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Rotation and Luminosity Variations in Post-Main Sequence Stars

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Abstract. Previous first-order analytic treatments of rotation acting upon stellar equilibria are extended to include later, post-Helium burning, stages of stellar evolution. Strong differential rotation is capable of substantially increasing the photon luminosities of post-main sequence stars, and thus accelerating their evolution. On the other hand, uniform rotation reduces the photon flux for a wide range of stellar interior types and conditions. Similar conclusions are drawn regarding the effects of rotation on the emission of neutrinos in pre-collapse phases of evolution. A brief discussion of the gravitational radiation emitted during these phases is also given.

Key words: rotating stars — post-main sequence evolution — neutrino emission — gravitational radiation

1. Introduction

Rotation is recognized as playing a very important role in the life of a star, and many aspects of the subject have been extensively examined. Much theoretical work has been done on the possible effects of rotation in star formation (e.g. Boss 1980; Wiita, Schramm and Symbalisty 1979) and on its ramifications for the appearance of main sequence stars (for reviews see Fricke and Kippenhahn 1972; Tassoul 1978) and white dwarfs (Tassoul 1978). Spurred on by the discovery of pulsars, the final stages of the collapse of rotating stars have also received considerable attention recently (*e.g.* Saenz and Shapiro 1979).

Still, the number of investigations into the effects of rotation in post-main sequence stars before the onset of collapse is quite small. Of course this is understandable: the great difficulties encountered in computing the evolution of

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complex, multi-layered spherical stars are considerably compounded by the introduction of angular momentum distributions as additional independent variables. Also, any uncertainties that arise during the calculations of rotating main sequence stellar models are amplified when one tries to go further, since various thermal, dynamical and secular timescales all change, sometimes dramatically (Fricke and Kippenhahn 1972). Because detailed numerical calculations have just recently been attempted in this regime (Endal and Sofia 1976, 1978), and because the physics involved in the transport of angular momentum is so uncertain (Kippenhahn 1974; Huppert and Spiegel 1977), it is clear that there is room for some simple analytical calculations that may point out a few basic conclusions and indicate directions for future numerical work. A useful beginning along these lines was made by Maeder (1974) when he used homology arguments to investigate the evolutionary tracks of the centres of rotating stars in the temperature-density plane.

The bulk of this paper is essentially an extension of Maeder's approach to later stages of stellar evolution. In Section 2 the approximations employed are summarized and the resulting equations relating rotation to temperatures and densities for various types of stars are presented. This technique is applied in Section 3 to the calculation of the photon and neutrino luminosities as functions of the amount and type of rotation present. Implications for evolution are outlined by considering some specific stellar structures. Finally, in Section 4, conclusions are drawn, and the great uncertainties still remaining are summarized.

2. An analytical approach

2.1 *Basic Equations and Approximations*

Following Schwarzschild (1958) and Maeder (1974) we average the effects of rotation and use a spherical approximation. Then the equation of hydrostatic equilibrium is

$$\frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4}(1-\chi), \tag{1}$$

where

$$\chi = \frac{2}{3}\frac{\Omega^2 r^3}{GM_r}, \tag{2}$$

And Ω is the angular velocity at point r. Maeder shows that if homologous contraction takes place in a star below a radius r_0, then for a radius $r \ll r_0$, we have (assuming local conservation of angular momentum)

$$\frac{1}{P}\frac{\partial P}{\partial t} = -\frac{(4-5\bar{\chi})}{(1-\bar{\chi})}\frac{1}{r}\frac{\partial r}{\partial t} \tag{3}$$

where the average effect of rotation is

$$\bar{\chi} = \left(\int_{M_r}^{M_{r_0}} \frac{GM_r}{4\pi r^4} \chi \, d M_r \right) \left(\int_{M_r}^{M_{r_0}} \frac{GM_r}{4\pi r^4} dM_r \right)^{-1}. \tag{4}$$

Other basic results are that during such a homologous contraction

$$d \text{ in } \rho = -3 \, d \text{ in } r \tag{5}$$

and

$$d \ln \bar{\chi} = \tfrac{1}{3} \, d \ln \rho. \tag{6}$$

Maeder (1974) uses these relations to argue (among other things) that if the contraction in the core is homologous and if angular momentum is conserved locally, that the mass of stars when carbon burns non-degenerately, M_{CD} may be raised from about 8 $M_\odot$ to as high as $20M_\odot$, while the mass limit for He flashes may be raised considerably from around $2M_\odot$ This is because, under these assumptions, a rotating star of a given mass would evolve to a lower central temperature at a given density. But some more recent calculations (Endal and Sofia 1978) show that local angular momentum conservation is probably not a good approximation in these regimes, and, since mixing tends to increase the duration of helium burning, a rotating star actually may have a higher central temperature near helium exhaustion. Later on, the pure rotational effects do start to dominate and eventually rotating models do evolve at lower central temperatures than their non-rotating counterparts. But Endal and Sofia find this effect to be smaller for more massive stars; therefore M_{CD} is unlikely to be increased very much, and we estimate that it is under 10 $M_\odot$. This question ought to be investigated more thoroughly because the numbers of Type II and Type I Supernovae, as well as the amount of heavy element enrichment they produce, may depend significantly upon this mass (Arnett and Schramm 1973; Hainebach, Norman and Schramm 1976).

