Analytical solutions for one-dimensional advection–dispersion equation of the pollutant concentration

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We present simple analytical solutions for the unsteady advection–dispersion equations describing the pollutant concentration $C(x, t)$ in one dimension. The solutions are obtained by using Laplace transformation technique. In this study we divided the river into two regions $x \leq 0$ and $x \geq 0$ and the origin at $x = 0$. The variation of $C(x, t)$ with the time t from $t = 0$ up to $t \to \infty$ (the steady state case) is taken into account in our study. The special case for which the dispersion coefficient $D = 0$ is studied in detail. The parameters controlling the pollutant concentration along the river are determined.

1. Introduction

It is well known in real situations, rivers are polluted by various kinds of pollutants coming from household and industrial sources (Shukla *et al.* 2008). Pollution of rivers has become a matter of concern for scientists working in environmental engineering, hydrology, chemical engineering, geology, soil physics, and mathematics. Analytical solutions of the mathematical models describing pollutant transport are rarely possible if important hydraulic and chemical processes are considered together (Massabo *et al.* 2006). Mathematical models have been used extensively to predict water quality and to provide reliable tools for water quality management in affected areas.

The particular river whose water quality was the motivation for this study is the Tha Chin River in Thailand (Pimpunchat *et al.* 2007). It is assumed that the pollutants are largely biological wastes, which undergo various biochemical and biodegradation processes using dissolved oxygen (Pimpunchat *et al.* 2007). Analytical solutions for one-dimensional transport in composite media are often derived with Laplace transforms (Carslaw and Jaeger 1959) and sometimes with Green's functions. Adjoin solution techniques and finite integral transform solutions are obtained by Mikhailov and Ozisik (1984) and Leij and Van Genuchten (1995). Van Genuchten and Alves (1982) and Ge and Lu (1996) provided analytical solutions for a physical system with zero initial concentration in a semi-infinite domain.

The explicit finite difference method is found to be effective and accurate for solving the one and two-dimensional advection–dispersion equations with variable coefficients in semi-infinite media. Also it can be used with arbitrary initial and boundary conditions, as well as with different variations of dispersion and velocity, for which analytical solutions are not available (Djordjevich and Savovic 2013; Savovic and Djordjevich 2012, 2013; Savovic and Caldwell 2003, 2009).

The objective of this study is to develop analytical solutions of one-dimensional unsteady

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state advection dispersion flow equation by using the method of Laplace transforms. Our results generalize the earlier solutions obtained by Pimpunchat *et al.* (2007), which form a subset of our solutions, for the limited case when $t \to \infty$.

2. Formulation of the problem

Consider the unsteady flow in a river as being one-dimensional characterized by a single spatial distance x (m) measured from the source of the river. The water pollution or the concentration of the pollutant $C(x,t)$ is assumed to vary with time t (days) along the length of the river and is treated as homogeneous across the river cross section (Dobbins 1964; Pimpunchat *et al.* 2009).

The river has been divided into two regions: upstream $x \le 0$ near the source, where it is assumed that the rate of pollutant addition along the river q (kg/m·day) vanishes, and downstream $0 \le x < L$ (m) (the polluted length of the river), where $q=$ const. The equations governing one-dimensional advective–dispersive transport can be written as (Pimpunchat *et al.* 2009):

$$\frac{\partial (AC_1)}{\partial t} = D\frac{\partial^2 (AC_1)}{\partial x^2} - \frac{\partial (vAC_1)}{\partial x} - k_1 \frac{X}{X+k} AC_1, \quad x \le 0,\ t > 0, \tag{1}$$

$$\frac{\partial (AC_2)}{\partial t} = D\frac{\partial^2 (AC_2)}{\partial x^2} - \frac{\partial (vAC_2)}{\partial x} - k_1 \frac{X}{X+k} AC_2 + q, \quad 0 \le x < L \le \infty,\ t > 0, \tag{2}$$

where C_1 and C_2 are the concentrations of the pollutant in the two regions respectively ($\mathrm{kg \cdot m^{-3}}$), A is the cross section area of the river ($\mathrm{m^2}$), D is the dispersion coefficient of pollutant in x direction ($\mathrm{m^2 \cdot day^{-1}}$), v is the water velocity in the x direction ($\mathrm{m \cdot day^{-1}}$), k_1 is the degradation rate coefficient for pollutant ($\mathrm{day^{-1}}$), k is the half-saturated oxygen demand concentration for pollutant decay ($\mathrm{kg \cdot m^{-3}}$), *X(x, t)* is the concentration of the dissolved oxygen within the river ($\mathrm{kg \cdot m^{-3}}$). For convenience, the stream reach is considered to be a homogeneous system, so that all parameters such as A, D, v, k_1, q hold constant values over time and space (Li 2006). In the general case when the half-saturated oxygen demand concentration for pollutant decay $k \neq 0$, it will be impossible to use Laplace transform to suggest an exact solution. In this paper by taking $k = 0$, we can apply the Laplace transformation and obtain the solution. The initial and boundary conditions associated with equations (1 and 2) are (Pimpunchat *et al.* 2009):

