The effect of inertia, viscous damping, temperature and normal stress on chaotic behaviour of the rate and state friction model

NITISH SINHA¹, ARUN K SINGH^{1,*} and TRILOK N SINGH²

¹*Department of Mechanical Engineering, Visvesvaraya National Institute of Technology, Nagpur 440 010, India.*

²*Department of Earth Sciences, Indian Institute of Technology Bombay, Mumbai 400 076, India.*

*Corresponding author. e-mail: aksinghb@gmail.com

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A fundamental understanding of frictional sliding at rock surfaces is of practical importance for nucleation and propagation of earthquakes and rock slope stability. We investigate numerically the effect of different physical parameters such as inertia, viscous damping, temperature and normal stress on the chaotic behaviour of the two state variables rate and state friction (2sRSF) model. In general, a slight variation in any of inertia, viscous damping, temperature and effective normal stress reduces the chaotic behaviour of the sliding system. However, the present study has shown the appearance of chaos for the specific values of normal stress before it disappears again as the normal stress varies further. It is also observed that magnitude of system stiffness at which chaotic motion occurs, is less than the corresponding value of critical stiffness determined by using the linear stability analysis. These results explain the practical observation why chaotic nucleation of an earthquake is a rare phenomenon as reported in literature.

Keywords. Rate and state friction; inertia; viscous damping; temperature; normal stress; chaotic motion of rock sliding.

List of notations and abbreviations

		c	$(1 - \beta_1 q_1)Q_a/RT^* + (1 - \beta_2 q_2)Q_a/RT^*$, $\rho = L_1/L_2$
τ	Dimensional frictional stress (P _a)	K	Stiffness of connecting spring (P _a m ⁻¹)
τ^*	Reference frictional stress (P _a)	k	$KL_1/\sigma_n a$, non-dimensional spring stiffness
σ_n	Effective normal stress (P _a)	r	Ratio of inertial time to frictional characteristic time
μ_*	Reference coefficient of friction	r	$\sqrt{mv_*^2}/\sigma_n a L_1$
θ_1, θ_2	State variable related to asperity contact of the sliding interface	$\hat{\gamma}$	$\gamma v_*/\sigma_n a$
a, b_1, b_2	Constants related to rate and state friction	$\hat{\gamma}$	Nondimensional viscous damping coefficient
Q_a, Q_{b_1}, Q_{b_2}	Activation energies corresponding to a, b_1 and b_2	m	Mass of the sliding block (kg)
c_1	Nondimensional term relates to frictional heating	β_1	b_1/a
c_2	Nondimensional term relates to heat conduction	β_2	b_2/a
		q_1	Q_{b_1}/Q_a

q_2	Q_{b_2}/Q_a
ψ	$\tau/\sigma a$
ϕ	$\ln(v/v_*)$
d_c	Critical slip distance (m)
v_0	Pulling velocity (ms^{-1})
R	Universal gas constant ($\text{J K}^{-1}\text{mol}^{-1}$)
T_*	Reference temperature (K)
T	tv_*/L_1
T_s	Temperature of the sliding interface (K)
$\hat{T}_s$	T_s/T_*

1. Introduction

In recent years, theoretical and experimental studies have established that friction along the earthquake faults plays a critical role in controlling their stability (Brace and Byerlee 1966; Dieterich 1979; Rice and Ruina 1983; Carlson and Langer 1989; Marone 1998). The rate and state friction (RSF) model is widely used to understand the dynamics of the earthquake process, for example, nucleation of earthquake, post-seismic relaxation process, etc. (Rice and Ruina 1983; Singh and Singh 2013; Rice and Tse 1986; Marone 1998). In recent times, the RSF model is also used for studying rock slope stability (Chau 1995; Helmstetter *et al.* 2004; Faillettaz *et al.* 2010; Singh *et al.* 2012). The RSF model is basically an extension of the classical Amontons-Coulombs (ACs) laws (Marone 1998). Based on the friction experiments at rock surfaces, Dieterich (1979), Ruina (1983) and Rice and Ruina (1983) proposed the RSF constitutive laws. Accordingly, frictional stress τ of rock surfaces depends on slip velocity v and the state variable θ for a constant normal stress σ_n , which is generally expressed as:

$$\tau = \sigma_n \left[\mu_* + a \ln \frac{v}{v_*} + b \ln \frac{v_* \theta}{L} \right] \quad (1)$$

where θ is a state variable signifying the history of the sliding surfaces (Ruina 1983). Further, a and b are the constants related to the ‘direct effect’ that is, increase in shear stress with slip velocity, while the ‘evolution effect’ causes gradual decrease in shear resistance of asperity contacts, respectively. The material constant L signifies the characteristic slip distance over which evolution of friction occurs due to direct effect (Ruina 1983). Furthermore, μ_* and v_* are reference frictional shear stress and shear velocity corresponding to the AC laws, respectively. Two basic

laws have originally been proposed for the ‘state variable’ θ namely the Ruina–Rice slip law/slip law $d\theta/dt = -v\theta \ln(v\theta/L)/L$ and the other law is known as Dieterich–Ruina aging law/aging law, that is, $d\theta/dt = 1 - \theta v/L$ (Ruina 1983; Marone 1998; Ranjith and Rice 1999). Under the steady sliding conditions, both friction laws result in the same expression for steady dynamic stress as well as critical stiffness (Ruina 1983; Ranjith and Rice 1999). However, a basic difference between these two laws is that, unlike the slip law, the aging law results in true time of stationary contact (Ruina 1983; Ranjith and Rice 1999). It has also been shown numerically that the RSF with the slip law results in chaos, but the same is not seen with the aging law (Liu 2007; Sinha and Singh 2016a,b). The reason for this contradicting behaviour is not known in literature (Liu 2007). Thus, further study is needed to unfold the mystery behind this issue.

Chaos is defined as ‘Aperiodic long-term behaviour in a deterministic system that exhibits sensitive dependence on initial conditions (SICs)’ (Strogatz 1994). Although chaos is a general manifestation of a non-linear dynamical system, it is not observed in all non-linear systems (Strogatz 1994). A necessary condition for chaos of a continuous dynamical system is that the solutions of the system of differential equations must show the dependency on SICs (Strogatz 1994; Lakshmanan and Rajasekar 2003). In addition, an important condition for the chaos of a such dynamical system is that it must possess at least three independent variables coupled with at least one non-linear term (Strogatz 1994; Lakshmanan and Rajasekar 2003). For instance, although the RSF with one state variable rate and state friction (1sRSF) is often used to explain stiffness dependence of stick-slip instability under both quasistatic and inertial conditions, 1sRSF does not result in chaotic motion (Ruina 1983; Rice and Ruina 1983; Gu *et al.* 1984; Marone 1998). However, the addition of one more state variable in the 1sRSF model results in the chaotic motion and this is known as the two state variables based rate and state dependent friction (2sRSF) model (Ruina 1983; Gu *et al.* 1984). Thus, equation (1) gets modified owing to two state variables θ_1 and θ_2 as following:

$$\tau = \sigma_n \left[\mu_* + a \ln \frac{v}{v_*} + b_1 \ln \frac{v_* \theta_1}{L_1} + b_2 \ln \frac{v_* \theta_2}{L_2} \right]. \quad (2)$$

