Application of Weibull model for redefined significant wave height distributions

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It is well accepted that the parent distribution for individual ocean wave heights follows the Weibull model. However this model does not simulate significant wave height which is the average of the highest one-third of some 'n' (n-varies) wave heights in a wave record. It is now proposed to redefine significant wave height as average of the highest one-third of a constant number (n-constant, say, $n = 100$) of consecutive individual wave heights. The Weibull model is suggested for simulating redefined significant wave height distribution by the method of characteristic function. An empirical support of 100.00% is established by χ^2-test at 0.05 level of significance for 3 sets of data at 0900, 1200 and 1500 hrs at Valiathura, Kerala coast. Parametric relations have been derived for the redefined significant wave height parameters such as mean, maximum one-third average, extreme wave heights, return periods of an extreme wave height and the probability of realising an extreme wave height in a time less than the designated return period.

1. Introduction

Detailed information on wave climatology is essential for all maritime activities including construction of coastal and offshore structures, harbours, offshore oil exploration and shipping activities (Draper 1973). Efficacy of marine structures and their cost analysis require a fairly good estimate of wave conditions (Ploeg 1968; Draper 1970). Often information on freak ocean waves and extreme wave conditions is needed in the design of offshore structures (Draper 1964; Thom 1971; Draper 1973). The only way to obtain reliable information on wave climate is to study them instrumentally and theoretically in all geographical areas.

Statistically meaningful understanding of random phenomena as waves requires a definite system of analyses through appropriate statistical techniques (Gouveia and Mahadevan 1983). The important statistical functions used to describe the basic properties of such random processes are

- probability distributions
- mean values
- auto correlation functions and
- spectral density functions.

Muraleedharan (1991) tried to identify the long-term wave height distribution pattern by probability density functions and derived therefrom parametric relations of certain important wave statistics. After bringing logical and experimental support for the validation of using the probability density functions for modelling waves, the wave statistics have been computed from visual wave observations. They are then compared with recorded wave measurements and utilised to infer the wave climate along the south-west coast of India (NIO 1982; NPOL 1978).

2. Materials and methods

The probability distributions used to model long-term distributions of wave heights are generally the log-normal, exponential, Weibull and Gumbel distributions. Studies have shown that none is superior to the

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others (Dattatri 1981; Baba 1985). For deep water conditions, the ratio of mean wave height (H_{mean}) to depth (d) of wave recording station tends to '0' ($(H_{\mathrm{mean}}/d) \rightarrow 0$) and the Gluhovski distribution assumes the form of Rayleigh distribution (Shahul Hameed and Baba 1985). The Rayleigh distribution has also been recommended for long-term wave height modelling by Dattatri *et al* (1976).

The theoretical supremacy of the Weibull model based on its shape parameter taking care of the sea conditions suitable for the Longuet-Higgins model (Longuet-Higgins 1952) and also for situations under more general conditions has been shown by Muraleedharan (1991) in terms of the intensity function for decaying of waves. The motivation for considering Weibull model for long-term distribution of wave heights has been established empirically (Muraleedharan *et al* 1993).

Significant wave height is not a single value but the average of some 'n' wave heights in a wave record (n – varies, t – constant). If the significant wave height is redefined as the average of the highest one-third of a constant number of consecutive wave heights (n – constant, say, $n = 100$, t – varies), and since the Weibull model is confirmed as the parent distribution for individual wave heights, the model is suggested for the simulation of the redefined significant wave height distribution by the method of characteristic function.

2.1 *Characteristic function of Weibull model*

The density function of Weibull model is given by

$$f(h)\mathrm{d}h = (b/a)(h/a)^{b-1}\exp-(h/a)^b\,\mathrm{d}h;\quad a,\ b>0,\ h>0,$$

where 'a' is the scale parameter, 'b' is the shape parameter and 'h' is the variable. The characteristic function of Weibull model is given by

$$E(\mathrm{e}^{ith}) = \int_0^\infty \mathrm{e}^{ith} f(h)\mathrm{d}h = \int_0^\infty \mathrm{e}^{ith}(b/a)(h/a)^{b-1}\exp-(h/a)^b\mathrm{d}h,$$

where 't' is an arbitrary real constant.

$$= (b/a)\int_0^\infty (\cos th + i\sin th)(h/a)^{b-1}\exp-(h/a)^b\,\mathrm{d}h$$

$$= (b/a)\left[\int_0^\infty \cos th(h/a)^{b-1}\exp-(h/a)^b\,\mathrm{d}h + \int_0^\infty i\sin th(h/a)^{b-1}\exp-(h/a)^b\mathrm{d}h\right]. \quad (A)$$

Integrating the first expression of (A) by parts,

$$(b/a)\left[\int_0^\infty \cos th(h/a)^{b-1}\exp-(h/a)^b\mathrm{d}h\right]$$

$$= (b/a)\left[\cos th\int (h/a)^{b-1}\exp-(h/a)^b\,\mathrm{d}h - \int\left[(\mathrm{d}(\cos th)/\mathrm{d}h)\int (h/a)^{b-1}\exp-(h/a)^b\mathrm{d}h\right]\mathrm{d}h\right].$$