$$C_1(x,0) = C_2(x,0) = 0, \tag{3}$$

$$C_1(0,t) = C_2(0,t), \quad t > 0, \tag{4}$$

$$\frac{dC_1(0,t)}{dx} = \frac{dC_2(0,t)}{dx}, \quad t > 0. \tag{5}$$

3. The analytical solution

Laplace transformation technique is defined by equation (6), and is used to get the analytical solutions. The Laplace transformation may be defined as:

If $f(x,t)$ is any function defined in $a \le x \le b$ and $t > 0$, then its Laplace transform with respect to t is denoted by:

$$L\{f(x,t)\} = F(x,p) = \int_0^\infty e^{-pt} f(x,t)\,dt, \quad p \succ 0 \tag{6}$$

where p is called the transform variable (Doetsch 1970; Kumar *et al.* 2011). The inverse Laplace transformation is denoted by $L^{-1}\{F(x,p)\} = f(x \cdot t)$ and defined by the complex variable:

$$L^{-1}\{F(x,p)\} = f(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(x,p) e^{-pt}\,dp \quad c \succ 0. \tag{6.1}$$

Applying Laplace transformation to equations (1 and 2) gives:

$$D\frac{d^2 \tilde{C}_1(x,p)}{dx^2} - v\frac{d\tilde{C}_1(x,p)}{dx} + C_1(x,0) - (k_1+p)\tilde{C}_1(x,p) = 0, \quad x \le 0,\ p > 0, \tag{7}$$

$$D\frac{d^2 \tilde{C}_2(x,p)}{dx^2} - v\frac{d\tilde{C}_2(x,p)}{dx} + C_2(x,0) - (k_1+p)\tilde{C}_2(x,p) + \frac{q}{Ap} = 0, \quad x \ge 0,\ p > 0. \tag{8}$$

The boundary conditions (4 and 5) in the Laplace domain are:

$$\tilde{C}_1(0,p) = \tilde{C}_2(0,p), \tag{9}$$

$$\frac{d\tilde{C}_1(0,p)}{dx} = \frac{d\tilde{C}_2(0,p)}{dx}. \tag{10}$$

By using equation (3), the solutions of equations (7 and 8) are given by:

$$\widetilde{C}_1(x,p)=\frac{D}{(k_1+p)}.\alpha_1\exp\left[x\left(\delta+\sqrt{(\alpha+p)/D}\right)\right]+\frac{D}{(k_1+p)}.\alpha_2\exp\left[x\left(\delta-\sqrt{(\alpha+p)/D}\right)\right], \tag{10.a}$$

$$\widetilde{C}_2(x,p)=\frac{q}{Ap(k_1+p)}+\frac{D}{(k_1+p)}.\alpha_3\exp\left[x\left(\delta+\sqrt{(\alpha+p)/D}\right)\right]+\frac{D}{(k_1+p)}.\alpha_4\exp\left[x\left(\delta-\sqrt{(\alpha+p)/D}\right)\right], \tag{10.b}$$

where $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4$ are constants. To determine the constants $\alpha_{1,}\alpha_2,\alpha_3,\alpha_4$, we use the boundary conditions (9) and (10). Hence, equations (10.a) and (10.b) are given by:

$$\widetilde{C}_1(x,p)=\frac{q}{Ap(k_1+p)}\left(\frac{\sqrt{(\alpha+p)/D}-\delta}{2\sqrt{(\alpha+p)/D}}\right)\times\left\{\exp\left[x\left(\delta+\sqrt{(\alpha+p)D}\right)\right]\right\},\quad x\le 0, \tag{11}$$

$$\widetilde{C}_2(x,p)=\frac{q}{Ap(k_1+p)}-\frac{q}{Ap(k_1+p)}\left(\frac{\delta+\sqrt{(\alpha+p)/D}}{2\sqrt{(\alpha+p)/D}}\right)\times\left\{\exp\left[x\left(\delta-\sqrt{(\alpha+p)/D}\right)\right]\right\},\quad x\ge 0, \tag{12}$$

where

$$\alpha=D\beta^2,\qquad \delta=\frac{v}{2D},\qquad \beta=\sqrt{\delta^2+\frac{k_1}{D}}. \tag{13}$$