We now come to the main topic of this paper, the effects of rotation upon stars in post-main sequence, but pre-collapse, phases. Several important approximations are made, and we shall mention most of them now.

First, we neglect magnetic fields completely; this is a somewhat dangerous simplification, and may be a major shortcoming, especially in the last phases of evolution (LeBlanc and Wilson 1970). Magnetic coupling between the core and the mantle could be the most efficient way to transfer angular momentum (Kippenhahn 1974) but magnetic energy is unlikely to be comparable to rotational energy, and very simple calculations show that under most circumstances the ratio of magnetic to rotational energy decreases as the core of the star contracts. Thus we join the vast majority of previous workers and shall henceforth ignore magnetic fields.

Second, we assume homology relations do hold, at least below a certain level in the star, and that any energy generation or neutrino losses outside of that zone are negligible. Homology relations do start to fail at temperatures around 10^9 K, when neutrino emission really dominates over photons, but the numerical calculations done so far do not yet allow us to set firm bounds on the errors so introduced.

A third limiting factor is our assumption that the star behaves spherically (equation 1). Once $\bar{\chi}$ grows too large the surface temperature and most other properties will vary considerably with latitude, but as long as we restrict $\bar{\chi}$ to less than 0·187, crudely corresponding to the secular stability limit for Maclaurin spheroids (see Section 2·4 below), we can hope that these changes do not vitiate our results.

Under these assumptions the basic equations of stellar structure are

$$\frac{dM_r}{dr} = 4\pi\, r^2 \rho, \tag{7}$$

$$\frac{dP_r}{dr} = -\frac{GM_r\,\rho}{r^2}(1-\chi), \tag{8}$$

$$\frac{dL_r}{dr} = 4\pi\, r^2\, \rho\, \epsilon \tag{9}$$

and

$$\frac{dT_r}{dr} = -\frac{3}{4ac}\,\frac{\kappa\rho}{T_r^3}\,\frac{L_r}{4\pi r^2} \tag{10a}$$

for a radiative zone, or

$$\frac{dT_r}{dr} = \frac{\Gamma_2 - 1}{\Gamma_2}\,\frac{T_r}{P_r}\,\frac{dP_r}{dr} \tag{10b}$$

for a convective zone, and the notation is standard.

2.2 *The Core*

For most of the post-main sequence phases of stellar evolution the star can be thought of as an inert core, covered with a shell, or multiple shells, of 'burning' material, and wrapped with a diffuse envelope. The core is usually degenerate, and this degeneracy will be assumed for the rest of this paper. As proportionalities between quantities are the only things of interest to us at this point, we can let the subscript *c* denote ' core ' values of quantities, so that

$$\rho_c \simeq M_c\, R_c^{-3}, \tag{11}$$

$$\frac{dP}{dr} \propto -\, M_c^2\, R_c^{-5}\,(1-\bar{\chi}). \tag{12}$$

But when assume a fixed polytropic index for the core,

$$P \propto \rho^{\Gamma} \propto M_c^{\Gamma} R_c^{-3\Gamma}, \tag{13}$$

We find (Meader 1974)

$$\frac{dP}{dr} \propto -M_c^{\Gamma} R_c^{-(3\Gamma+1)}. \tag{14}$$

Combining equations (12) and (14) leads to

$$R_c \propto M_c^{(2-\Gamma)/(4-3\Gamma)} (1-\bar{\chi})^{1/(4-3\Gamma)} \tag{15}$$

and

$$\rho_c \propto M_c^{-2/(4-3\Gamma)} (1-\bar{\chi})^{-3/(4-3\Gamma)}. \tag{16}$$

This last relation can be used to show how effective rotation is in supporting additional mass, for under our linear treatment of rotation, and assuming homology we have, for a given p_c.

$$M_c \propto (1-\bar{\chi})^{-3/2}$$

Independent of the value of Γ in the core.

