A generalized solution for groundwater head fluctuation in a tidal leaky aquifer system

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A new analytical solution is developed for describing groundwater level fluctuations in a coupled leaky confined aquifer system which consists of an unconfined aquifer, confined aquifer, and an aquitard in between. The aquifer system has a tidal boundary at the seashore, a no flow boundary at remote inland side, and a confined aquifer extending under the sea and terminated with an outlet-capping. This new solution has shown to be a generalisation of most existing analytical solutions for a tidal aquifer system which includes single confined and leaky confined aquifers. In addition, the solution is used to explore the influences of the dimensionless leakance of the outlet-capping, the dimensionless hydraulic diffusivities, and the leakages of the inland and offshore aquitards on the head responses in the leaky confined aquifer.

1. Introduction

In most of the coastal areas, groundwater and seawater are hydraulically connected. The tidal dynamics in coastal aquifers is an interesting topic for hydrologists and plays an important role in numerous environmental issues in coastal areas. Jacob (1950) introduced a simple coastal aquifer system that considered a vertical beach, straight coastline and one-dimensional flow in a coastal confined aquifer. Since then, the dynamic interaction between groundwater and seawater has attracted much attention. The coastal subsurface formation is usually complex; the Jacob's solution was in fact too simple to be applied to many real-world problems. Therefore, the analytical solutions for complicated tidal aquifer configurations were developed. Those include a three-layered coastal aquifer system (Jiao and Tang 1999; Li and Jiao 2001a; Jeng *et al* 2002; Li *et al* 2007; Chuang *et al* 2010), a confined aquifer extending under the sea (Van der Kamp 1972; Li and Jiao 2001b; Chuang and Yeh 2007, 2008; Li *et al* 2008) and a coastal confined aquifer with an outlet-capping (Guo *et al* 2007; Xia *et al* 2007; Geng *et al* 2009). Previous studies showed that dynamic effect of the unconfined aquifer on the head fluctuations in the confined aquifer plays an active role in solving coastal leaky confined aquifer problems.

This paper develops a new analytical solution for describing groundwater level fluctuation in a heterogeneous leaky confined aquifer extending finite distance under the sea and terminated at an outlet-capping. Most of the existing analytical solutions of the tidal aquifer systems are shown as special cases of the present solution. These solutions include one-dimensional flow in a coastal confined aquifer (Jacob 1950; Ferris 1951; Van der Kamp 1972; Geng *et al* 2009) and a variety of coastal leaky confined aquifer systems (e.g., Jiao and Tang 1999; Li and Jiao 2001b; Chuang and Yeh 2007, 2008; Li *et al* 2007; Xia *et al* 2007; Jeng *et al* 2002).

Keywords. Aquitard; coastal aquifer; leaky aquifer; analytical solution.

In addition, the present solution can be considered as an extension of Xia *et al*'s (2007) solution with differences in two following aspects:

(1) the offshore and inland parts of the aquifer have different hydraulic properties, and
(2) the water table in the unconfined aquifer fluctuates with tide.

The influence of those two situations on the behaviour of the groundwater level fluctuation in the inland part of the confined aquifer is investigated. The joint dynamic effects of water table fluctuation, the leakage through its submarine outlet-capping, and the leakages of the inland and offshore aquitards on the head fluctuations in the inland part of the leaky confined aquifer are also examined.

2. Problem formulation

Consider a coastal aquifer system with an unconfined aquifer, a confined aquifer, and an aquitard between them as displayed in figure 1. The origin of the x-axis is located at the intersection of the mean sea surface and the beach face. The x-axis is horizontal, positive landward and perpendicular to the coastal line. Tidal fluctuations in both unconfined and confined aquifers are considered. These two aquifers interact with each other through leakage. The unconfined aquifer terminates at the coast while the aquitard and confined aquifer extend over a finite distance (l) under the sea with an outlet-capping (Xia *et al* 2007). The bottom of the confined aquifer is impermeable and the leakages of the offshore and inland aquitards are different. Consider that the hydraulic parameters of the unconfined aquifer as well as the offshore and inland confined aquifers are all different. In addition, the thickness of the unconfined aquifer is very large when compared to the amplitude of the tidal fluctuation. The flow in the confined aquifer is essentially horizontal and there is a vertical leakage through the aquitard. The initial hydraulic head in the whole system is uniform and equals h_{MSL}, which is the distance from the groundwater level to any arbitrary location. Assume that elastic storage of the aquitard is negligible and the leakage is linearly proportional to the difference in head between the unconfined aquifer and its underlain confined one.

2.1 *Groundwater flow equations*

Under above assumptions, the governing equations of the head fluctuations for the inland unconfined and confined aquifers ($x > 0$) are respectively (Chuang and Yeh 2008)

$$S_1 \frac{\partial h_1}{\partial t} = T_1 \frac{\partial^2 h_1}{\partial x^2} + L_i \left(h_2 - h_1\right) \tag{1a}$$

$$S_2 \frac{\partial h_2}{\partial t} = T_2 \frac{\partial^2 h_2}{\partial x^2} + L_i \left(h_1 - h_2\right) \tag{1b}$$

and for the offshore aquifer ($x < 0$) is

$$S_3 \frac{\partial h_2}{\partial t} = T_3 \frac{\partial^2 h_2}{\partial x^2} + S_3 T_e \frac{dh_s}{dt} + L_o \left(h_s - h_2\right) \tag{1c}$$

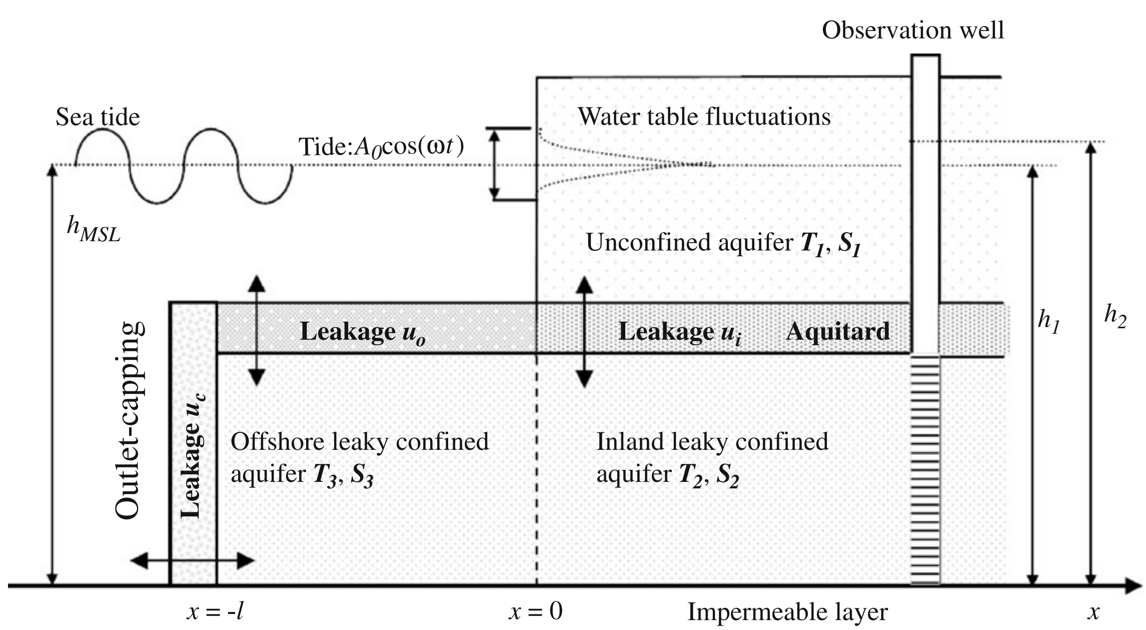

Figure 1. Schematic diagram of a tidal leaky aquifer system.

where h_1 and h_2 are the hydraulic heads in the unconfined and confined aquifers, respectively; h_s is the water level of the sea tide; T_e, the tidal efficiency, reflects the fluctuation of groundwater level caused by compression of both the aquifer skeleton and groundwater due to the tidal loading above the offshore aquitard (Li and Jiao 2001b); S_1 is the specific yield of the unconfined aquifer. S_2 and S_3 are the storativities of the inland and offshore confined aquifers, respectively. T_1, T_2 and T_3 are the transmissivities of the unconfined, inland and offshore confined aquifers, respectively. The leakage is defined as the ratio of the hydraulic conductivity of the aquitard to the thickness of the aquitard and L_o and L_i are the leakages of the offshore and inland aquitards, respectively. The hydraulic conductivity and/or thickness of the inland aquitard may differ from those of the offshore aquitard due to different depositional sediment faces.