The inverse of equations (11 and 12) have been obtained by taking the inverse of each of them using the inverse Laplace transformation and applying the convolution theorem and shift theorem (Roberts and Kaufman 1969). Thus, $C_1(x,t)$ and $C_2(x,t)$ are given by:

$$\begin{aligned}C_1(x,t)&=\frac{q}{4k_1A}\{\exp[(\delta-\beta)x]\}\operatorname{erfc}\left(\frac{-x}{2\sqrt{Dt}}+\sqrt{\alpha t}\right)\\&+\frac{q}{4k_1A}\{\exp[(\delta+\beta)x]\}\operatorname{erfc}\left(\frac{-x}{2\sqrt{Dt}}-\sqrt{\alpha t}\right)\\&-\frac{q}{2k_1A}\{\exp[-k_1t]\}\operatorname{erfc}\left(\frac{-x}{2\sqrt{Dt}}+\sqrt{(\alpha-k_1)t}\right)\\&-\frac{q}{4k_1A}\frac{\delta}{\beta}\{\exp[(\delta+\beta)x]\}\operatorname{erfc}\left(\frac{-x}{2\sqrt{Dt}}-\sqrt{\alpha t}\right)\\&+\frac{q}{4k_1A}\frac{\delta}{\beta}\{\exp[(\delta-\beta)x]\}\operatorname{erfc}\left(\frac{-x}{2\sqrt{Dt}}+\sqrt{\alpha t}\right),\\&x\le 0,\end{aligned} \tag{14}$$

$$\begin{aligned}C_2(x,t)&=\frac{q}{k_1A}-\frac{q}{k_1A}\{\exp[-k_1t]\}\\&-\frac{q}{4k_1A}\frac{\delta}{\beta}\{\exp[(\delta-\beta)x]\}\operatorname{erfc}\left(\frac{x}{2\sqrt{Dt}}-\sqrt{\alpha t}\right)\\&+\frac{q}{4k_1A}\frac{\delta}{\beta}\{\exp[(\delta+\beta)x]\}\operatorname{erfc}\left(\frac{x}{2\sqrt{Dt}}+\sqrt{\alpha t}\right)\\&+\frac{q}{2k_1A}\{\exp[-k_1t]\}\operatorname{erfc}\left(\frac{x}{2\sqrt{Dt}}-\sqrt{(\alpha-k_1)t}\right)\\&-\frac{q}{4k_1A}\{\exp[(\delta+\beta)x]\}\operatorname{erfc}\left(\frac{x}{2\sqrt{Dt}}+\sqrt{\alpha t}\right)\\&-\frac{q}{4k_1A}\{\exp[(\delta-\beta)x]\}\operatorname{erfc}\left(\frac{x}{2\sqrt{Dt}}-\sqrt{\alpha t}\right),\\&x\ge 0.\end{aligned} \tag{15}$$

It is generally more convenient to work with models written in dimensionless variables, by employing the following definitions:

$$t^*=k_1t,\quad x^*=\frac{k_1x}{v},\quad C_1^*=\frac{C_1}{(q/k_1A)},\quad {C_2}^*=\frac{C_2}{(q/k_1A)},\quad D^*=\frac{k_1D}{v^2},\quad \delta^*=\delta\left(\frac{v}{k_1}\right),\quad \beta^*=\beta\left(\frac{v}{k_1}\right),$$

where $(*)$ denotes dimensionless quantity. Hence, equations (14) and (15) take the form:

$$\begin{aligned}C_1^*(x^*,t^*)&=\frac{1}{4}\{\exp[(\delta^*-\beta^*)x^*]\}\operatorname{erfc}\left(\frac{-x^*+2D^*\beta^*t^*}{2\sqrt{D^*t^*}}\right)\\&+\frac{1}{4}\{\exp[(\delta^*+\beta^*)x^*]\}\operatorname{erfc}\left(\frac{-x^*-2D^*\beta^*t^*}{2\sqrt{D^*t^*}}\right)\\&-\frac{1}{2}\{\exp[-t^*]\}\operatorname{erfc}\left(\frac{-x^*+2D^*\delta^*t^*}{2\sqrt{D^*t^*}}\right)\\&-\frac{\delta^*}{4\beta^*}\{\exp[(\delta^*+\beta^*)x^*]\}\operatorname{erfc}\left(\frac{-x^*-2D^*\beta^*t^*}{2\sqrt{D^*t^*}}\right)\\&+\frac{\delta^*}{4\beta^*}\{\exp[(\delta^*-\beta^*)x^*]\}\operatorname{erfc}\left(\frac{-x^*+2D^*\beta^*t^*}{2\sqrt{D^*t^*}}\right),\\&x^*\le 0,\end{aligned} \tag{16}$$