Similarly, both the state variable laws also get modified as $d\theta_i/dt = -v\theta_i \ln(v\theta_i/L_i)/L_i$ and $d\theta_i/dt = 1 - \theta_i v/L_i$, where $i = 1, 2$. The 2sRSF was proposed to match the step velocity experiments and has also been reported its chaotic behaviour (Ruina 1983). Gu *et al.* (1984) have studied numerically the chaotic behaviour of the 2sRSF model in detail and reported that the 2sRSF with the slip law indeed results in the chaotic motion via period doubling bifurcation. Rice (1993, 1996) has also discussed the complexity involved during an earthquake with the rate and state friction. Gu and Wong (1994) used a slightly modified form of the 2sRSF model to study the chaotic behaviour of sliding rock surfaces at higher sliding velocity and they have also confirmed that the period doubling route to chaos with the experiments. Niu and Chen (1994) have also investigated the period doubling behaviour of the 2sRSF in detail using the different numerical tools for instance, Lyapunov Exponents, fractal dimensions, etc. Becker (2000) concluded through the numerical simulations that irregular behaviour of the 2sRSF can be reduced to one-dimensional unimodal mapping; thus the route of chaos is period doubling. Liu (2007) compared the linear and non-linear behaviour of the 2sRSF with the slip law as well as the aging law and reported the chaotic behaviour with slip law, but not with the aging law. Xuejun (2013) has also carried out linear and non-linear stability analysis of the 2sRSF using the numerical tools such as bifurcation diagram, phase diagram and Lyapunov exponents. Recently, we have also studied the chaotic behaviour of three state variables rate and state dependent friction (3sRSF) and observed that the system becomes more chaotic than the 2sRSF model (Sinha and Singh 2016a, b). Burridge and Knopoff (1967) have proposed a chaotic model for seismicity along the earth fault using the modified multiple spring mass system which is known as Burridge and Knopoff (B-K) model. This promising model has extensively been studied since its proposal. Brittany *et al.* (2011) have used the RSF to study the chaotic behaviour of the B-K model considering both discrete as well as corresponding continuum model. Kawamura *et al.* (2017) have also done the statistical analysis of the B-K model in the light of rate and state friction.

Despite all these numerical studies, it is not known in literature how viscous damping, inertia, normal stress, temperature, etc., related to the RSF affects its chaotic behaviour. In the present article, we have investigated numerically

the effect of aforementioned parameters on the chaotic behaviour of the 2sRSF with the slip law. We have employed the time series and phase plot to discuss the results.

2. Modelling of governing differential equations

We use a standard spring-mass slider having mass m , viscous damping γ and spring stiffness K per unit area of the sliding mass. The free end of the spring is being pulled with a constant velocity V_0 and the other end of spring is connected to the sliding mass with velocity of center of mass v (Gu *et al.* 1984; Ranjith and Rice 1999). The governing equation for the slider is given by

$$K(V_0 - v) - \dot{\tau} - \gamma\dot{v} = m\ddot{v}, \quad (3)$$

and the dimensionless form of equation (3) is obtained

$$\frac{d^2\phi}{dT^2} = r^{-2}e^{-\phi} \left[k(v_0 - e^\phi) - \frac{d\psi}{dT} - \hat{\gamma}e^\phi \frac{d\phi}{dT} \right] - \left(\frac{d\phi}{dT} \right)^2. \quad (4)$$

Further, the non-dimensional form of equation (2) is obtained in terms of equation (5) as:

$$\psi = \phi + \beta_1\hat{\theta}_1 + \beta_2\hat{\theta}_2. \quad (5)$$

Also the non-dimensional form of the slip law for both the state variables is obtained as:

$$\begin{aligned} \dot{\hat{\theta}}_1 &= -e^\phi(\phi + \hat{\theta}_1), \\ \dot{\hat{\theta}}_2 &= -\rho e^\phi(\phi + \hat{\theta}_2). \end{aligned} \quad (6)$$

In order to derive the system of governing differential equation, we have eliminated $\hat{\theta}_1$ and in light of equations (2–6)

$$\left\{ \begin{aligned} \frac{d^2\phi}{dT^2} &= r^{-2}e^{-\phi} [k(v_0 - e^\phi) - \dot{\phi}(1 + \hat{\gamma}e^\phi) \\ &\quad + e^\phi\{(\beta_1 - 1)\phi + \psi - \beta_2\hat{\theta}_2\} \\ &\quad + \rho\beta_2e^\phi(\phi + \hat{\theta}_2)] \\ \frac{d\psi}{dT} &= \dot{\phi} - e^\phi[(\beta_1 - 1)\phi + \psi - \beta_2\hat{\theta}_2] \\ &\quad - \rho\beta_2e^\phi(\phi + \hat{\theta}_2) \\ \frac{d\hat{\theta}_2}{dT} &= -\rho e^\phi(\phi + \hat{\theta}_2) \end{aligned} \right. \quad (7)$$

Figure 1. Phase diagrams concerning ψ vs. ϕ for viscous damping $\hat{\gamma} = 0$ and $\hat{\gamma} = 0.009$ at fixed $\beta_1 = 1.0$, $\beta_2 = 0.84$, $\rho = 0.048$, $k = 0.068843$ and initial condition $[0.005, 0, 0]$.