2.3 *The Burning Shell*

To obtain constraints on the temperature and luminosity we now generalize the approach of Refsdal and Weigert (1970) and Maeder (1974). In the shell, assumed to be thin relative to the core in both mass and radial co-ordinates, the basic parameters essentially depend on M_c (Refsdal and Weigert 1970), and including rotation (Maeder 1974) we can write

$$\rho \propto M_c^{\alpha} (1-\tilde{\chi})^{\alpha_1}, \quad P \propto M_c^{\beta} (1-\tilde{\chi})^{\beta_1},$$

$$T \propto M_c^{\gamma} (1-\tilde{\chi})^{\gamma_1} \text{ and } L \propto M_c^{\delta} (1-\tilde{\chi})^{\delta_1} \tag{18}$$

where we have used the average density weighted rotational support in *the shell* defined via

$$\tilde{\chi} = \int_{\text{shell}} \rho \chi \, d(1/r) \Big/ \int_{\text{shell}} \rho \, d(1/r). \tag{19}$$

In order to determine the exponents in equations (18) we must introduce an equation of state and the constitutive equations. For simplicity, we shall assume that the opacity does not vary substantially across the shell; also, because the shell is assumed to be (and usually is) thin, we take $M_r = M_c$, a constant, within the shell. While the bulk of the energy is carried off by photons, the great majority of the energy is generated by nuclear fusion, and we can approximate

$$L_r \propto \rho^u T^n r^2 \, dr, \tag{20}$$

Where u and n depend on the dominant type of nuclear reaction (*cf.* Clayton 1968). We are then led to distinguish different cases determined by the physical conditions in the shell.

Case 1: *Gas pressure dominated, radiative transport*

In this case, the only one considered by Maeder (1974), we take

$$P \propto \rho T, \tag{21a}$$

$$dP \propto - M_c \rho \frac{dr}{r^2}(1 - \chi) \tag{21b}$$

and

$$dT^4 \propto \rho L \frac{dr}{r^2} \tag{21c}$$

for the shell and the region above it. Comparing equations (18), (20) and (21) leads to the relations

$$\beta = a + \gamma, \quad \beta = a + 1, \quad 4\gamma = a + \delta, \quad \delta = u a + n \gamma,$$

which can be solved to yield

$$a - \frac{4-n}{u+1}, \quad \beta = 1 + \frac{4-n}{u+1}, \quad \gamma = 1, \quad \delta = 4 - \frac{4-n}{u+1}. \tag{22}$$

(in this, and in all other cases as well, it turns out that $a_1 = a$, $\beta_, = \beta$, $\gamma_1 = \gamma$, and $\delta_1 = \delta$.)

When we have a low mass star, so that a H-shell surrounding a He-core is processsing material through the CNO cycle, $u=2$ and n =16 Then, for the variable of most interest to us, we obtain

$$L \propto M_c^8 (1 - \tilde{\chi})^8. \tag{23}$$

When we examine the star at fixed core density we have in this instance, using equation (17),

$$L \propto (1 - \bar{\chi})^{-12} (1 - \tilde{\chi})^8. \tag{24}$$

Case 2 : Gas pressure dominated, convective transport

While equations (21a, b) still hold, in place of (21c) we must use

$$dT \propto T\,(1-\Gamma_2^{-1})\frac{dP}{P}\,. \tag{25}$$

Using equations (20), (21a, b) and (25) we solve for the exponents in (18) and discover that

$$a=\frac{1}{\Gamma_2-1},\quad \beta=\frac{\Gamma_2}{\Gamma_2-1},\quad \gamma=1,\quad \delta=\frac{u}{\Gamma_2-1}+n, \tag{26}$$

so that the adiabatic index of the convective shell is very important in determining most of these scaling laws. Since, however, $4/3 < \Gamma_2 \lesssim 5/3$ for stable shells, and η is substantially greater than $3u/2$ (the variation in δ allowed by that range in Γ_2), δ is not as relatively dependent on Γ2 and $\delta \simeq n + 3u/2$ *This* is always much greater than in Case 1 and larger variations with rotation may therefore be expected if the burning shell is convective.

Case 3: *Radiation pressure dominated*

If the shell is able to become radiation-pressure dominated we utilize equations (21b, c) but replace (21a) with

$$P \propto T^4, \tag{27}$$

so that the exponents in equations (18) are

$$\alpha=\frac{4-n}{4u+n},\quad \beta=1+\frac{4-n}{4u+n},\quad \gamma=\frac{u+1}{4u+n},\quad \delta=1. \tag{28}$$

This implies much more gradual changes in all properties, and since equation (17) still holds in our approximations, we have, at a given ρ_c,

$$L \propto (1-\bar{\chi})^{-3/2}\,(1-\tilde{\chi}). \tag{29}$$

Case 4: *Degenerate shells*

If the shell ever approaches degeneracy, neither of the basic transport mechanisms we have been using so far is a very good approximation and our approach becomes hopeless. This means that in some cases, once we proceed beyond C-burning (Arnett 1972b, 1973a) where the shells can become somewhat degenerate, none of our valid cases may hold. Bat even though O-burning cores are quite degenerate for $M_\alpha \leqslant 8\,M_\odot$ (M_a is the original mass of the He star, which corresponds to the maximum He core mass built up just prior to He ignition (Arnett 1973b), we calculate that the He and C-burning shells are not very degenerate then, so that this restriction may not be a serious drawback. Also note that Ne-burning cores in stars with $M_\alpha \leqslant 16\ M_\odot$ are degenerate (Arnett 1974a), so, unless we are dealing with *very* massive stars, our key assumption that the core approximates a white dwarf while the overlaying burning shell is not degenerate is usually reasonable.