2.2 *Boundary and continuity conditions*

The tidal boundary at $x = 0$ may be written as:

$$h_1(0, t) = h_s(t) = h_{\mathrm{MSL}} + A_0 \cos(\omega \cdot t) \tag{2a}$$

where $h_1(0, t)$ is the hydraulic head at $x = 0$, A_0 is the amplitude of the tidal change, and ω is the tidal frequency. Also $\omega = 2\pi/t_0$ where t_0 is the tidal period. The leakage rate at the outlet-capping is expressed as:

$$-K_3 \frac{\partial h_2(x, t)}{\partial x} = \frac{K'(h_s - h_2)}{m} \quad \text{when } x = -l, \tag{2b}$$

where l is the distance extended under the sea, K_3 is the hydraulic conductivity of the offshore confined aquifer, and K' and m are the permeability and thickness of the outlet-capping, respectively. The continuity conditions of the hydraulic head and flux at $x = 0$, respectively require

$$\lim_{x\uparrow 0} h_2(x, t) = \lim_{x\downarrow 0} h_2(x, t), \tag{2c}$$

$$T_3 \lim_{x\uparrow 0} \frac{\partial h_2(x, t)}{\partial x} = T_2 \lim_{x\downarrow 0} \frac{\partial h_2(x, t)}{\partial x}. \tag{2d}$$

The boundary conditions for equations (1a) and (1b) on the inland side may respectively be expressed as:

$$\lim_{x\to\infty} \frac{\partial h_1(x, t)}{\partial x} = 0 \tag{2e}$$

$$\lim_{x\to\infty} \frac{\partial h_2(x, t)}{\partial x} = 0 \tag{2f}$$

which states that the slopes of the hydraulic head approach zero at the infinite boundary.

3. Present solution and special cases

Some normalized parameters used in Xia *et al* (2007) are also adopted hereinafter for the convenience of comparison. The tidal wave propagation parameter is defined as $a_1 = \sqrt{\omega S_1/2T_1}$ for the unconfined aquifer, $a_2 = \sqrt{\omega S_2/2T_2}$ for the inland confined aquifer, and $a_3 = \sqrt{\omega S_3/2T_3}$ for the offshore confined aquifer. The dimensionless leakage is $u_i = L_i/\omega S_2$ for the inland aquitard and $u_o = L_o/\omega S_3$ for the offshore aquitard. The dimensionless leakance of outlet-capping is $u_c = K'/(a_3 m K_3)$. In addition, the dimensionless storativity for inland aquifer is introduced as $S_i = S_1/S_2$ and that for offshore aquifer as $S_o = S_3/S_2$, the dimensionless transmissivity for inland aquifer is defined as $T_i = T_1/T_2$ and that for offshore aquifer as $T_o = T_3/T_2$, and the dimensionless hydraulic diffusivity for inland aquifer is denoted as $D_i = T_i/S_i$ and that for offshore aquifer as $D_o = T_o/S_o$. The solutions of $h_1(x, t)$ and $h_2(x, t)$ for the inland unconfined and confined aquifers ($x > 0$) are, respectively, expressed as:

$$h_1(x, t) = h_{\mathrm{MSL}} + \mathrm{Re}\left[A_0\left(\alpha_1 e^{-\lambda_1 x} + \alpha_2 e^{-\lambda_2 x}\right)e^{-i\omega t}\right], \tag{3a}$$

$$h_2(x, t) = h_{\mathrm{MSL}} + \mathrm{Re}\left[A_0\left(\alpha_1 \beta_1 e^{-\lambda_1 x} + \alpha_2 \beta_2 e^{-\lambda_2 x}\right)e^{-i\omega t}\right]. \tag{3b}$$

The solution for offshore aquifer ($x < 0$) is:

$$h_2(x, t) = h_{\mathrm{MSL}} + \mathrm{Re}\left[A_0\left(\alpha_3 e^{\lambda_3 x} + \alpha_4 e^{-\lambda_3 x} + \beta_3\right)e^{-i\omega t}\right] \tag{3c}$$

where Re denotes the real part of the complex expression and the variables α_1, α_2, α_3, α_4, β_1, β_2, β_3, λ_1, λ_2 and λ_3 are defined by equations (A6a–o), respectively, in Appendix A. Derivations of the other solutions considered as special cases of the present solution are given in Appendix B and the discussions of special cases are as follows:

3.1 *Ignoring water table fluctuations in upper unconfined aquifer*

If $T_o \to 1$ and $D_i \to 0$, the water table of the unconfined aquifer can then be considered as maintained constant. Accordingly, Xia *et al*'s (2007) solution with neglecting the water table fluctuation is indeed a special case of the present solution. Xia *et al* (2007) mentioned that their solution can reduce to the head fluctuation solution of Li *et al* (2007) if the semi-permeable layer is replaced by an impermeable one. Therefore, the solution of Li *et al* (2007) is also a special case of the present solution. In addition, the solution of Geng *et al* (2009) is a special case of the present solution when the leakage of aquitard is zero.

3.2 *Ignoring outlet-capping effect in offshore aquifer*

When $m \rightarrow 0$, the thickness of the outlet-capping is very thin and negligible. The effect of outlet-capping is negligible and the extended roof is under the free flow condition. Under this circumstance, the present solution is equal to the solution presented in Chuang and Yeh (2008). It is interesting to note that the solution of Li and Jiao (2001b) is a special case of the one presented in Chuang and Yeh (2008). Therefore, the solutions for the coastal confined aquifer with a roof extending over a certain distance under the tidal water, such as Li and Jiao (2001b) and Chuang and Yeh (2008), are all special cases of the present solution.

3.3 *Extending roof length of offshore aquifer to infinity*

If the roof length of the offshore aquifer extends to infinity, i.e., $l \rightarrow \infty$, the new solution (equations 3a–c), will reduce to the solution for head responses in a coupled coastal confined aquifer system consisting of a semi-permeable layer and a confined aquifer extending over an infinite distance under the sea. Then the present solution is equal to the one of Chuang and Yeh (2007). The solution for the leaky confined aquifer (i.e., equations 3b and 3c) should reduce to the solution presented in Van der Kamp (1972) when both leakages of the offshore and inland aquitards are equal to zero. Therefore, the solutions presented in Van der Kamp (1972) and Chuang and Yeh (2007) for confined aquifers extending over an infinite distance under the sea are also special cases of the newly developed solution.

3.4 *Ignoring both outlet-capping and roof length*

When the conditions $l \rightarrow 0$ and $m \rightarrow 0$ are held, the roof does not extend under the sea and the effect of outlet-capping on the head fluctuation in the confined aquifer is negligible. These two conditions are also applied to Jeng *et al* (2002). Equations (3a) and (3b) are exactly the same as the ones of Jeng *et al* (2002, equations 12 and 13) except that the variables of β_1 and β_2 are in terms of dimensionless parameters. Jeng *et al* (2002) demonstrated that their solution reduces to that of Jiao and Tang (1999) when the water table is uniform. Jiao and Tang (1999) mentioned that their solution is the same as that given by Ferris (1951) if there is no leakage from the above unconfined aquifer. In addition, Li and Jiao (2001b) showed that the solution of confined aquifer is equal to the one developed by Jacob (1950) when both the leakage term and roof length are zero. Obviously, those solutions of Jacob (1950), Ferris (1951), Jiao and Tang (1999), and Jeng *et al* (2002) for the tidal confined aquifer with zero offshore length can also be considered as our special cases.