Figure 2. The phase diagrams concerning the effect of inertia r on chaotic behaviour of the 2sRSF model with inertia ($r = 0.009$) and without inertia ($r = 0$) for fixed value of $\beta_1 = 1.0$, $\beta_2 = 0.84$, $\rho = 0.048$ and $k = 0.068843$ at initial condition $[0.005, 0, 0]$.

In the above derivations from equations (2–7), the dimensionless terms are defined as:

$$\begin{aligned}\psi &= (\tau - \tau^*)/(\sigma_n a), \quad \phi = \ln v/v_*, \\ \hat{\theta}_1 &= \ln(v_* \theta_1/L_1), \quad \hat{\theta}_2 = \ln(v_* \theta_2/L_2), \\ T &= v_* t/L_1, \quad \rho = L_1/L_2, \\ \beta_1 &= b_1/a, \quad \beta_2 = b_2/a, \quad k = K L_1/(\sigma_n a), \\ r^2 &= m v_*^2/(\sigma_n a L_1), \quad \hat{\gamma} = \gamma v_*/(\sigma_n a).\end{aligned}$$

The role of dimensionless inertia and viscous damping are also investigated on chaotic behaviour of the 2sRSF model, where r is the ratio of inertial time to frictional characteristic time, which is generally expressed as $r^2 = m v_*^2/(\sigma_n a L_1)$ and viscous damping $\hat{\gamma} = \gamma v_*/(\sigma_n a)$ (Ranjith and Rice 1999). It is to be noted that one gets the quasi-static condition of sliding upon neglecting the inertial term r and equation (7) reduces to the original 2sRSF model (Niu and Chen 1994, 1995; Becker 2000; Xuejun 2013). The 2sRSF friction also becomes

1sRSF law, if one of the state variables is neglected (Ruina 1983; Rice and Ruina 1983). In the next section, we have simulated the system of ordinary differential equation in equation (7) using *ode 23s* solver of MATLAB[®].

3. Results and discussion

3.1 Effect of viscous damping and inertia on chaotic behaviour of 2sRSF law

The phase diagrams (figure 1) show that chaotic behaviour of the sliding system reduces to periodic oscillation as magnitude of viscous damping increases from $\hat{\gamma} = 0$ to $\hat{\gamma} = 0.009$ for a fixed value of spring stiffness $k = 0.068843$. This result is expected as viscous damping, in general, eliminates the oscillatory behaviour of a vibrating system (Lakshmanan and Rajasekar 2003). Similarly, the phase diagrams (figure 2) also show that even a small change in inertial term r in equation

(7) causes chaotic motion to the period doubling. Further increase in r results in complete elimination of the chaos. This result is again consistent with the Duffing system in which a slight increase in mass eliminates its chaotic behaviour (Lakshmanan and Rajasekar 2003). Gu and Wong (1994) have also pointed out that increase in inertia or mass of the sliding system results in period redoubling of the 2sRSF model.

3.2 Effect of temperature on chaotic behaviour of the 2sRSF model

We have also investigated numerically the effect of temperature on chaotic behaviour of the 2sRSF model. Singh and Singh (2016) studied numerically the linear stability of the rate, state and temperature with one state variable that is, 1sRSTF model. They have followed the Segall and Rice (2006) approach to model an earthquake fault using the spring-mass system. The numerical simulations of the 1sRSTF have established that it is the relative competition between velocity weakening effect and thermal strengthening effect that controls the frictional stability of the sliding surfaces (Singh and Singh 2016). In this paper, we have derived the governing system of differential equations for two state variables based rate, state and temperature friction that is, 2sRSTF. The governing differential equation of the 2sRSTF with the spring-mass slider is derived the following the same procedure as discussed in the modelling section of this paper. The final system of governing differential equation is given by

$$\begin{cases} \frac{d\phi}{dT} = k(v_0 - e^\phi) + e^\phi[(\beta_1 - 1)\phi + \psi - \hat{\mu}_* - \beta_2\hat{\theta}_2 \\ \quad - c(\hat{T}_s^{-1} - 1)] + \rho\beta_2e^\phi(\phi + \hat{\theta}_2) \\ \quad + c\hat{T}_s^{-2}[c_1e^\phi\psi - c_2(\hat{T}_s - \hat{T}_{\text{inf}})] \\ \frac{d\psi}{dT} = k(v_0 - e^\phi) \\ \frac{d\hat{\theta}_2}{dT} = -\rho e^\phi(\phi + \hat{\theta}_2) \\ \frac{d\hat{T}_s}{dT} = c_1e^\phi\psi - c_2(\hat{T}_s - \hat{T}_{\text{inf}}) \end{cases} \quad (8)$$