Case 5: *Predominantly gas pressure, but non-negligible radiation pressure*

Following Refsdal and Weigert (1970) we introduce

$$\psi = 1 - P_R/P \tag{30}$$

with P_R the radiation pressure , and then, in lieu of equation (21a) we take

$$P \propto \rho T / \psi, \tag{31}$$

where we assume that ψ is not too different from unity. Then

$$L \propto [M_c (1 - \tilde{\chi})\, \psi]^\delta, \tag{32}$$

where δ is given in equation (22); similar relations hold for other shell variables. For a non-rotating star we have

$$\psi_0 = 1 - K' T_0^3/\rho_0 \equiv 1 - K, \tag{33}$$

where $K' = a m_p\, \mu/3k$, as long as $P_R \ll P$ Using equation (33), the modifications to T and ρ analogous to that for L in (32), and the scaling laws given in equations (18) and (22), we eventually obtain, to lowest order,

$$\psi \simeq 1 - K\, [M_c (1 - \tilde{\chi})]^{3-\alpha}. \tag{34}$$

Then employing equation (17) in (32) and using (34) produces the relation (at constant ρ_c),

$$L \propto (1 - \bar{\chi})^{-3\delta/2}\ (1 - \tilde{\chi})^\delta\ \{1 - K\, [(1 - \bar{\chi})^{-3/2}\ (1 - \tilde{\chi})]^{3-\alpha}\}^\delta. \tag{35}$$

To this order, if $\bar{\chi} = \tilde{\chi}$ (see Section 2.4 below), this reduces to $L \propto (1 + \frac{1}{2}\, \delta\, \tilde{\chi} - \delta\, K)$. As δ is always positive, we see that the effect of radiation pressure is to decrease the photon luminosity. This provides a negative feedback mechanism, since for a given ρ an increase in T raises L, but concomitantly drops ψ and moderates the flux.

2.4 *Type and Strength of Rotation*

Our results to this point have been characterized by the two parameters $\bar{\chi}$ and $\tilde{\chi}$, representative of the average degree of rotational support in the core and the shell, respectively. Let us now consider what the reasonable ranges for these quantities are. First we note, as mentioned previously, that an upper limit is set by the requirement that our cores be stable against non-axisymmetric perturbations on a secular (dissipative) timescale. Translating the $T/|W|$ creations of Bodenheimer and Ostriker (1973) leads us to an approximate upper limit of χ_{max} = 0·187 for $\bar{\chi}$ This bound is applied to $\bar{\chi}$, not $\tilde{\chi}$, since it is the average degree of rotation in dense core that is of interest in setting instability timescales (Wiita and Press 1976). Since the burning shells are usually much less dense as well as much less massive (Arnett 1973b), they do not dominate the total ratio of rotational to gravitational energy.

Now we follow Maeder (1974) in analyzing the ratio $\bar{\chi}/\tilde{\chi}$, which is representative of the degree of differential rotation in the inner portions of the star. Since the shell is thin and its density low, the weighting factor used in equation (19) falls off quickly, we are safe in evaluating $\tilde{\chi}$ at the surface of the core, using equation(2).if $\bar{r}$ and $M_{\bar{r}}$ correspond to an ' average ' point where $\chi\,(\bar{r})$ = $\bar{\chi}$ inside the core, we can write

$$\eta \equiv \frac{\bar{\chi}}{\tilde{\chi}} = \frac{\Omega^2\,(\bar{r})\,\bar{r}^3\,M_c}{\Omega^2\,(R_c)\,R_c^3\,M_{\bar{r}}}\,. \tag{36}$$

As a simple approximation let us set M_r = ½ M_c, $\bar{r}$ =½ R_c, and then the ratio η is fixed by the rotation law. Solid body rotation, Ω = const., yields $\eta \simeq 0{\cdot}25$. On the other hand, extreme differential rotation, with $\Omega \propto (r^2 + a^2)^{-2}$, allows for $\eta \simeq 4$ if $a \ll R_c$ A compromise choice would be $\eta = 1$, equivalent to moderate differential rotation and the law $\Omega \propto (r+a)^{-1}$ This last relation is close to the one found in the rotating white dwarf models of Ostriker and Bodenheimer (1968) and has the additional advantage of simplifying the algebra. Thus, unless stated otherwise, we shall illustrate our results by taking $\eta = 1$ in the rest of the paper.