$$
\begin{aligned}
&C_2^*(x^*,t^*)\\
&=1-\exp[-t^*]\\
&\quad-\frac{\delta^*}{4\beta^*}\{\exp[(\delta^*-\beta^*)x^*]\}\mathrm{erfc}\left(\frac{x^*-2D^*\beta^*t^*}{2\sqrt{D^*t^*}}\right)\\
&\quad+\frac{\delta^*}{4\beta^*}\{\exp[(\delta^*+\beta^*)x^*]\}\mathrm{erfc}\left(\frac{x^*+2D^*\beta^*t^*}{2\sqrt{D^*t^*}}\right)\\
&\quad+\frac{1}{2}\{\exp[-t^*]\}\mathrm{erfc}\left(\frac{x^*-2D^*\delta^*t^*}{2\sqrt{D^*t^*}}\right)\\
&\quad-\frac{1}{4}\{\exp[(\delta^*+\beta^*)x^*]\}\mathrm{erfc}\left(\frac{x^*+2D^*\beta^*t^*}{2\sqrt{D^*t^*}}\right)\\
&\quad-\frac{1}{4}\{\exp[(\delta^*-\beta^*)x^*]\}\mathrm{erfc}\left(\frac{x^*-2D^*\beta^*t^*}{2\sqrt{D^*t^*}}\right),\\
&\quad x^*\geq 0. \qquad (17)
\end{aligned}
$$

From equations (16 and 17), it is clear that both C_1^* and C_2^* depend on only two parameters q/k_1A and δ^*, where

$$\beta^*=\sqrt{\delta^{*^2}+2\delta^*}, \qquad D^*=\frac{1}{2\delta^*}.$$

3.1 *Special cases*

The steady state solution is obtained from equations (14) and (15) by taking the limit $t\to\infty$. Hence, in this case $C_1(x)$ and $C_2(x)$ take the form:

$$C_1(x)=\frac{q}{k_1A}\left(\frac{\beta-\delta}{2\beta}\right)\{\exp[(\delta+\beta)x]\}, \quad x\leq 0, \qquad (18)$$

$$C_2(x)=\frac{q}{k_1A}\left(1-\left(\frac{\delta+\beta}{2\beta}\right)\right)\{\exp[(\delta-\beta)x]\}, \quad x\geq 0, \qquad (19)$$

equations (18 and 19) are the same as those obtained by Pimpunchat *et al.* (2009), (equation 3.7).

3.2 *Model including time and zero dispersion*

In practice, for pollutants such as insoluble carbon sized much larger than 0.45 μm and with an anomalous shape indicating that the radius of the diffusing pollutant $r\gg 0$, then D may be approximately zero (Angelakis and Rolston 1985), for this case equation (2) reduces to:

$$\frac{\partial(AC_3)}{\partial t}=-\frac{\partial(vAC_3)}{\partial x}-k_1AC_3+q, \quad x\geq 0, \qquad (20)$$

the initial and boundary conditions are:

$$C_3(x,0)=p_1, \quad x\geq 0, \qquad (21)$$

$$C_3(0,t)=p_2, \quad t>0, \qquad (22)$$

where $C_3(x,t)$ is the pollutant concentration for the special case when the dispersion coefficient $D=0$, p_1 and p_2 are the initial rate of pollution (kg m^{-3}) along the river and the rate of pollution (kg m^{-3}) at the origin respectively. For this case, there is no pollution in upstream for the absence of dispersion, i.e., ($C_3(x,t)=0$ for $x<0$). We will use Laplace transform technique to solve equation (20) subject to initial condition (21) and boundary condition (22). Applying Laplace transformation to governing equation (20) and incorporating the boundary condition (22) gives:

$$p\,\tilde{C}_3(x,p)-C_3(x,0)=-v\frac{\partial\,\tilde{C}_3(x,p)}{\partial x}-k_1\,\tilde{C}_3(x,p)+\frac{q}{Ap}, \quad p>0, \qquad (23)$$

$$\tilde{C}_3(0,p)=\frac{p_2}{p}. \qquad (24)$$

By using equation (21), equation (23) can be written as:

$$\frac{\partial\,\tilde{C}_3(x,p)}{\partial x}+\left(\frac{k_1+p}{v}\right)\tilde{C}_3(x,p)=\frac{1}{v}\left(p_1+\frac{q}{Ap}\right), \quad p>0, \qquad (25)$$

where p is the Laplace transform variable. The general solution of equation (25) is:

$$\tilde{C}_3(x,p)=\left(\frac{q}{Ap}+p_1\right)\frac{1}{(k_1+p)}+\alpha_5\left\{\exp\left[-\left(\frac{k_1+p}{v}\right)x\right]\right\}, \qquad (26)$$