The non-dimensional terms in equation (8) are defined as $\beta_1 = b_1/a$, $\beta_2 = b_2/a$, $\hat{\mu}_* = \mu_*/a$, $q_1 = Q_{b1}/Q_a$, $q_2 = Q_{b2}/Q_a$ and $c = Q_a(1 - \beta_1q_1)/RT_* + Q_a(1 - \beta_2q_2)/RT_*$, where Q_a , Q_{b1} and Q_{b2} are activation energies corresponding to the physical parameters a , b_1 and b_2 , respectively. Further, c_1 and c_2 are the constants related to heat generation and heat dissipation respectively (Singh and Singh 2016). Equation (8) also includes dimensionless terms namely temperature of the sliding interface $\hat{T}_s$ and surrounding temperature $\hat{T}_{\text{inf}}$. It is important to mention that the 2sRSTF in equation (8) results in the original 2sRSF for $q_1 = (q_{cr} = \beta_1^{-1})$ and $q_2 = (q_{cr} = \beta_2^{-1})$.

The numerical simulations of equation (8) reveal that, in general, a slight change in any parameter results in diminishing the chaotic behaviour of the sliding system. For instance, figure 3 shows that as q_1 increases from $q_1 = 1.00$ (which corresponds to the 2sRSF model) to $q_1 = 1.005$, the chaotic behaviour reduces to the regular motion. The similar observation is also seen in the case of change in q_2 for a fixed $q_1 = 1.00$. The reason

Figure 3. The phase diagrams (ψ vs. ϕ) related to the effect of temperature strengthening parameter q_1 on chaotic behaviour of the 2sRSF system for $q_1 = 1.0 = (q_{cr} = \beta_1^{-1})$ and $q_1 = 1.01$ but fixed values of $q_2 = 1.1905 = (q_{cr} = \beta_2^{-1})$, $\beta_1 = 1.0$, $\beta_2 = 0.84$, $\rho = 0.048$, $c_1 = 1.30 \times 10^{-4}$, $c_2 = 2.4 \times 10^{-5}$, $k = 0.068843$ and initial condition $[0.005, 50, 0, 1]$.

Figure 4. The phase diagrams concerning the chaotic behaviour of the 2sRSF model for varying $\alpha = 0^\circ$, $\alpha = 0.0053^\circ$, $\alpha = 0.0055^\circ$ and $\alpha = 0.0056^\circ$ but fixed $\beta_1 = 1.0$, $\beta_2 = 0.84$, $\rho = 0.048$ and $k = 0.068843$ and initial condition $[0.005, 50, 0, 1]$. Time series plot is also presented for $\alpha = 0.0055^\circ$ for showing stress dependence of sensitive to initial conditions.

for this observation may be attributed to the temperature strengthening effect of q_1 and q_2 , which reduce the velocity weakening effect thereby the chaotic motion as well. Moreover, other parameters related to the 2sRSF such as c_1 and c_2 have also been found to be the reducing effect on the chaotic behaviour. These observations are independent of initial temperature $\hat{T}_s$ of the sliding surfaces that is, $\hat{T}_{\text{inf}} > \hat{T}_s$ or $\hat{T}_{\text{inf}} < \hat{T}_s$. It is also concluded that the stiffness value at which irregular motion

occurs is less than the critical stiffness obtained (Appendix).

