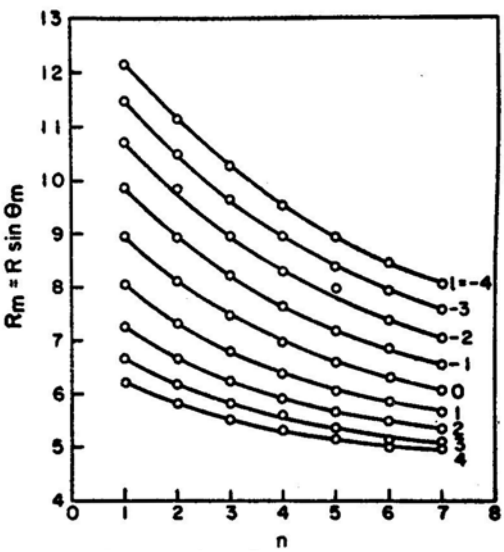

Figure 6. The distance $R_m = R \sin \theta_m$ of the point where the thickness is maximum, as a function of n for different values of l for a disk with $a = 4$, $b = 20$.

of the disks, the maximum thickness occurs nearer to the inner edge as n increases. Fig. 7 shows the variation of the maximum thickness with the density distribution for different values of n. The maximum thickness rises slowly as l increases, attaining a maximum around $l = -2$ to $l = 0$ for $n = 1$ to 7, and then falls off rapidly as l increases, which is consistent with normal distributions.

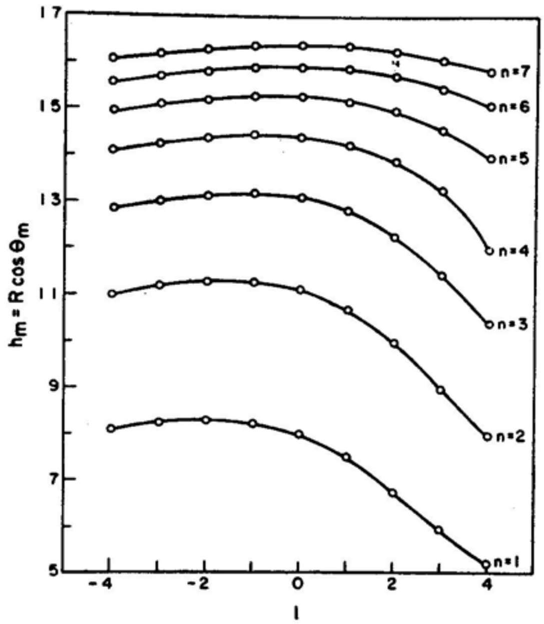

Figure 7. Maximum height $h_m = R \cos \theta_m$ as a function of *l* for different values of *n* for a disk with $a = 4$, $b = 20$.

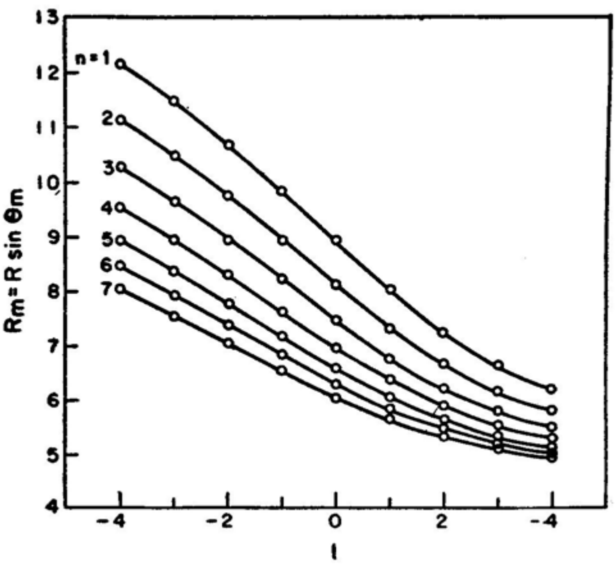

Figure 8. The distance $R_m = R \sin \theta_m$ as a function of *l* for different values of *n* for a disk with $a = 4$, $b = 20$.