3. Rotational effects on luminosities

3.1 *Photon Luminosities*

The results of the previous section are brought together in Table 1, where the effects of rotation on various core/shell structures are summarized. The temperatures and densities picked are characteristic (for the later stages) of the M_α=4 $M_\odot$ non-rotating models of Arnett (1973a, 1974a), corresponding to main-sequence masses of about 15 $M_\odot$ (Arnett 1972a). The burning shells may be either radiative (Case 1), or convective (Case 2), depending on the exact phase of evolution and the mass of the core. For example, the non-rotating $M_a = 4M_\odot$ star investigated by Arnett has a He-burning shell which starts off as convective, and later switches over to radiative; but by the time a C-burning shell is established the He shell becomes marginally convective again (Arnett 1972b). That C-burning shell also appears to start off as convective but eventually becomes stable. Later shell-burning stages for this type of star are predominantly convective (Arnett 1974a, b). The general trend is for the He and C shells to be convective for a greater fraction of the time as the mass of the star increases, thus increasing the effect of rotation; however, during Ne and O-burning stages the heavier stars allow for radiative shells (Arnett 1972b, 1974a, b). For a few of the later stages some of the shells can become radiation pressure dominated (Case 3) for very massive stars (M_α = 32 $M_\odot$). In attempting somewhat more detailed calculations, particularly for stars with $M_\alpha \geqslant 16$ $M_\odot$, the effects of radiation pressure (Case 5) should be included.

Table 1. Rotation and photon luminosities.

Core/Shell	$\log \rho_c$	$\log T_{shell}$	u	n	Case	α	β	γ	δ	L_R/L_0 at χ_{max} $\eta = 0{\cdot}25$	$\eta = 1{\cdot}00$	$\eta = 4{\cdot}00$
He/H (CNO)	6·0	7·40	2	16·7	(1)*	−4·23	−3·23	1·00	8·23	1·53 (−4)	2·34	8·69
					(2)† $\Gamma_2 = 5/3$	1·50	2·50	1·00	19·70	7·32 (−10)	7·68	1·77 (+2)
					(3)‡	−0·51	0·49	0·12	1·00	3·94 (−1)	1·11	1·30
C/He (3α)	7·5	8·23	3	22·0	(1)	−4·50	−3·50	1·00	8·50	1·14 (−4)	2·41	9·32
					(2) $\Gamma_2 = 5/3$	1·50	2·50	1·00	26·50	5·14 (−13)	15·54	1·05 (+3)
					(3)	−0·53	0·47	0·12	1·00	3·94 (−1)	1·11	1·30
O, Ne, Mg/C	8·0	8·95	2	12·0	(1)	−2·67	−1·67	1·00	6·67	8·10 (−4)	1·99	5·76
					(2) $\Gamma_2 = 1{\cdot}57$	1·75	2·75	1·00	15·51	6·42 (−8)	4·98	58·78
					(3)	−0·40	0·60	0·15	1·00	3·94 (−1)	1·11	1·30
S, Si/O	9·0	9·30	2	33.0	(1)	−9·67	−8·67	1·00	13·67	4·58 (−7)	4·12	36·25
					(2) $\Gamma_2 = 1{\cdot}55$	1·82	2·82	1·00	36·64	1·02 (−17)	44·38	1·51 (+4)
					(3)	−0·71	0·29	0·07	1·00	3·94 (−1)	1·11	1·30

Notes:

*Case 1 is gas pressure dominated with radiative transport.

†Case 2 is gas pressure dominated with convective transport; the value of Γ_2 is noted.

‡Case 3 is radiation pressure dominated and is purely illustrative because, for the models of this table, the gas pressure is always larger.

Temperatures and densities are taken from Clayton (1968) and Arnett (1972b, 1973b, 1974a, b) for a $M_\alpha = 4\ M_\odot$ core.

For illustrative purposes, let us stay with a $M_\alpha = 4\ M_\odot$ model, and note that when it is undergoing He-shell burning and is radiative (Case 1), $\delta \simeq 8.5$, so using equations (17) and (18) along with the assumption that $\eta = 1$ yields

$$L \propto (1 - \bar{\chi})^{-4\cdot 25}. \tag{37}$$

Other Case 1 conditions produce exponents between –3·33 and –6·83. Hence, for moderate differential rotation, the photon luminosity during shell burning stages increases, in accord with the results of Kippenhahn, Meyer-Hofmeister and Thomas (1970) for the earliest post-main sequence phases. But when we recall the stability and linearity limits for our assumptions, such an increase is not terribly dramatic; using $\chi_{\max}$ in equation (37) leads to a rotating to non-rotating luminosity ratio

$$L_R \lesssim 2\cdot 4\ L_0. \tag{38}$$

This would imply an effective speeding up of the evolution rate by roughly a factor of two during these later stages. Of course an exact value cannot be obtained from this crude analytic approach, as the detailed rotational and chemical history of the star are of great importance. Such variations in the luminosity and in evolutionary speeds, therefore densities on the H-R diagram, are, in principle, detectable. But because this technique cannot make any predictions concerning the surface temperatures we cannot really estimate the shifts on the H-R diagram that would be produced. Two other points militate against observation of these types of perturbations: first, the changes brought about will be hidden in the scatter of the diagram, particularly for these rapidly evolving stars; second, the effects of rotation may cause a star of a given mass to mimic a non-rotating star of a different mass (*e.g.* for main sequence stars see Sackmann 1970).