where α_5 is arbitrary constant. Applying the condition (24) to equation (26), we get:

$$
\begin{aligned}
\tilde{C}_3(x,p)&=\frac{q}{Ap(k_1+p)}+\frac{p_1}{k_1+p}\\
&\quad-\frac{q}{Ap(k_1+p)}\left\{\exp\left[-\left(\frac{k_1+p}{v}\right)x\right]\right\}\\
&\quad-\frac{p_1}{k_1+p}\left\{\exp\left[-\left(\frac{k_1+p}{v}\right)x\right]\right\}\\
&\quad+\frac{p_2}{p}\left\{\exp\left[-\left(\frac{k_1+p}{v}\right)x\right]\right\}. \qquad (27)
\end{aligned}
$$

The inverse of Laplace transform of equation (27) is:

$$
\begin{aligned}
C_3(x,t)&=\frac{q}{A}\left(\frac{1}{k_1}-\frac{1}{k_1}\{\exp[-(k_1t)]\}\right)\\
&\quad+p_1\{\exp[-(k_1t)]\}
\end{aligned}
$$

$$-\frac{q}{A}\left(\frac{1}{k_1}-\frac{1}{k_1}\{\exp[-(k_1 t)]\}\right)$$
$$\otimes\left(\exp\left[-\left(\frac{k_1}{v}\right)x\right]H\left(t-\frac{x}{v}\right)\right)$$
$$-p_1\left\{\exp\left[-\left(\left(\frac{k_1}{v}\right)x+k_1 t\right)\right]H\left(t-\frac{x}{v}\right)\right\}$$
$$+p_2\left\{\exp\left[-\left(\frac{k_1}{v}\right)x\right]H\left(t-\frac{x}{v}\right)\right\}, \qquad (28)$$

where $H(t-x/v)$ is Heaviside function, defined by:

$$H\left(t-\frac{x}{v}\right)=1 \quad \text{if } t>\frac{x}{v},$$
$$0 \quad \text{if } t<\frac{x}{v}, \qquad (29)$$

and $\otimes$ denotes the multiplication operation in the convolution theorem (Sudicky and Neville 2008). By using the convolution theorem equation (28) gives:

$$C_3(x,t)=\frac{q}{k_1 A}-\frac{q}{k_1 A}\{\exp[-(k_1 t)]\}$$
$$+p_1\{\exp[-(k_1 t)]\}-\frac{q}{k_1 A}\left\{\exp\left[-\left(\frac{k_1}{v}\right)x\right]\right\}$$
$$+\frac{q}{k_1 A}\left\{\exp\left[-\left(\frac{k_1}{v}x+k_1 t\right)\right]\right\}$$
$$-p_1\left\{\exp\left[-\left(\frac{k_1}{v}x+k_1 t\right)\right]\right\}$$
$$+p_2\left\{\exp\left[-\left(\frac{k_1}{v}\right)x\right]\right\}, \quad t>\frac{x}{v}. \qquad (30)$$

To write equation (30) in dimensionless form we use the following dimensionless variables:

$$x^*=\frac{k_1}{v}x, \quad t^*=k_1 t, \quad C_3^*(x^*,t^*)=\frac{C_3(x,t)}{(q/k_1 A)},$$
$$p_1^*=\frac{p_1}{(q/k_1 A)}, \quad p_2^*=\frac{p_2}{(q/k_1 A)}, \qquad (31)$$

where (v/k_1) scale for length and $q/k_1 A$ scale for concentration. Hence, equation (30) becomes:

$$C_3^*(x^*,t^*)=1-\exp[-t^*]+p_1^*\exp[-t^*]$$
$$-\exp[-x^*]+\exp[-(x^*+t^*)]$$
$$-p_1^*\{\exp[-(x^*+t^*)]\}$$
$$+p_2^*\{\exp[-x^*]\}. \qquad (32)$$

From equation (32), it is obvious that $C_3^*(x^*, t^*)$ depends on two dimensionless parameters p_1^* and p_2^* only.

3.3 *Steady-state case*

Equation (30) for the steady state when $t\to\infty$, gives:

$$C_3(x)=\frac{q}{k_1 A}-\frac{q}{k_1 A}\left\{\exp\left[-\left(\frac{k_1}{v}\right)x\right]\right\}$$
$$+p_2\left\{\exp\left[-\left(\frac{k_1}{v}\right)x\right]\right\}. \qquad (33)$$

For the special case, when $p_2=0$, equation (33) will be the same as that obtained by Pimpunchat *et al.* (2009).