3.3 Effect of normal stress on chaotic behaviour of the 2sRSF model

Linker and Dieterich (1992) proposed the role of varying normal stress on frictional dynamics with the RSF model and have also introduced an additional term in the state variable laws to match the

experimental data. In the present study, we have not considered that additional term to avoid further complexity in the system of the differential equation. Dupont and Bapna (1994) studied theoretically the effect of normal stress on frictional stability with one state variable law. Following the approach of these researchers, we have also derived the governing differential equations for varying normal stress and two state variables which may be stated as the two state variables based rate, state and normal stress friction (2sRSNF) model. The non-dimensional form of the system of governing differential equations is derived as following the same procedure as mentioned earlier

$$\begin{cases} \frac{d\phi}{dT} = \hat{\sigma}_n^{-1}k(v_0 - e^\phi) - \hat{\sigma}_n^{-2}k(v_0 - e^\phi)\psi \tan(\alpha) \\ \quad + \rho\beta_2 e^\phi(\phi + \hat{\theta}_2) + e^\phi[(\beta_1 - 1)\phi \\ \quad - \beta_2\hat{\theta}_2 - \hat{\mu}_* + \psi/\hat{\sigma}_n] \\ \frac{d\psi}{dT} = k(v_0 - e^\phi) \\ \frac{d\hat{\theta}_2}{dT} = -\rho e^\phi(\hat{\theta}_2 + \phi) \\ \frac{d\hat{\sigma}_n}{dT} = -k(v_0 - e^\phi)\tan(\alpha) \end{cases} \quad (9)$$

where parameter α is basically angular inclination of the connecting spring from the horizontal in the spring-mass sliding system, that acts also like coupling parameter which converts the normal stress into the shear stress (Linker and Dieterich 1992; Dupont and Bapna 1994). Non-dimensional effective normal stress $\hat{\sigma}_n$ is defined as $\hat{\sigma}_n = \sigma_n/\sigma_*$, where σ_* and σ_n are reference normal stress and fixed normal stress, respectively. While the remaining non-dimensional terms are defined the same as earlier in this article. Noting that similar to equation (8), equation (9) also reduces to the 2sRSF for $\alpha = 0^\circ$.

The numerical simulations of the 2sRSNF model in equation (9) are also presented in figure 4. It is seen that the chaotic tendency of the sliding system decreases as α increases slightly from $\alpha = 0^\circ$ to $\alpha = 0.0053^\circ$. However, there is a sudden appearance of chaotic motion at $\alpha = 0.0055^\circ$ before it disappears again for $\alpha > 0.0055^\circ$. The time series plot in figure 4 confirms the evidence of chaos at $\alpha = 0.0055^\circ$. For example, it is seen in figure 4 that a slight change in magnitude of α from $\alpha = 0.0055^\circ$ to $\alpha = (0.0055 + 10^{-9})^\circ$ results in deviation of stress with time. The possible reason for the appearance of chaos in the 2sRSNF is due to decrease in

adhesive contacts between asperities at the sliding interface. This results in decrease of the effective normal stress and increases in frictional stress for $\alpha > 0$. In the case of negative spring angle, adhesive contact among asperities, in contrast, increases because of increase in effective normal stress thus suppressing any irregular motion or chaos at the sliding surfaces (Tolstoi 1967; Cochard *et al.* 2003). In the case of 2sRSNF, it is again found that critical stiffness ($k_{cr} = 0.080275$) is larger than the stiffness value ($k = 0.068848$) at which chaos occurs. The details of linear stability of the 2sRSNF are also described in Appendix.

Present study has shown that in addition to viscous damping, inertia, temperature and normal stress also diminish the chaotic motion of the sliding surfaces. At the same time, the chaotic motion is seen for very specific values of the friction parameters in case of normal stress. On the basis of these results, it is concluded that the chaotic motion may not be always a cause for the nucleation and propagation of an earthquake. This also explains the practical observation why chaotic nucleation and propagation of an earthquake is a rare phenomenon as reported in literature (Huang and Turcotte 1990; Shelly 2010).