4. Results and discussion for present solution

Equations (3a–c) are the solutions for the groundwater heads in the inland unconfined aquifer, inland confined aquifer, and offshore part of the confined aquifer, respectively. Since most field studies on coastal aquifer systems focus on the inland part of aquifer and the observation of groundwater heads in the offshore aquifer is usually unavailable. Thus, only the groundwater heads in inland part of the aquifers are addressed herein (Li and Jiao 2001b). Some lumped parameters are introduced for convenience of discussion. The amplitude coefficient (C_e) is defined as $C_e = \mathrm{Re}\,[\alpha_1\beta_1 + \alpha_2\beta_2]$ to represent the maximum amplitude of groundwater fluctuation. In addition, the constant phase shift (φ) of the inland confined aquifer is defined as $\varphi = \mathrm{Re}\,[\alpha_1\beta_1 + \alpha_2\beta_2]\,/\,\mathrm{Im}\,[\alpha_1\beta_1 + \alpha_2\beta_2]$, where Im represents the imaginary part of the complex expression, to represent the phase angle when the groundwater fluctuation has the maximum amplitude. Reasonable ranges of aquifer parameter values reported in the literature (e.g., Li *et al* 2001b; Jeng *et al* 2002) are listed in table 1 for the case study. The formation materials of the offshore confined aquifer and its outlet-capping are considered to be fine sand and silt, respectively, in this study. The thickness of the outlet-capping is assumed to be 10 m. Thus, the hydraulic conductivity of fine sand formation ranges from 2×10^{-2} to 20 m/day and that of silt formation ranges from 8×10^{-5} to 2 m/day (Domenico and Schwartz 1997). In addition, the values of the tidal wave propagation parameter a_2 and a_3 are in the range of $10^{-1} \sim 10^{-3}$ m^{-1} (Li and Jiao 2001b). The leakance of outlet-capping, $u_c = K'/(a_3 m K_3)$, may therefore fall in the range from 4×10^{-6} to 10^4. Accordingly, the consideration for the value of the logarithm of the leakance of outlet-capping, lg u_c, varied from -3 to 3 seems to be reasonable for coastal aquifers. The

Table 1. *Input data for the case study (Li et al 2001b; Jeng et al 2002).*

Parameter	Value
Amplitude of tide A_0	0.65 m
Mean sea level h_{MSL}	0 m
Out-capping thickness m	0 or varying m
Specific yield S_1	$0.1 \sim 10^{-3}$
Storativity of inland leaky confined aquifer S_2	$10^{-3} \sim 10^{-5}$
Storativity of offshore leaky confined aquifer S_3	$10^{-3} \sim 10^{-5}$
Tidal frequency ω	2π rad/day
Tidal propagation parameter of aquifer $a_{1\sim3}$	$10^{-1} \sim 10^{-3}$/m
Transmissivity of unconfined aquifer T_1	2000 m^2/day or varying
Transmissivity of inland leaky confined aquifer T_2	2000 m^2/day or varying
Transmissivity of offshore leaky confined aquifer T_3	2000 m^2/day or varying
Dimensionless inland leakage $u_i = L_i/\omega S_2$	0.1 or 10
Dimensionless offshore leakage $u_o = L_o/\omega S_3$	0.1 or 10
Dimensionless out-capping leakage $u_c = K'/(a_3 m K)$	0 or varying
Dimensionless tidal propagation parameter $a_3 l$	0 or varying (l = roof length)
Dimensionless inland storativity $S_i = S_1/S_2$	1 or 10^4
Dimensionless offshore storativity $S_o = S_3/S_2$	$10^{-2} \sim 10^2$
Dimensionless inland transmissivity $T_i = T_1/T_2$	1 or varying
Dimensionless offshore transmissivity, $T_o = T_3/T_2$	1 or varying
Dimensionless inland hydraulic diffusivity $D_i = T_i/S_i$	10^{-4} or 1 or varying
Dimensionless offshore hydraulic diffusivity $D_o = T_o/S_o$	10^{-2} or 1 or varying

effects of dimensionless out-capping, aquitard leakages and hydraulic diffusivities on the groundwater heads in the inland confined aquifer are addressed in the following sections.

4.1 *The effect of dimensionless inland leakage on head fluctuation*

Figure 2 shows the curves of the amplitude coefficient C_e and phase shift φ *versus* the logarithm of dimensionless outlet-capping leakage lg u_c when the dimensionless inland leakage u_i varies from 0.1 to 10 with parameters $a_3 l = 0.1$, $T_e = 0.5$, $D_o = 1$ and $u_o = 1$. The figure indicates that both C_e and φ are close to constant when lg $u_c < -3$ or >3 and the behaviours of C_e and φ are therefore not discussed in these two regions. The solid lines denote the present solution with $D_i = 0.1$ while the dashed lines stand for the solution with $D_i = 0$ implying that the water table fluctuation in the unconfined aquifer is negligible. The solid lines in the figure display that C_e increases with u_c for all u_i in the range $-1 <$ lg $u_c < 2$. In addition, the solid lines also indicate that the influence of u_i is large for the groundwater heads in the inland confined aquifer and the value of C_e increases with u_i when u_c is relatively small (say lg $u_c < 0.5$). On the contrary, the influence of u_i is relatively small for the groundwater head and C_e decreases with increasing u_i when u_c is large (lg $u_c > 0.5$). The dashed lines of figure 2(a) display that the C_e decreases with increasing u_i for all u_c. The influence of water table fluctuation on the groundwater head in the inland confined aquifer increases significantly with u_i for all u_c and decreases with increasing u_c for all u_i as compared the solid and dashed lines shown in figure 2(a). Both solid and dashed lines in figure 2(b) show that the influence of u_i on φ decreases significantly with increasing u_c. Moreover, the solid lines display that φ decreases with increasing u_c when $u_i \geq 1$. However, φ initially increases with u_c, reaches a peak value at lg $u_c = -0.5$, and then decreases as u_c increases for the case of $u_i = 0.1$. The dashed lines in figure 2(b) show that the φ has a peak value near lg $u_c = 0.5$ and decreases when away from the peak value when $u_i = 0.1$ or 1, but the φ increases with increasing u_c when $u_i = 10$. This phenomenon is caused by the combined effects of the outlet-capping and aquitard leakages. The dashed lines also display that φ increases with decreasing u_i for all u_c. In addition, the value of φ depends on the water table fluctuation in the unconfined aquifer when u_c is relatively small and the effect of the water table fluctuation is relatively small when u_c is large as demonstrated in figure 2(b).