Regarding the onset of instability as remarked in Section 3, we evaluate the critical adiabatic index γ_c by setting $\sigma^2 = 0$ for different values of *a, b, n* and *l.* Table 1 shows the values of γ_c for different *l, n, a* and *b.* The table also shows the value of the ratio of the kinetic energy to the potential energy ($I\Omega^2/|W|$) for each disk. Normally, in the case of a self-gravitating fluid sphere, if there is rotation, then the criterion for

Table 1. The ratio of kinetic energy to potential energy $I\Omega^2/|W|$ and y, as a function of n for disks with different values of a, b and l. γ_{c1} refers to the case when $\xi^r = q + \alpha q^2$, and $\xi^\theta = q+\beta q^2$ while γ_{c_2} refers to the case when $\xi^r = R+ \alpha q$, $\xi_\theta = R+ \beta q$.

n	$I\Omega^2/\lvert W\rvert$	γ_{c_1}	γ_{c_2}	n	$I\Omega^2/\lvert W\rvert$	γ_{c_1}	γ_{c_2}
$a = 4{\cdot}0$, $b = 100{\cdot}0$, $l = -3$				$a = 4{\cdot}0$, $b = 100{\cdot}0$, $l = 0$			
1	0·037	1·07	1·06	1	0·63	0·89	1·04
2	0·012	1·04	1·17	2	0·24	0·99	1·16
3	0·005	1·02	1·10	3	0·10	0·97	1·15
4	0·003	0·96	0·99	4	0·052	0·92	1·11
5	0·002	0·89	0·90	5	0·031	0·87	1·07
6	0·001	0·83	0·82	6	0·021	0·81	1·04
$a = 4{\cdot}0$, $b = 100{\cdot}0$, $l = -2$				$a = 4{\cdot}0$, $b = 100{\cdot}0$, $l = 1$			
1	0·15	1·05	1·10	1	1·18	0·62	0·93
2	0·050	1·06	1·17	2	0·49	0·88	1·14
3	0·021	1·02	1·11	3	0·21	0·92	1·15
4	0·011	0·95	1·04	4	0·11	0·89	1·13
5	0·006	0·89	0·97	5	0·066	0·85	1·09
6	0·004	0·83	0·92	6	0·045	0·80	1·06
$a = 4{\cdot}0$, $b = 100{\cdot}0$, $l = -1$				$a = 20{\cdot}0$, $b = 116{\cdot}0$, $l = 0$			
1	0·35	1·00	1·09	1	1·01	0·47	0·54
2	0·13	1·04	1·17	2	0·70	0·76	0·94
3	0·054	1·00	1·11	3	0·49	0·85	1·04
4	0·028	0·94	1·08	4	0·34	0·86	1·05
5	0·017	0·88	1·04	5	0·25	0·84	1·04
6	0·011	0·82	0·99	6	0·19	0·81	1·02

stability differs from $\gamma = 4/3$ to $\gamma - 4/3 = -2\, I\Omega^2/9\,|W|$ (Lebovitz 1970) showing an increase in the range of stability. However, we find that there is no such simple relation connecting γ_c and the energy ratio for the case of rotating disks. But the critical γ is always less than 4/3 indicating that all the cases considered here correspond to stable configurations under axisymmetric perturbations.

We have thus found that the ordinary perfect fluid ($\gamma = 5/3$) disks rotating around massive objects are stable under radial pulsations with frequency $(MG/r^3)^{1/2}$. In this case the boundary could be expanding or contracting as given by the amplitude function $\xi^r = nr$. On the other hand if the perturbations are axisymmetric the critical value of γ is much less than 4/3, thus indicating stability of such disks. From this detailed study, it appears that the dynamical configuration of a rotating disk around a massive object is similar to that of a self-gravitating fluid sphere. However, in the above analysis, as the consideration of axisymmetric perturbations are restricted only to a specific choice of θ-dependence, the criterion of stability is not very general. In fact, it is at least qualitatively known that the general axisymmetric perturbations do lead to instabilities that would result in either ring structures or spiral structures caused by anisotropies in either temperature or density. In fact it is very much necessary to study the above analysis for more general 0-dependent perturbations, which would perhaps be associated with more general initial configurations. It is also essential to compare the results obtained above with the case of disks when self-gravitation is not neglected.

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