4. Conclusions

The present numerical simulations establish that viscous damping, inertia, temperature and normal stress, in general, eliminate the chaotic behaviour of the sliding system. However, in case of change in the coupling parameter $\alpha = 0.0053^\circ$ to 0.0056° related to the normal stress, results in appearance of the chaotic motion for $\alpha = 0.0055^\circ$. The critical stiffness k_{cr} obtained with the linear stability analysis of the 2sRSF model is $k_{cr} = 0.088$ which is larger in magnitude than the stiffness $k = 0.068843$ at which chaotic motion occurs. It is also concluded that why chaotic motion of the sliding surfaces is a rare phenomenon in crustal earthquakes and failure of the rock slopes. Since most of the parameters of the RSF results in reducing the chaotic effect of the sliding system.

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Appendix

A1. Linear stability of 2sRSTF model

In this section, we have carried out linear stability of the 2sRSTF model for predicting the critical stiffness at which sliding behaviour changes from stable to unstable. The system of differential equations in equation (8) is linearized about the equilibrium point or fixed point. This is obtained by equating the time derivatives in equation (8) equal to zero and solving the system of algebraic equations numerically. Moreover, the corresponding Jacobian matrix of equation (8) is obtained as:

$$J_0 = \begin{bmatrix} -ke^{\phi_{ss}} + (\beta_1 - 1)e^{\phi_{ss}} + \rho\beta_2 e^{\phi_{ss}} + cc_{11}e^{\phi_{ss}}\psi_{ss}\hat{T}_{ss}^2, & e^{\phi_{ss}} + cc_{11}e^{\phi_{ss}}\hat{T}_{ss}^2, & \dots \\ -\beta_2 e^{\phi_{ss}} + \rho\beta_2 e^{\phi_{ss}}, & ce^{\phi_{ss}}\hat{T}_{ss}^2 - cc_{22}\hat{T}_{ss}^2 & \\ -ke^{\phi_{ss}}, & 0, & 0, \\ -\rho e^{\phi_{ss}}, & 0, & -\rho e^{\phi_{ss}}, \\ c_1 e^{\phi_{ss}}\psi_{ss}, & c_1 e^{\phi_{ss}}, & 0, \\ & & & -c_2 \end{bmatrix}_{(\phi_{ss}, \psi_{ss}, \hat{T}_{ss})}$$

The characteristic polynomial is obtained by expanding the above Jacobian matrix and this is given by $s_0\lambda^4 + s_1\lambda^3 + s_2\lambda^2 + s_3\lambda + s_4 = 0$, where the co-efficient of the polynomial are expressed as:

$$s_0 = 16,$$

$$s_1 = 8 \left(c_2 + e^\phi - \beta_1 e^\phi + ke^\phi + \rho e^\phi - \rho\beta_2 e^\phi - cc_1 e^\phi \psi / \hat{T}_s^2 \right),$$

$$s_2 = 4 \left(c_2 e^\phi - \beta_1 c_2 e^\phi + c_2 k e^\phi + k e^{2\phi} + c_2 \rho e^\phi - \beta_2 c_2 \rho e^\phi + \rho e^{2\phi} - \rho\beta_1 e^\phi - \rho\beta_2 e^{2\phi} + \rho k e^{2\phi} + cc_1 k e^{2\phi} / \hat{T}_s^2 \dots - cc_1 e^{2\phi} \psi / \hat{T}_s^2 - cc_1 \rho e^{2\phi} \psi / \hat{T}_s^2 \right),$$

$$s_3 = 2 \left(c_2 k e^{2\phi} + c_2 \rho e^{2\phi} - \beta_1 c_2 \rho e^{2\phi} - \beta_2 c_2 \rho e^{2\phi} + c_2 \rho k e^{2\phi} + \rho k e^{3\phi} + cc_1 k e^{3\phi} / \hat{T}_s^2 + cc_1 \rho k e^{3\phi} / \hat{T}_s^2 \dots - cc_1 \rho e^{3\phi} \psi / \hat{T}_s^2 \right)$$

$$s_4 = \left(c_2 k \rho e^{3\phi} + cc_1 k \rho e^{4\phi} / \hat{T}_s^2 \right),$$

The Routh–Hurwitz criterion is applied on above characteristic polynomial to obtain the governing algebraic equations for stability, namely, $s_1 s_2 -$

$s_0 s_3 = 0$ and $s_1 s_2 s_3 - s_0 s_3^2 - s_4 s_1^2 = 0$. These two coupled non-linear algebraic equations are, in turn, solved numerically for critical stiffness.