4.2 *The effect of dimensionless offshore leakage on head fluctuation*

Figure 3 displays the curves of the C_e and φ versus lg u_c denoted by the solid lines for $D_i = 0.1$ and the

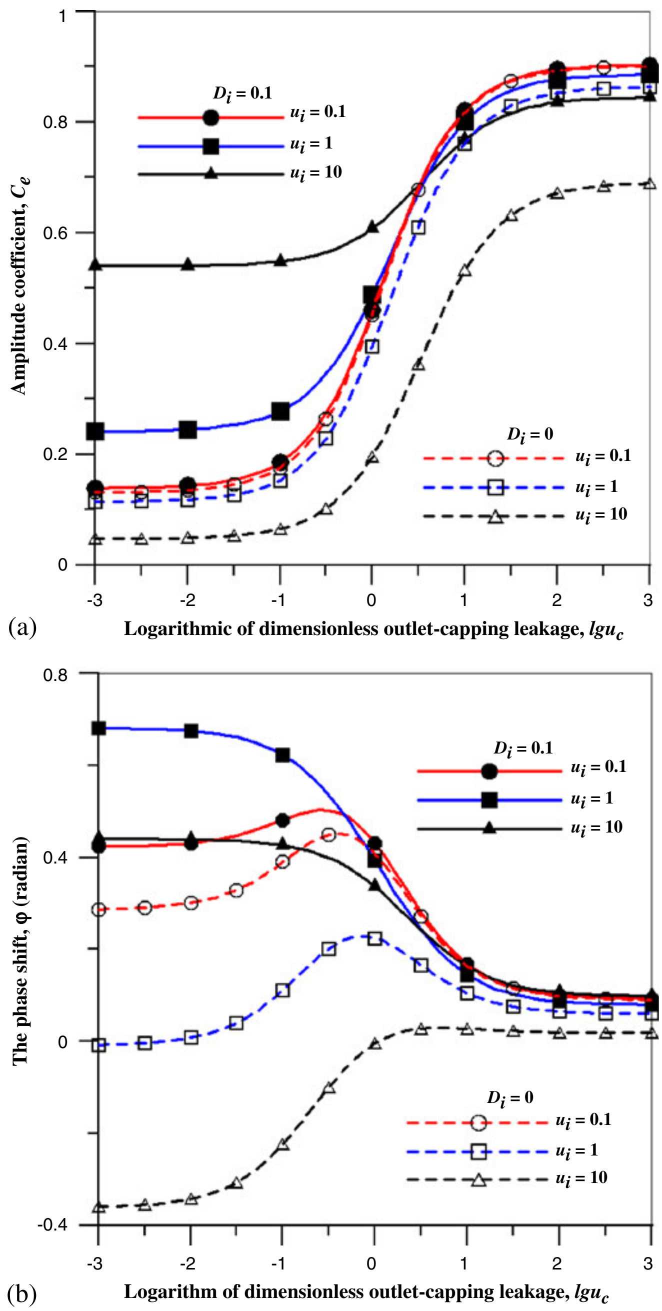

Figure 2. The curves for **(a)** amplitude coefficient (C_e) and **(b)** the phase shift (φ) *versus* logarithm of dimensionless outlet-capping leakage ($\lg u_c$) when the dimensionless inland leakage (u_i) varies from 0.1 to 10 with parameters $a_3 l = 0.1$, $T_e = 0.5$, $D_i = 0$ or 0.1, $D_o = 1$ and $u_o = 1$.

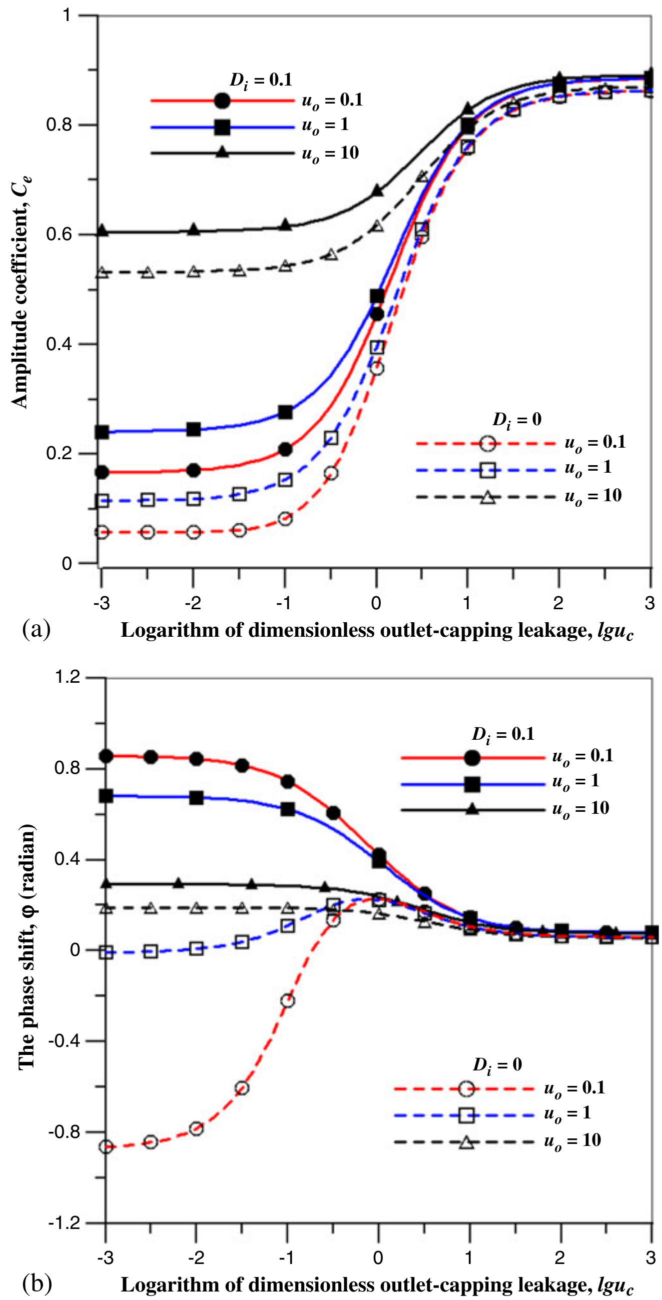

Figure 3. The curves for **(a)** amplitude coefficient (C_e) and **(b)** the phase shift (φ) *versus* logarithm of dimensionless outlet-capping leakage ($\lg u_c$) when the dimensionless offshore leakage (u_o) varies from 0.1 to 10 with parameters $a_3 l = 0.1$, $T_e = 0.5$, $D_i = 0$ or 0.1, $D_o = 1$ and $u_i = 1$.

dashed line for $D_i = 0$ when the u_o varies from 0.1 to 10 with parameters $a_3 l = 0.1$, $T_e = 0.5$, $D_o = 1$ and $u_i = 1$. Both figures 3(a) and 3(b) indicate that the influence of u_o on the C_e and φ decreases gradually with increasing u_c. The dynamic effect of water table fluctuation on the head response in the confined aquifer increases with decreasing u_c for all u_o. Figure 3(a) shows that the C_e increases with u_c for all u_o. In addition, the influence of u_c on C_e is large and this influence decreases as u_o increases when $-1 < \lg u_c < 1$. These phenomena reflect the joint effects of u_c and u_o on the groundwater heads in inland part of the aquifer are very significant. The influence of u_o is large when u_c is small; on the other hand, the influence of u_o is small when u_c is large. The solid lines in figure 3(b) exhibit that the φ decreases with increasing u_c. The influence of u_c on φ decreases with increasing u_o when

$-2 < \lg u_c < 1$. The dashed lines display that the φ has a peak value near $\lg u_c = 0$ and decreases when away from the peak value indicating that the effect of u_c on φ is large when $u_o = 0.1$ or 1, but the φ decreases slightly with increasing u_c when $u_o = 10$. On the other hand, the effect of u_c on φ is almost negligible when $\lg u_c > 1$. Obviously, the effect of u_o on φ is small when u_c is large as demonstrated in the figure.