4. Results and discussions

We present simple analytical solutions for the unsteady advection dispersion equations describing the pollutant concentration $C(x, t)$ in one dimension using Laplace transformation method. From solutions we obtained variation of C_1, C_2, and C_3 with time and space. The time is given in days and the values of the concentration of pollutant are measured in kg m^{-3}. In general, the pollutant concentration is given by equations (14, 15 and 30).

The analytical solutions obtained previously by Pimpunchat *et al.* (2007) ignored the solution in the important interval of the time t, where $0\leq t<\infty$. The importance of the solution in this period is due to the fact that the remediation by aeration occurs in this period of time t, before the pollutant concentration reaches its maximum value as $t\to\infty$. The main achievement of our paper is that we obtained the pollutant concentration at any instant of the time t in the period $0\leq t\leq\infty$.

Figure 1 shows the variation of (C_1, C_2) in the range $-10\leq x\leq 20$ (m) with the time t for the

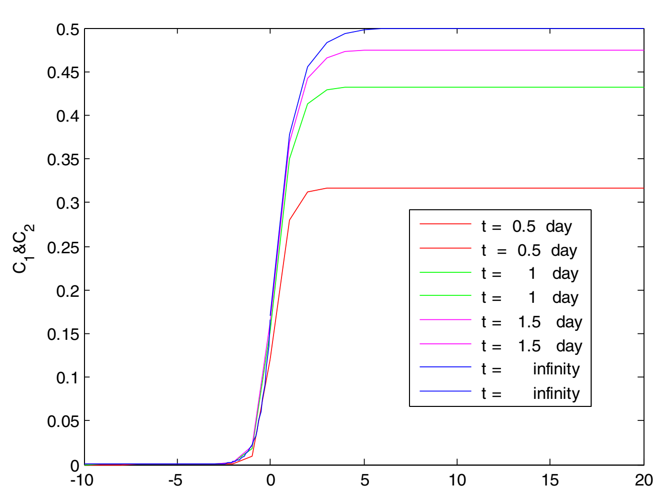

Figure 1. Shows the analytical unsteady state solution with dispersion for $C_1(x,t)$ and $C_2(x,t)$ at different time described by equations (14, 15).

case when $D \neq 0$, described by equations (14) and (15) for $t = 0.5, 1.0, 1.5$ (days) and $t \to \infty$. To test our model, the parameters A, v, q and D are taken to be equal to 1 and k_1 is equal to 2 (Pimpunchat *et al.* 2007). From figure 1, it is clear that:

For $x \leq 0$, the time t has very small effect on C_1. For $x \geq 0$ as t increases C_2 increases and reaches its maximum value as $t \to \infty$.

The values of C_1 and C_2 at the steady state as $t \to \infty$, coincide with that obtained by Pimpunchat *et al.* (2007).

In general, the concentration of pollutant increases as x increases from upstream to downstream. This result is in good agreement with that reported by Kumar *et al.* (2009) and Mourad *et al.* (2013).

Figure 2 shows the variation of $C_1^*(x^*, t^*)$ and $C_2^*(x^*, t^*)$ given by equations (16), (17) with δ^* where $\delta^* = 0.1$, 0.5 and 1 and $t^* = 3$ ($k_1 = 2$ day^{-1}, $t = 1.5$ day). From figure 2, it is clear that:

For $x > 0$ as δ^* decreases the values of C_2^* decreases. Hence, to obtain minimum values for C_2^* the values of D or k_1 must increase. Opposite effects are noticed for the region $x < 0$.

The pollutant concentration in zero dispersion case is given by equation (30). This means that the pollutant concentration is caused only by pure convection and rate of pollutant addition along the river. Hence, the downstream limit as $x \to \infty$ from equation (30) is given by:

$$C_3(x \to \infty, t) = \frac{q}{k_1 A}\{1 - \exp[-k_1 t]\} + p_1\{\exp[-k_1 t]\}. \tag{34}$$

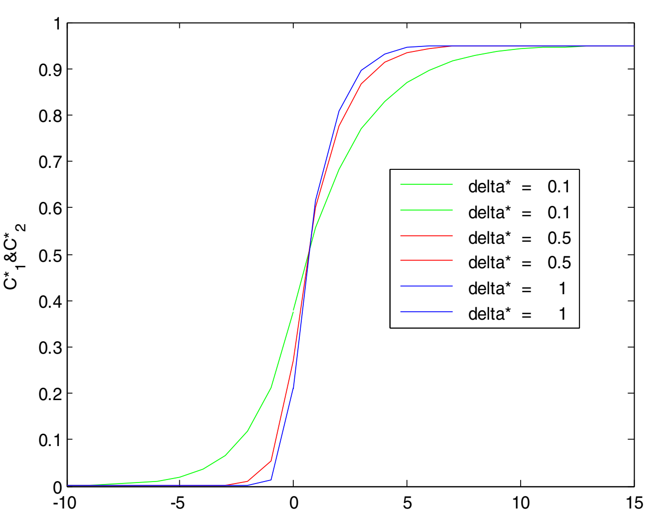

Figure 2. Shows the analytical unsteady state solution with dispersion for $C_1^*(x^*, t^*)$ and $C_2^*(x^*, t^*)$ at different values of delta* (δ^*) described by equations (16, 17).