A2. Linear stability of 2sRSNF model

In this section, we have also carried out linear stability of the 2sRSNF model for predicting the critical stiffness of the system. We followed the same procedure for evaluating critical stiffness of the 2sRSTF friction model as is done in the case of 2sRSTF. The system of equilibrium point of equation (9) is obtained about the steady sliding as $\phi_{ss} = e^{v_0}$, $\hat{\theta}_{2ss} = -\phi_{ss}$, $\hat{\sigma}_{nss} = 1$ and $\psi_{ss} = \hat{\sigma}_{nss}[\hat{\mu}_* + (1 - \beta_1 - \beta_2)\phi_{ss}]$. This is now used to obtain the Jacobian matrix as following:

$$J_0 = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ J_{21} & J_{22} & J_{23} & J_{24} \\ J_{31} & J_{32} & J_{33} & J_{34} \\ J_{41} & J_{42} & J_{43} & J_{44} \end{bmatrix}_{(\phi_{ss}, \psi_{ss}, \hat{\theta}_{2ss}, \hat{\sigma}_{nss})}$$

where abbreviations are defined as:

$$J_{11} = -ke^{\phi_{ss}} / \hat{\sigma}_{nss} - ke^{\phi_{ss}} \psi_{ss} \tan \alpha / \hat{\sigma}_{nss}^2 + \rho\beta_2 e^{\phi_{ss}} + (\beta_1 - 1)e^{\phi_{ss}},$$

$$J_{12} = e^{\phi_{ss}} / \hat{\sigma}_{nss}, \quad J_{13} = (\rho - 1)\beta_2 e^{\phi_{ss}},$$

$$J_{14} = -e^{\phi_{ss}} \psi_{ss} / \hat{\sigma}_{nss}^2,$$

$$J_{21} = -ke^{\phi_{ss}}, \quad J_{22} = 0,$$

$$J_{23} = 0, \quad J_{24} = 0$$

$$J_{31} = -\rho e^{\phi_{ss}}, \quad J_{32} = 0,$$

$$J_{33} = -\rho e^{\phi_{ss}}, \quad J_{34} = 0$$

$$J_{41} = ke^{\phi_{ss}} \tan \alpha, \quad J_{42} = 0,$$

$$J_{43} = 0, \quad J_{44} = 0.$$

The characteristic equation is given $|J_0 - \lambda I| = 0$, where λ and I are eigen value and identity matrix, respectively. The characteristic equation is expanded for getting the characteristic polynomial as $a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$. The co-efficient of the polynomial is defined as the following

$$a_0 = 16,$$

$$\begin{aligned}
a_1 &= 8 \left(e^\phi - \beta_1 e^\phi + k e^\phi / \hat{\sigma}_n + \rho e^\phi \right. \\
&\quad \left. - \beta_2 \rho e^\phi + \psi e^\phi k \tan \alpha / \hat{\sigma}_n^2 \right), \\
a_2 &= 4 \left(k e^{2\phi} / \hat{\sigma}_n + \rho e^{2\phi} - \beta_1 \rho e^{2\phi} \right. \\
&\quad \left. - \beta_2 \rho e^{2\phi} + k \rho e^{2\phi} / \hat{\sigma}_n + k \psi e^{2\phi} \tan \alpha / \hat{\sigma}_n^2 \right. \\
&\quad \left. + k \psi \rho e^{2\phi} \tan \alpha / \hat{\sigma}_n^2 \right), \\
a_3 &= 2 \left(k \rho e^{3\phi} / \hat{\sigma}_n + k \psi \rho e^{3\phi} \tan \alpha / \hat{\sigma}_n^2 \right), \\
a_4 &= 0.
\end{aligned}$$

Finally, the Routh–Hurwitz criterion is used to obtain critical stiffness k_{cr} at which sliding behaviour of the system changes from stable to unstable or vice versa. The Routh and Hurwitz criterion leads to the following system of coupled algebraic equations as $a_1 a_2 - a_0 a_3 = 0$ and $a_1 a_2 a_3 - a_0 a_3^2 - a_4 a_1^2 = 0$. These equations are, in turn, solved numerically for critical stiffness k_{cr} of the sliding system.

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