4.3 *The simultaneous effect of dimensionless leakages on head fluctuation*

If the dimensionless inland leakage (u_i) and offshore leakage (u_o) are set equal, then the dimensionless leakage u is $u = u_i = u_o$. The curves of C_e and φ *versus* the logarithm of the dimensionless out-capping leakage, $\lg u_c$, in the range between -3 and 3 are demonstrated in figure 4 for $u_i = u_o = u = 1$ and 10, $a_3 l = 0.1$, $T_e = 0.5$, $D_i = 0.1$, and $D_o = 1$. In figure 4, the solid lines denote the present solution while the dashed lines represent the solution of Xia *et al* (2007) in which the water table fluctuation is not considered. Figure 4(a) shows that C_e increases with u_c for all u. However, the C_e predicted by the present solution is obviously greater than that of Xia *et al*'s (2007) solution for all u_c and their difference increases with u and decreases with increasing u_c. In other words, the C_e will be underestimated if the water table fluctuation in the unconfined aquifer is neglected. Figure 4(b) displays a similar result that φ in the present solution is obviously greater than that of Xia *et al*'s (2007) solution for all u_c. The φ decreases with increasing u_c in the present solution and the influence of u_c on φ increases as u decreases. The effects of u_c on φ predicted by the present solution and Xia *et al*'s (2007) solution are very different. In figure 4(b), the dashed lines display that the φ increases slightly with u_c when $\lg u_c < 0.5$ and decreases slightly with increasing u_c when $\lg u_c > 0.5$ for $u = 10$. On the other hand, the φ increases with u_c when $\lg u_c < 0$ and decreases with increasing u_c when $\lg u_c > 0$ for $u = 1$. Figure 4 indicates that the influence of the water table fluctuation in the unconfined aquifer on the C_e and φ when $\lg u_c = -3$ is obviously larger than that when $\lg u_c = 3$. Note that the outlet-capping might be considered as no-flow boundary when $\lg u_c = -3$ while the capping is considered as free-flow boundary when $\lg u_c = 3$. The predicted results from Xia *et al*'s (2007) solution are significantly different from those of the present solution. The use of Xia *et al*'s (2007) solution should therefore be cautious if the water table fluctuation is significant.

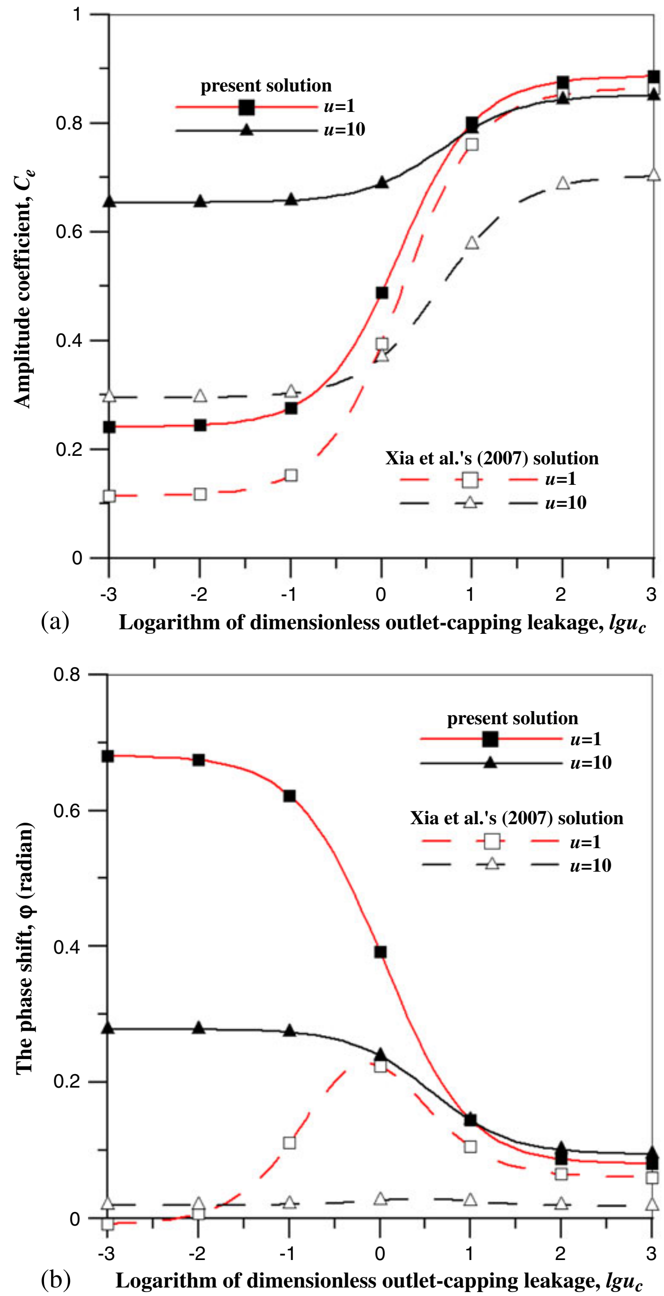

Figure 4. The curves for **(a)** amplitude coefficient (C_e) and **(b)** the phase shift (φ) *versus* logarithm of dimensionless outlet-capping leakage ($\lg u_c$) when the dimensionless leakage ($u = u_i = u_o$) is equal to 1 and 10 with parameters $a_3 l = 0.1$, $T_e = 0.5$, $D_i = 0.1$, and $D_o = 1$.

4.4 *The effects of dimensionless hydraulic diffusivities on head fluctuation*

Figure 5 shows that the curves of C_e and φ *versus* $\lg u_c$ when D_i varies from 0 to 1 with parameters $a_3 l = 0.1$, $D_o = 1$, $T_e = 0.5$, $u_i = 1$ and $u_o = 1$. Figure 5(a) displays that the C_e increases dramatically with u_c when $-1 < \lg u_c < 1$ for all D_i. In addition, a larger D_i gives a larger C_e. In other

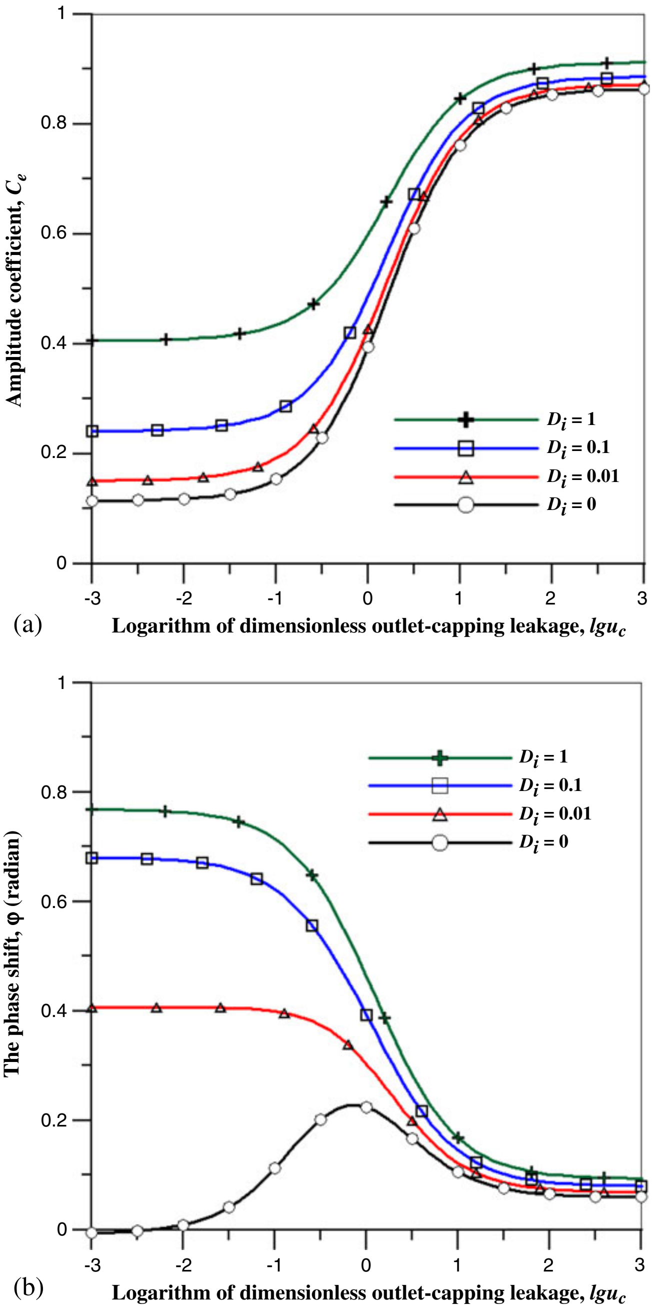

Figure 5. The curves for **(a)** amplitude coefficient (C_e) and **(b)** the phase shift (φ) *versus* logarithm of dimensionless outlet-capping leakage (lg u_c) when the dimensionless hydraulic diffusivity (D_i) varies from 0 to 1 with parameters $a_3 l = 0.1$, $D_o = 1$, $T_e = 0.5$, $u_i = 1$ and $u_o = 1$.