In all the three major cases of the last section, comparisons of stars at equivalent core temperatures (instead of densities) lead to no changes in the photon luminosities. This is because equation (18) invariably shows that $M_c \propto (1-\bar{\chi})^{-1}$ at fixed T, and that proportionality yields no change in L, independent of y, δ and η.Comparisons at fixed core mass, which although of less physical significance than density may be computationally convenient, always lead to decreases in L, since δ is always found to be positive.

A convective shell (Case 2) allows more latitude in energy transport, and we intuitively expect that rotation will allow for more dramatic effects with regard to the flux permitted to pass. However, changes in density, pressure and temperature are likely to be smaller, due to the mixing over larger regions of the star. These ideas are supported by the results for the exponents listed in Table 1, as the absolute values of α, β and γ are always less than for radiative transport, while that for δ is much higher. Again assuming moderate differential rotation ($\eta = 1$) and setting $\bar{\chi} = \chi_{\max}$ the increase in L at given densities is at least a factor of five for our $M_\alpha = 4M_\odot$ conditions. Changes of this magnitude, and the concomitant speeding up of the evolution, might be detectable and could conceivably provide some badly needed insight into the question of the coupling of convection and rotation.

In the event that the star were to evolve through a radiation pressure dominated phase in the shell (Case 3), the effects of rotation would be quite small and almost certainly undetectable. The essential physical reason for this behaviour is that rotation couples much more tightly to the fluid than to the radiation, and when radiation pressure is large the coupling is weak and all effects are reduced.

As we allow the rotation law to deviate from moderate differential motion we can expect to find more substantial changes. An important result is that solid body rotation throughout the core and shell always leads to a drop in the luminosity (as has long been known on the main sequence, *e.g.* Sackmann 1970). Note that the numbers quoted for η = 0·25 in Table 1 are almost certainly too small, since setting $\bar{\chi} = \chi_{max}$ allowed us to take $\tilde{\chi}$ = 0·748, which, although dynamically possible in the shell, clearly vitiates our basic assumption of sphericity. But even if $\tilde{\chi}$ is forced to stay under 0·2, non-negligible drops in luminosity, and therefore significantly lengthened evolution times, are to be expected if the viscosity and other angular momentum redistribution mechanisms are sufficiently rapid to ensure solid body rotation (Durisen 1975). Under these circumstances, for example, the He-core/H-shell structure can allow L_R/L_0 = 0·3. Such a drop in photon flux may significantly change the internal structure of the star by favouring the growth of convection zones in or above the burning shell. Alternatively, large increases in the luminosity are possible, even for Case 1, if extreme differential rotation can be achieved. While this may be possible in stages where free-fall is approached, as in proto-stellar or supernova collapse phases, in the non-dynamical phases we are concentrating on, such a rotation law does not appear very likely (Endal and Sofia 1978). In this case our comparisons at constant central density lead to results that differ from standard main sequence models, for they compare at constant mass, and as mentioned above, this always leads to a drop in luminosity.

3.2 *Neutrino Luminosities*

Starting from the time when core carbon-burning becomes the dominant source of nuclear energy, neutrino emission becomes the dominant mode for removing energy from a star. Thus the calculations of the previous subsection for the last burning phases are seen to be even less accurate than our many approximations would imply, for equation (20) is no longer nearly exact. It should also be recognized that these neutrino losses imply that the assumption of essentially homologous contraction breaks down, which some calculations (*e.g.* Endal and Sofia 1977) neglect to consider.

Now the relevant question is how rotation affects the total neutrino emission. At this point the distinction between the core and the shell is not so important since the neutrino flux is basically due to the ‘ thermal ’ processes (pair, plasma and photo production) in these stages of stellar evolution (Beaudet, Petrosian and Salpeter 1967; Tubbs *et al.* 1981) and comes from the entire central region. Our analytical treatment is not applicable to the very late stages, when densities get above 10^{10}g cm^{-3} or temperatures rise above 10^{10} K; at that point URCA and other non-thermal processes dominate and homology breaks down completely.