For the special case when $t \to \infty$, equation (34) gives:

$$C_3(x \to \infty, t \to \infty) = \frac{q}{k_1 A}. \tag{35}$$

This result agrees with that obtained by Pimpunchat *et al.* (2007). In figures 3–5, the values of the parameters q, k_1, A and v are taken from Pimpunchat *et al.* (2007).

Figure 3 shows the variation of $C_3^*(x^*, t^*)$ along the river from source up to sink at different times t^*=6.616 (t=0.8 day), 7.443 (t=0.9 day), 8.27 (t=1 day) and 9.097 (t=1.1 day) where $k_1 = 8.27$ day^{-1}, $p_1^* = 2894.5$ and $p_2^* = 28.945$. From figure 3, it is clear that:

As x^* increases the value of C_3^* decreases for any time t^*, it reaches a constant value near the sink. The effect of the time t^* is very small near the

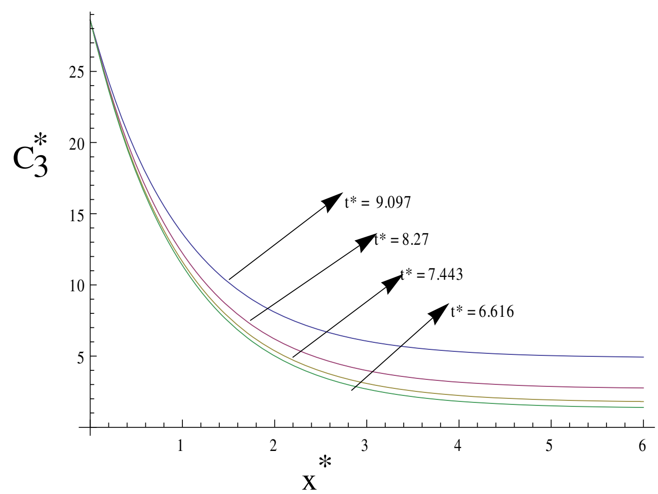

Figure 3. Shows $C_3^*(x^*, t^*)$ along the river from source up to sink at different t^* where p_1^* and p_2^* are constant as described by equation (32).

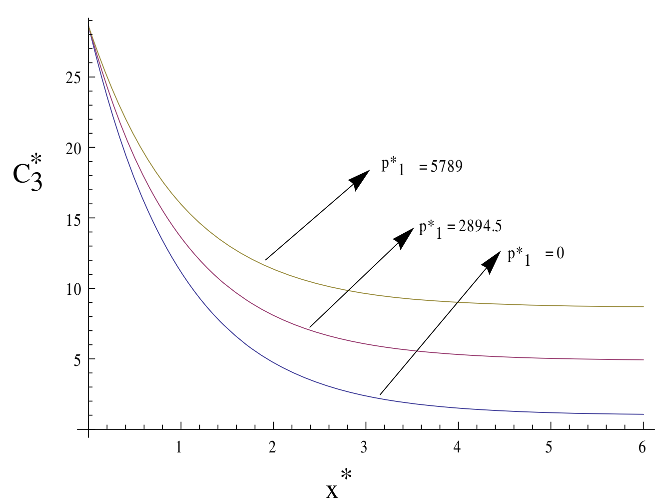

Figure 4. Shows $C_3^*(x^*, t^*)$ as a function of space and time along the river for different values of p_1^* where p_2^* and t^* are constant described by equation (32).

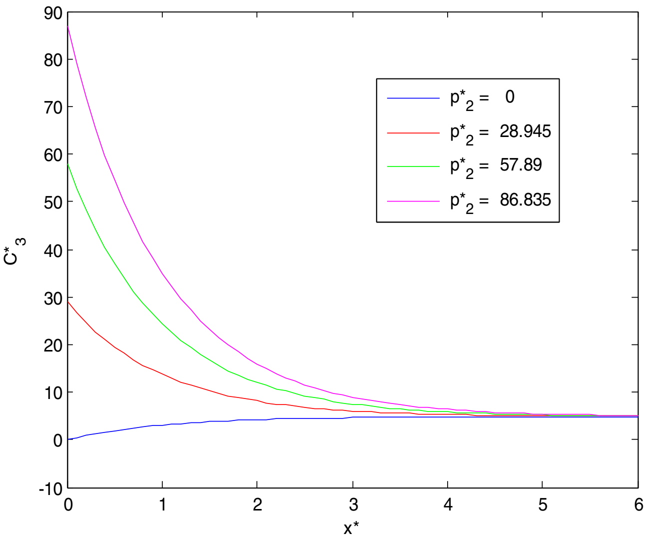

Figure 5. Shows $C_3^*(x^*, t^*)$ as function of space and time along the river for different values of p_2^* where p_1^* and t^* are constant described by equation (32).

upstream and dominant near the downstream. As t^* increases the value of C_3^* increases at any cross-section of the river.