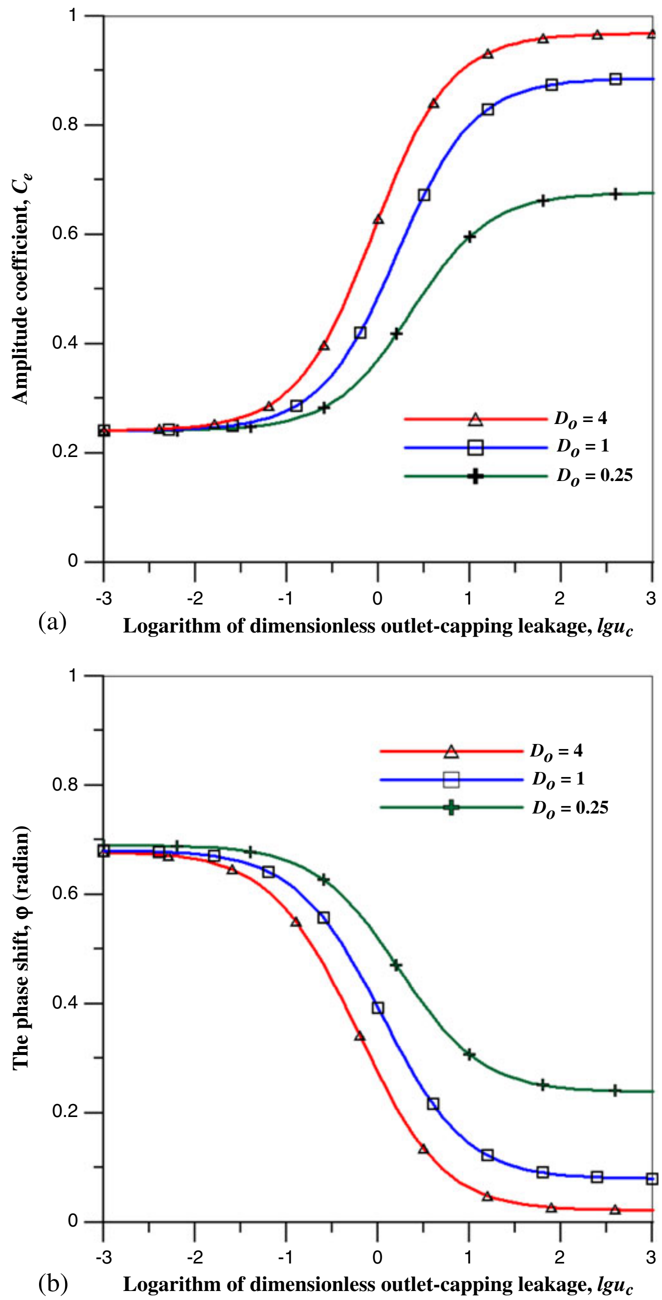

Figure 6. The curves for **(a)** amplitude coefficient (C_e) and **(b)** the phase shift (φ) *versus* logarithm of dimensionless outlet-capping leakage (lg u_c) when the dimensionless hydraulic diffusivity (D_o) varies from 0.25 to 4 with parameters $a_3 l = 0.1$, $D_i = 0.1$, $T_e = 0.5$, $u_i = 1$ and $u_o = 1$.

words, the effect of D_i is large when u_c is relatively small. By contrast, the influence of D_i is small when u_c is relatively large. The figure also shows that the discrepancy caused by neglecting the water table fluctuation in the unconfined aquifer increases with D_i and decreases with increasing u_c. Figure 5(b) shows that the φ decreases as u_c increases for $D_i > 0.01$ and the decrease of φ is very rapid for lg u_c in the range from -1 to 1. The figure also shows that the φ has a peak value near lg $u_c = 0$ when $D_i = 0$ and the effect of D_i on φ in the range of lg $u_c < 0$ is larger than that in the range of lg $u_c > 0$.

Figure 6 displays the curves for C_e and φ *versus* logarithm of dimensionless outlet-capping leakage when D_o varies from 0.25 to 4 with parameters $a_3 l = 0.1$, $D_i = 0.1$, $T_e = 0.5$, $u_i = 1$ and $u_o = 1$. Figure 6(a) shows that the C_e increases with u_c for all D_o and the influence of u_c on C_e increases with D_o. The figure also shows that the effect of

D_o increases with u_c for C_e. Figure 6(b) displays that the φ decreases with increasing u_c for all D_o and the influence of u_c on φ increases with D_o. In addition, the figure also displays the effect of D_o on φ increases with u_c.

5. Conclusion

A new analytical solution has been developed for a coastal aquifer system consisting of an unconfined aquifer, an aquitard, and a leaky confined aquifer. The unconfined aquifer ends at the coast while the aquitard and confined aquifer extend over a finite distance under the sea and terminate with an outlet-capping. The solution has been demonstrated to be a generalization of most existing analytical solutions for various types of coastal aquifer systems. In addition this solution can be used to explore the influences of the outlet-capping leakage, the hydraulic diffusivities, and the leakages of the inland and offshore aquitards on the head response in the leaky confined aquifer. It is found that the interaction between the outlet-capping leakage and the dynamic effect of water table fluctuation on the head response in the leaky confined aquifer is very significant for the aquifer with an outlet-capping in a coupled coastal aquifer system.

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Appendix A.
Derivation of solutions to equations (1a–c) subject to equations (2a–f)

Let $H_1(x,t)$ and $H_2(x,t)$ be complex functions of the real variables x and t that satisfy the governing equations (1a–c) and the conditions (2a–f). Assume that the solutions to equations (1a–c) subject to equations (2a–f) are $h_1(x,t)$ and $h_2(x,t)$ and expressed, respectively, as:

$$h_1(x,\ t) = h_{\mathrm{MSL}} + \mathrm{Re}\left[H_1(x,\ t)\right] \quad \text{(A1a)}$$

$$h_2(x,\ t) = h_{\mathrm{MSL}} + \mathrm{Re}\left[H_2(x,\ t)\right] \quad \text{(A1b)}$$

where Re denotes the real part of the complex expression. Consider that

$$H_1(x,t) = A_0 X_1(x)\, e^{-i\omega t} \quad \text{(A2a)}$$

$$H_2(x,t) = A_0 X_2(x)\, e^{-i\omega t} \quad \text{(A2b)}$$

where $X_1(x)$ and $X_2(x)$ are unknown functions of x and $i=\sqrt{-1}$. Substituting equations (A2a) and (A2b) into following nine equations (1a–c) and (2a–f), which $H_1(x,t)$ and $H_2(x,t)$ satisfy, and dividing the results by $A_0 e^{-i\omega t}$ yield the results for the inland aquifers ($x>0$) as:

$$X_1''(x) + 2a_1^2\left(i - \frac{u_i}{S_i}\right) X_1(x) + \frac{2a_1^2 u_i}{S_i} X_2(x) = 0 \quad \text{(A3a)}$$

$$X_2''(x) + 2a_2^2\left(i - u_i\right) X_2(x) + 2a_2^2 u_i X_1(x) = 0 \quad \text{(A3b)}$$

and the result for the offshore aquifer ($x<0$) as:

$$X_2''(x) + 2a_3^2\left(i - u_o\right) X_2(x) = 2a_3^2\left(T_e i - u_o\right). \quad \text{(A3c)}$$