At any given temperature and density we can abstract the temperature and density dependences from the total neutrino (plus anti-neutrino) emission rate, and write $Q \propto \rho^{\nu} T^{k}$, where Q is the emissivity in erg cm^{-3} s^{-1}. Equivalently, the rate S, in erg g^{-1} s^{-1} is

$$S \propto \rho^{\nu-1} T^{k}. \tag{39}$$

In order to estimate the effects of rotation we assume first that the bulk of the neutrino-antineutrino pairs come from the core region, as it is the hottest and the densest region, and we reasonably neglect the contributions of the mantle and envelope. Second we use an average temperature and density characteristic of the core (*cf.* Arnett 1973a) to estimate the flux; this is adequate for our proportionality arguments. Third, we scale this temperature as the shell temperature in equation (18); as long as the core is degenerate this scaling is reasonable, although, of course, the actual average core temperature is higher than that in the surrounding zone. It is important to notice that spherical models show only small temperature gradients in these regimes (Arnett 1974b), so even this—the weakest of our assumptions—is a viable one.

We write the ratio of rotating to non-rotating neutrino luminosities as a product

$$\frac{L_R^\nu}{L_0^\nu} = \left(\frac{M_R}{M_0}\right)\left(\frac{S_R}{S_0}\right) = \left(\frac{M_R}{M_0}\right)\left(\frac{\rho_R}{\rho_0}\right)^{v-1}\left(\frac{T_R}{T_0}\right)^k. \tag{40}$$

If we again compare cores of the same average density we still have equation (17), so $M_R/M_0 = (1 - \chi)^{-3/2}$ Case 3 is no longer amenable to our treatment, but for both Cases 1 and 2 we still may take $T \propto M_c\,(1 - \tilde{\chi})$, as this result is independent of equation (20) which no longer holds. When comparing rotating and non-rotating cores of equal density we then obtain

$$\frac{L_R^\nu}{L} \simeq (1 - \bar{\chi})^{-3(k+1)/2}(1 - \tilde{\chi})^k. \tag{41}$$

In Table 2 this ratio is evaluated for several stages of evolution and for several values of η. The basic conclusions turn out to be similar to the case of photon emission, in that differential rotation can increase the neutrino flux, while solid body rotation suppresses it. Because of the similar k exponents from C-burning through O-burning stages, we find that moderate differential rotation tends to raise the neutrino luminosity in that entire region by a factor of about four or five. Again this will accelerate the evolution through these already short-lived phases. Note that the results for solid body rotation are again almost certainly too low (for the reason discussed above) but it is clear that significant drops in L_ν are possible in this case.

Table 2. Rotation and neutrino luminosities.

Core	$\log \rho$	$\log T$	v	k	L_R/L_0 at χ_{max} $\eta = 0{\cdot}25$	$\eta = 1{\cdot}00$	$\eta = 4{\cdot}00$
He	3·0	8·2	3·0	2·4	0·10	1·76	2·59
C	5·7	8·9	0·0	11·5	6·0(−6)	4·51	28·34
Ne	7·2	9·1	−0·2	11·8	4·7(−6)	4·61	30·03
O	7·3	9·2	−0·7	11·7	5·0(−6)	4·59	29·63

Notes:

Average core values for ρ and T are from Arnett (1972b, 1973b, 1974a, b) for $M_\alpha = 4\,M_\odot$ cores. The neutrino emissivities are from Tubbs *et al.* (1981), assuming that the heavy lepton (τ particle) contributes.

3.3 *Gravitational Radiation*

We now turn to a brief discussion of one aspect of a topic that has excited much recent speculation and calculation: the emission of gravitational radiation by collapsing, rotating cores. One approach has been to concentrate on the star and to attempt to use detailed evolutionary models carried till the late stages in order to model a *somewhat* realistic core collapse (Meier *et al.* 1976; Endal and Sofia 1977, 1978). The other route has been to concentrate on the gravitational waves and to use simple equations of state and models for hypothetical collapsing cores that can be more easily followed numerically (Shapiro 1979; Saenz and Shapiro 1979, and references therein). These latter calculations have yielded an upper limit to energy lost via gravitational radiation of $\Delta E \lesssim 10^{-2}\ Mc^2$ in a time $\lesssim 10^{-2}$ if the collapse is cold, but the more likely initial state of a hot, high entropy core (Schramm and Arnett 1975) implies a slower collapse and yet less emission. However Endal and Sofia (1977) claim a minimum energy loss of the order of 2 per cent through gravitational waves.

No new calculations are presented in this section; we merely wish to point out that the above discrepancy can be understood as arising from several unlikely assumptions made by Endal and Sofia. Their work (Endal and Sofia 1976, 1978) on evolving rotating stars through some post-main sequence phases uses an equipotential technique, which, when coupled with fast computers, allows for significant improvements over earlier work (*e.g.* Kippenhahn, Meyer-Hofmeister and Thomas 1970; Sackmann and Weidemann 1972). Their diffusion technique is a major advance in treating secular angular momentum redistribution mechanisms, but their choice of the size of the diffusion constant is a rather arbitrary one, and slight variations will probably have important—and so far unexplored—consequences (*cf.* the mixing length treatment of convection). They, like most previous workers, probably exaggerate the effectiveness of chemical composition barriers (Huppert and Spiegel 1977).