Figure 4 shows the variation of C_3^* along the river with different values of

$$p_1^*(q = 0.06\ \text{kg}\cdot\text{m}^{-1}\cdot\text{day}^{-1},\ k_1 = 8.27\ \text{day}^{-1},\ A = 2100\ \text{m}^2,\ p_1 = 0.0,\ 0.01,\ 0.02\ \text{kg}\cdot\text{m}^{-3})$$

where $p_2^* = 28.945$ and $t^* = 6.616$. From figure 4, it is clear that:

As expected the minimum value of C_3^* occurs for $p_1^* = 0$. As p_1^* increases C_3^* increases at any cross-section of the river ($x^* =$ constant).

Figure 5 shows the variation of C_3^* along the river with different values of

$$p_2^*(q = 0.06\ \text{kg}\cdot\text{m}^{-1}\cdot\text{day}^{-1},\ k_1 = 8.27\ \text{day}^{-1},\ A = 2100\ \text{m}^2,\ p_2 = 0,\ 0.0001,\ 0.0002,\ 0.0003\ \text{kg}\cdot\text{m}^{-3})$$

where $p_1^* = 2894.5$ and $t^* = 6.616$. From figure 5, it is clear that:

For $p_2^* = 0$ (for water without pollution) from the upstream of the river $x^* = 0$, as x^* increases C_3^* increases, this is due to the presence of p_1^*. For the other values of p_2^* as x^* increases C_3^* decreases. At any cross-section where $x^* =$ constant, C_3^* increases as p_2^* increases.

5. Notations and symbols

A	Cross-section area of the river (m^2).
$C(x,\ t)$	C_1, C_2 and C_3 are the pollutant concentrations ($\text{kg}\cdot\text{m}^{-3}$).
D	Dispersion coefficient of pollutant in the x direction ($\text{m}^2\cdot\text{day}^{-1}$).
k	Half-saturated oxygen demand concentration for pollutant decay ($\text{kg}\cdot\text{m}^{-3}$).
k_1	Degradation rate coefficient for pollutant (day^{-1}).
L	Polluted length of river (m).
P	Laplace transform variable.
p_1 and p_2	Initial rate of pollution ($\text{kg}\cdot\text{m}^{-3}$) along the river and the rate of pollution ($\text{kg}\cdot\text{m}^{-3}$) at the origin respectively.
q	Added pollutant rate along the river ($\text{kg}\cdot\text{m}^{-1}\cdot\text{day}^{-1}$).
t	Time (day).
v	Water velocity in the x direction ($\text{m}\cdot\text{day}^{-1}$).
x	Position (m).
X	Dissolved oxygen concentration ($\text{kg}\cdot\text{m}^{-3}$).
$(*)$	Dimensionless quantity.
$\otimes$	Multiplication operation in the convolution theorem.

6. Conclusion

Analytical solutions of unsteady pollutant concentration $C(x,\ t)$ of one-dimensional advection–dispersion equation were derived by using Laplace transformation method. The solutions continue the earlier work of Pimpunchat *et al.* (2007) (steady state case). From the solution we predicted the pollutant concentration as function of space and time and obtained the variation of pollutant concentration with the parameters of the flow. By writing the solution for the pollutant concentration in the dimensionless form $C_1^*(x^*, t^*)$ and $C_2^*(x^*, t^*)$, we found that both $C_1^*(x^*, t^*)$ and $C_2^*(x^*, t^*)$ depend only on the two dimensionless parameters q/k_1A and δ^*. We found that $C_3^*(x^*, t^*)$ in general, increases with the increase of one of the dimensionless parameters, t^*- time or p_1^* $-$ the initial pollution of the river, or p_2^* $-$ the pollution at the source $x^* = 0$, keeping the two other parameters constant. Numerical studies show that the variation of C_3^* with time t^* is negligible for the special case when $p_1^* = p_2^* = 0$. From this study, the condition $\frac{\partial C}{\partial x} \to 0$ as $x \to \infty$ is satisfied automatically.

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