The tidal boundaries, equations (4a) and (4b) may be respectively written as:

$$X_1(0) = 1, \quad \text{(A4a)}$$

$$X_2'(-l) - a_3 u_c X_2(-l) + a_3 u_c = 0. \quad \text{(A4b)}$$

In addition, the continuity conditions of equations (4c) and (4d) may be respectively expressed as:

$$\lim_{x\uparrow 0} X_2(x) = \lim_{x\downarrow 0} X_2(x), \quad \text{(A4c)}$$

$$T_o \lim_{x\uparrow 0} X_2'(x) = \lim_{x\downarrow 0} X_2'(x). \quad \text{(A4d)}$$

The boundary conditions, equations (4e–f) may also be respectively written as:

$$X_1'(+\infty) = 0, \quad \text{(A4e)}$$

$$X_2'(+\infty) = 0. \quad \text{(A4f)}$$

Thus, the general solutions to equations (A3a–c) for inland aquifers ($x>0$) are:

$$X_1(x) = \alpha_1 e^{-\lambda_1 x} + \alpha_2 e^{-\lambda_2 x} \quad \text{(A5a)}$$

$$X_2(x) = \alpha_1\beta_1 e^{-\lambda_1 x} + \alpha_2\beta_2 e^{-\lambda_2 x} \quad \text{(A5b)}$$

and for offshore aquifer ($x < 0$) is:

$$X_2(x) = \alpha_3 e^{\lambda_3 x} + \alpha_4 e^{-\lambda_3 x} + \beta_3 \tag{A5c}$$

where variables α_1, α_2, α_3, β_1, β_2, β_3, λ_1, λ_2, and λ_3 are defined, respectively as:

$$\alpha_1 = \frac{D_1}{D} \tag{A6a}$$

$$\alpha_2 = \frac{D_2}{D} \tag{A6b}$$

$$\alpha_3 = \frac{D_3}{D} \tag{A6c}$$

$$\alpha_4 = \frac{D_4}{D} \tag{A6d}$$

$$\begin{aligned} D = {} & (\lambda_3 - a_3 u_c) e^{-\lambda_3 l} \\ & \times (T_o \beta_1 \lambda_3 - T_o \beta_2 \lambda_3 - \beta_1 \lambda_1 + \beta_2 \lambda_2) \\ & + (\lambda_3 + a_3 u_c) e^{\lambda_3 l} \\ & \times (-T_o \beta_1 \lambda_3 + T_o \beta_2 \lambda_3 - \beta_1 \lambda_1 + \beta_2 \lambda_2) \end{aligned} \tag{A6e}$$

$$\begin{aligned} D_1 = {} & (\lambda_3 - a_3 u_c) e^{-\lambda_3 l} \\ & \times (-T_o \beta_2 \lambda_3 + T_o \beta_3 \lambda_3 + \beta_2 \lambda_2) \\ & + (\lambda_3 + a_3 u_c) e^{\lambda_3 l} \\ & \times (T_o \beta_2 \lambda_3 - T_o \beta_3 \lambda_3 + \beta_2 \lambda_2) \\ & - 2\lambda_3 a_3 u_c T_o (1 - \beta_3) \end{aligned} \tag{A6f}$$

$$\begin{aligned} D_2 = {} & (\lambda_3 - a_3 u_c) e^{-\lambda_3 l} \\ & \times (T_o \beta_1 \lambda_3 - T_o \beta_3 \lambda_3 - \beta_1 \lambda_1) \\ & + (\lambda_3 + a_3 u_c) e^{\lambda_3 l} \\ & \times (-T_o \beta_1 \lambda_3 + T_o \beta_3 \lambda_3 - \beta_1 \lambda_1) \\ & + 2\lambda_3 a_3 u_c T_o (1 - \beta_3) \end{aligned} \tag{A6g}$$

$$\begin{aligned} D_3 = {} & (\lambda_3 + a_3 u_c) e^{\lambda_3 l} \\ & \times (-\beta_1 \beta_2 \lambda_1 + \beta_1 \beta_3 \lambda_1 + \beta_1 \beta_2 \lambda_2 - \beta_2 \beta_3 \lambda_2) \\ & - a_3 u_c (1 - \beta_3) \\ & \times (T_o \beta_1 \lambda_3 - T_o \beta_2 \lambda_3 - \beta_1 \lambda_1 + \beta_2 \lambda_2) \end{aligned} \tag{A6h}$$

$$\begin{aligned} D_4 = {} & (\lambda_3 - a_3 u_c) e^{-\lambda_3 l} \\ & \times (-\beta_1 \beta_2 \lambda_1 + \beta_1 \beta_3 \lambda_1 + \beta_1 \beta_2 \lambda_2 - \beta_2 \beta_3 \lambda_2) \\ & - a_3 u_c (1 - \beta_3) \\ & \times (T_o \beta_1 \lambda_3 - T_o \beta_2 \lambda_3 + \beta_1 \lambda_1 - \beta_2 \lambda_2) \end{aligned} \tag{A6i}$$

$$\beta_1 = 1 - \frac{S_i B_1}{2a_1^2 u_i} - \frac{S_i i}{u_i} \tag{A6j}$$

$$\beta_2 = 1 - \frac{S_i B_2}{2a_1^2 u_i} - \frac{S_i i}{u_i} \tag{A6k}$$

$$\beta_3 = \frac{T_e i - u_o}{i - u_o} \tag{A6l}$$

$$\lambda_1 = \sqrt{B_1} \tag{A6m}$$

$$\lambda_2 = \sqrt{B_2} \tag{A6n}$$

$$\lambda_3 = 2a_3 \sqrt{\frac{u_o - i}{2}} \tag{A6o}$$

with variables B_1 and B_2 respectively defined as:

$$B_1 = -c_1 - \sqrt{c_1^2 - c_2} \tag{A6p}$$

$$B_2 = -c_1 + \sqrt{c_1^2 - c_2} \tag{A6q}$$

and variables c_1 and c_2 respectively defined as:

$$c_1 = -\left(a_1^2/S_i + a_2^2\right) u_i + \left(a_1^2 + a_2^2\right) i, \tag{A6r}$$

$$c_2 = -4a_1^2 a_2^2 (1 + u_i i/S_i + u_i i). \tag{A6s}$$

Appendix B. Derivation of other solutions considered as special cases

B1. Ignoring water table fluctuations in upper unconfined aquifer

If $T_o \to 1$, $D_i \to 0$ and $u = u_i = u_o$, the variables α_3 and α_4 become α_{3a} and α_{4a}, respectively. Based on equations (A6a–s), one can obtain

$$\begin{aligned} \lambda_1 = \lambda_2 & = \lambda_3 a_2 (2u - 2i)^{0.5} \\ & = a(p - qi) \end{aligned} \tag{B1a}$$

$$\beta_3 = \frac{u - iT_e}{u - i} = \lambda - i\mu \tag{B1b}$$

$$X_2(x) = -\left(\alpha_{3a} e^{\lambda_3 x} + \alpha_{4a} e^{-\lambda_3 x} + \beta_3\right) \tag{B1c}$$

for offshore aquifer ($-l < x < 0$) and

$$\begin{aligned} X_2(x) & = -(\alpha_1 \beta_1 + \alpha_2 \beta_2) e^{-\lambda_1 x} \\ & = -\gamma e^{-\lambda_1 x} \end{aligned} \tag{B1d}$$

for inland aquifer ($x > 0$) with the variables γ, α_{3a} and α_{4a} defined, respectively as:

$$\alpha_{4a} = \frac{u_c(1-\beta_3)}{(u_c+(p-iq))}e^{-\lambda_3 l} + \frac{\beta_3(u_c-(p-iq))}{2(u_c+(p-iq))}e^{-2\lambda_3 l} = C_1 \quad \text{(B1e)}$$

$$\alpha_{3a} = -\frac{\beta_3}{2} = C_2 \quad \text{(B1f)}$$

$$\gamma = \frac{u_c(1-\beta_3)}{(u_c+(p-iq))}e^{-\lambda_3 l} + \frac{\beta_3(u_c-(p-iq))}{2(u_c+(p-iq))}e^{-2\lambda_3 l} + \frac{\beta_3}{2} = C_1 - C_2 = C_3. \quad \text{(B1g)}$$