In Endal and Sofia's (1977) estimate of gravitational radiation emission the key assumption is that their last evolved 10 $M_\odot$ model with central density of $1{\cdot}0 \times 10^6$ g cm^{-3} and central temperature of $5{\cdot}8\times10^8$ K, can be homologously extrapolated down to the much higher central density of 1.9×10^9 g cm^{-3}, while the angular momentum distribution of their last computed model remains the same. But it is exactly during these post-carbon burning phases that neutrino losses break down the homology arguments (Arnett 1972b, 1974a, b), and while the nuclear evolutionary timescales are much shorter by this time, some of the redistribution timescales also drop and the second half of the above assumption is also dubious. Using conservation of specific angular momentum they claim to show that non-axisymmetric, fissioning cores develop before the collapse phase (*e.g.* $\rho = 10^{10}$ g cm^{-3}, taken as the initial state in Saenz and Shapiro 1979); the subsequent collapse of such a triaxial or multiple core could lead to efficient ($\Delta E/Mc^2 > 10^{-2}$) emission of gravitational radiation (Miller 1974; Clark and Eardley 1977).

While this scenario may be possible, we think it highly unlikely for the following reasons. First, observed white dwarfs (as well as neutron stars) never exhibit the distorted shapes or extreme rotational velocities expected if the cores do evolve this way (Hardorp 1974). Although the low values of observed angular momentum for neutron stars may reflect a great deal of dissipation during their violent births, no such mechanism appears to be available for white dwarfs, and the Endal and Sofia scenario, unlike others, seems to imply rapidly rotating dwarfs. Second, we do not trust either the assumption of specific angular momentum conservation or that of homologous contraction in this critical phase, as explained above. And it should be noted that the question as to whether critical angular velocities (and thus substantial angular momentum dissipation) or high T/ |W| values (and thus axisymmetric evolution) are reached first, is extremely sensitive to these assumptions. Third, this approach (as well as ours) totally neglects magnetic fields, which may put a substantial break on the contraction and spin up. For even if the values of Leblanc and Wilson (1970) and Meier *et al.* (1976) for the field energy are high, and their assumption of pinned field lines overemphasizes the growth of the fields, the fields probably still have significance at this stage. If strong differential rotation does develop, some amplification of fields, removing energy from the rotational mode, is to be expected, for the work of Weiss (1966) is not applicable to this compressible situation. Finally, even if we were to accept the dubious arguments of Endal and Sofia, we must note that the competition between rotational effects is also very sensitive to the initial conditions, *i.e.* the rotation law found for their last computed model. And because of the assumptions that went into their detailed calculations, a large uncertainty in those initial conditions must exist (see Tassoul 1978, p. 352).

4. Conclusions

Rotation can affect the later stages of stellar evolution in several substantial ways. Using an approximate analytical treatment we have shown that moderate differential rotation tends to increase both photon and neutrino fluxes during the post-main sequence phases. If the burning shells are convective these increases can be quite large. But, if the core and the shell somehow manage to rotate rigidly, substantial decreases in luminosity are likely. This great uncertainty emphasizes the gaps in our current understanding of the problem.

We have also argued that a star evolving through the multiple shell burning stages is unlikely to fission or even become substantially axisymmetric. However, when the star reaches the final pre-supernova phase and the core starts to collapse, rotation may produce dramatic effects. One of these is the coupling of rotation with the emission of neutrinos; Kazanas and Schramm (1979) have recently shown that some fraction of the angular momentum can be carried off in this way. But, as convincingly argued by Shapiro (1979) these effects probably do not include the loss of more than 2 per cent of the rest mass energy in the form of gravitational radiation.

Some progress is being made in extending detailed rotating models beyond the main sequence and helium core burning stages (Endal and Sofia 1976, 1978). But a mere listing of the effects that must be included in such calculations underscores their difficulty. Among them are: convection, Solberg-Hoiland instabilities, dynamical and secular shear instabilities, Eddington currents, μ-currents and barriers, Goldreich-Schubert-Fricke instabilities, photon and neutrino viscosities, and magnetic fields. The coupling of the last with rotation is probably the most important question inadequately treated as yet for this stage of evolution. But basic physical questions regarding the interaction of convection and rotation, the shear instabilities (Zahn 1974), and μ-currents also require more analysis. While most of the other processes appear to be basically understood, their application to evolution has almost always been in terms of drastically simplified approximations.

Future numerical work must strive to clear up many of these problems by either using more sophisticated treatments, or else by demonstrating that the approximations employed are really valid. Until that kind of analysis is performed, we feel that the crude treatment given here is useful, as it serves to illustrate important effects and places bounds on the results expected from more careful numerical calculations.

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