Note that the variables $a, p, q, \lambda, \mu, C_1, C_2$ and C_3 are defined the same as those in Xia *et al* (2007). Equations (B1c) and (B1d) are identical to the corresponding terms in the head solutions of Xia *et al* (2007, equations A9 and A10). Note that the complex expression used in their paper is $\mathrm{Re}(e^{i\omega t})$ while that used in this study is $\mathrm{Re}(e^{-i\omega t})$.

B2. Ignoring outlet-capping effect in offshore aquifer

When $m \to 0$ and $T_o \to 1$, the variables α_1, α_2, α_3, and α_4 become α_{1b}, α_{2b}, α_{3b}, and α_{4b}, respectively. Based on equations (A6a–s), one can obtain

$$\alpha_{1b} = \frac{e^{-\lambda_3 l}(-\beta_2\lambda_2+\beta_2\lambda_3-\beta_3\lambda_3)+e^{\lambda_3 l}(\beta_2\lambda_2+\beta_2\lambda_3-\beta_3\lambda_3)-2\lambda_3(1-\beta_3)}{\Gamma_1} \quad \text{(B2a)}$$

$$\alpha_{2b} = \frac{e^{-\lambda_3 l}(\beta_1\lambda_1-\beta_1\lambda_3+\beta_3\lambda_3)+e^{\lambda_3 l}(-\beta_1\lambda_1-\beta_1\lambda_3+\beta_3\lambda_3)+2\lambda_3(1-\beta_3)}{\Gamma_1} \quad \text{(B2b)}$$

$$\alpha_{3b} = \frac{e^{\lambda_3 l}(-\beta_1\beta_2\lambda_1+\beta_1\beta_3\lambda_1+\beta_1\beta_2\lambda_2-\beta_2\beta_3\lambda_2)-(1-\beta_3)(-\beta_1\lambda_1+\beta_2\lambda_2+\beta_1\lambda_3-\beta_2\lambda_3)}{\Gamma_1} \quad \text{(B2c)}$$

$$\alpha_{4b} = \frac{e^{-\lambda_3 l}(\beta_1\beta_2\lambda_1-\beta_1\beta_3\lambda_1-\beta_1\beta_2\lambda_2+\beta_2\beta_3\lambda_2)-(1-\beta_3)\Gamma_2}{\Gamma_1} \quad \text{(B2d)}$$

where variables Γ_1 and Γ_2 are respectively defined as:

$$\Gamma_1 = e^{-\lambda_3 l}(\beta_1\lambda_1-\beta_2\lambda_2-\beta_1\lambda_3+\beta_2\lambda_3) - \Gamma_2 e^{\lambda_3 l} \quad \text{(B2e)}$$

and

$$\Gamma_2 = \beta_1\lambda_1-\beta_2\lambda_2+\beta_1\lambda_3-\beta_2\lambda_3. \quad \text{(B2f)}$$

Therefore, equations (3a) and (3b) can be respectively written as:

$$h_1(x,t) = h_{\mathrm{MSL}} + \mathrm{Re}\left[A_0\left(\alpha_{1b}e^{-\lambda_1 x} + \alpha_{2b}e^{-\lambda_2 x}\right)e^{-i\omega t}\right] \quad \text{(B2g)}$$

$$h_2(x,t) = h_{\mathrm{MSL}} + \mathrm{Re}\left[A_0\left(\alpha_{1b}\beta_1 e^{-\lambda_1 x} + \alpha_{2b}\beta_2 e^{-\lambda_2 x}\right)e^{-i\omega t}\right] \quad \text{(B2h)}$$

and equation (3c) becomes

$$h_2(x,t) = h_{\mathrm{MSL}} + \mathrm{Re}\left[A_0\left(\alpha_{3b}e^{\lambda_3 x} + \alpha_{4b}e^{-\lambda_3 x} + \beta_3\right)e^{-i\omega t}\right]. \quad \text{(B2i)}$$

Equations (B2g–i) are exactly the same as the solutions presented in Chuang and Yeh (2008, equations 10a–c).

B3. Extending roof length of offshore aquifer to infinity

If $T_o \rightarrow 1$ and the roof length of the offshore aquifer extends to infinity (i.e., $l \rightarrow \infty$) the variables α_1, α_2, α_3, and α_4 of equations (A6a–d) reduce to α_{1c}, α_{2c}, α_{3c}, and α_{4c}, respectively, and

$$\alpha_{1c} = \frac{-\beta_2\lambda_2 - \beta_2\lambda_3 + \beta_3\lambda_3}{\Gamma_2} \quad \text{(B3a)}$$

$$\alpha_{2c} = \frac{\beta_1\lambda_1 + \beta_1\lambda_3 - \beta_3\lambda_3}{\Gamma_2} \quad \text{(B3b)}$$

$$\alpha_{3c} = \frac{\beta_1\beta_2\lambda_1 - \beta_1\beta_3\lambda_1 - \beta_1\beta_2\lambda_2 + \beta_2\beta_3\lambda_2}{\Gamma_2} \quad \text{(B3c)}$$

and

$$\alpha_{4c} = 0. \quad \text{(B3d)}$$

Equations (B3a–c), expressed as the dimensionless parameters, are in fact the same as the corresponding terms defined in Chuang and Yeh (2007, equations 13–15).

B4. Ignoring both outlet-capping and roof length

If $l \rightarrow 0$ and $m \rightarrow 0$, the effect of outlet-capping is negligible and the roof does not extend under the sea. These two conditions are also used in Jeng *et al* (2002). Accordingly, the variables α_1 and α_2 become α_{1d} and α_{2d}, respectively. Based on equations (B2a) and (B2b), one can obtain

$$\alpha_{1b} = \frac{\beta_2 - 1}{\beta_2 - \beta_1} = -\frac{T_1\lambda_2^2 + iS_1\omega}{T_1(\lambda_1^2 - \lambda_2^2)} = \alpha_{1d} \quad \text{(B4a)}$$

$$\alpha_{2b} = \frac{1 - \beta_1}{\beta_2 - \beta_1} = \frac{T_1\lambda_1^2 + iS_1\omega}{T_1(\lambda_1^2 - \lambda_2^2)} = \alpha_{2d} \quad \text{(B4b)}$$

$$\beta_1 = 1 - \frac{SB_1}{2a_1^2 u_i} - \frac{Si}{u_i} = 1 - \frac{T_1\lambda_1^2}{L_i} - \frac{iS_1\omega}{L_i} \quad \text{(B4c)}$$

$$\beta_2 = 1 - \frac{SB_2}{2a_1^2 u_i} - \frac{Si}{u_i} = 1 - \frac{T_1\lambda_2^2}{L_i} - \frac{iS_1\omega}{L_i}. \quad \text{(B4d)}$$

Equations (B4a) and (B4b) can be found in Jeng *et al* (2002, equation 14). In addition, equations (3a) and (3b) are exactly the same as the ones of Jeng *et al* (2002, equations 12 and 13) except that the variables of β_1 and β_2 are in terms of dimensionless parameters.

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