Put $(h/a)^b = y$

$$= 1 - t\int \sin th\exp-(h/a)^b\,\mathrm{d}h. \quad (1)$$

Now integrating the second expression of (A) by parts

$$(b/a)\left[\int i\sin th(h/a)^{b-1}\exp-(h/a)^b\mathrm{d}h\right]$$

$$= (ib/a)\left[\sin th\int (h/a)^{b-1}\exp-(h/a)^b\,\mathrm{d}h - \int\left[(\mathrm{d}(\sin th)/\mathrm{d}h)\int (h/a)^{b-1}\exp-(h/a)^b\mathrm{d}h\right]\mathrm{d}h\right].$$

Put $(h/a)^b = y$

$$= it\int \cos th\exp(-(h/a)^b)\mathrm{d}h. \quad (2)$$

Adding (1) and (2) $\Rightarrow 1 - t\int \sin th\exp-(h/a)^b\,\mathrm{d}h + it\int\cos th\exp-(h/a)^b\,\mathrm{d}h$

$$= 1 + it\left[\sum((it)^r/r!)\int_0^\infty h^r\exp-(h/a)^b\,\mathrm{d}h\right];\quad r = 0, 1, 2, \ldots, \infty$$

put $(h/a)^b = y$

$$= 1 + (it/b)\left[\sum((it)^r/r!)a^{r+1}\int_0^\infty y^{((r+1)/b)-1}\mathrm{e}^{-y}\mathrm{d}y\right]$$

$$\int_0^\infty \mathrm{e}^{-y}y^{((r+1)/b)-1}\mathrm{d}y$$

is the well known gamma function.

$$\therefore \phi(t) = 1 + \sum((it)^{r+1}/r!)(a^{r+1}/b)\Gamma((r+1)/b)$$

$\phi(t)$ is the characteristic function of the Weibull model.

2.2 *Sampling distribution for redefined significant wave heights*

It is to be noted that the characteristic function of the sum of 'n' independent variables is the product of their C. F. This simple property enables us to find the sampling distribution of the significant wave heights. (Maurice *et al* 1964). If we have a sample of 'n' values from a population whose characteristic function is $\phi(t)$, the characteristic function of their sum is ϕ^n.

The characteristic function of the Weibull model has been derived to be

$$\phi(t) = 1 + \sum((it)^{r+1}/r!)(a^{r+1}/b)\Gamma((r+1)/b).$$

Therefore the characteristic function of the sum of 'n' values is given by

$$\phi^n(t) = \left[1 + \sum((it)^{r+1}/r!)(a^{r+1}/b)\Gamma((r+1)/b)\right]^n.$$

Therefore the C.F. of the mean of 'n' values is given by

$$\phi^n(t)=\left[1+\sum((it/n)^{r+1}/r!)(a^{r+1}/b)\Gamma((r+1)/b)\right]^n.$$

2.3 *Approximation of the characteristic function*

The characteristic function of the Weibull model is given by

$$\phi(t)=1+\sum((it)^{r+1}/r!)(a^{r+1}/b)\Gamma((r+1)/b).$$

$\therefore$ The C.F. of the sum of 'n' values is given by

$$\phi^n(t)=\left[1+\sum((it)^{r+1}/r!)(a^{r+1}/b)\Gamma((r+1)/b)\right]^n.$$

When 't' is sufficiently small then each term in this series < 1 and it will be a decreasing series and approximation to the first two terms are valid.

Therefore

$$\phi^n(t)=1+n\sum((it)^{r+1}/r!)(a^{r+1}/b)\Gamma((r+1)/b).$$

Therefore the characteristic function of the mean ($1/n$ of that sum) is given by

$$\Psi(t)=1+n\sum((it/n)^{r+1}/r!)(a^{r+1}/b)\Gamma((r+1)/b).$$

By inspection the density function of the mean could be found. Generally the characteristic function of the statistic and the characteristic function of the parent model are compared and the modifications that have occurred in the former are noticed and appropriate changes are made in the density function of the parent model for providing the density function of the statistic. Here such a direct comparison is difficult as the characteristic function is expressed as a summation series in an infinite form with a gamma factor. Hence an approximation to the density function for the statistic is made by inspection and it is seen to be,

$$f(h)=(b/an)(h/an)^{b-1}\exp-(h/an)^b.$$

The proposed density function poses all the moments and characteristic function of the parent Weibull model for individual wave heights. The interval of the variable is '$1/n$' instead of unity and hence this modified version of the Weibull model is the density function of the redefined significant wave heights (H_{rs}).

An approximation to the density function of the mean was found to be $\mathrm{d}F=(b/an)(h/an)^{b-1}\exp-(h/an)^b\,\mathrm{d}h$, which is the proposed model for redefined significant wave heights.

2.4 *Testing procedures for characteristic function*

Looking to the definition of $\phi(t)$ as $\int_{-\infty}^{\infty}e^{ith}\,\mathrm{d}F$, we see that the necessary conditions for $\Psi(t)$ to be a characteristic function are:

- That $\Psi(t)$ must be continuous in t,

  $$\Psi(t)=1+n\sum((ita/n)^{r+1}/br!)\Gamma((r+1)/b)$$

  is clearly continuous in t.
- That $\Psi(t)$ is defined in every finite t-interval,

  $$\Psi(t)=1+n\sum((ita/n)^{r+1}/br!)\Gamma((r+1)/b),$$

  is defined in every finite t-interval.
- That $\Psi(0)=1$,

  $$\Psi(0)=1+n\sum((i0a/n)^{r+1}/br!)\Gamma((r+1)/b)=1,$$

  which satisfies the condition.
- That $\Psi(t)$ and $\Psi(-t)$ shall be conjugate quantities,

  $$\Psi(t)=1+n\sum((ita/n)^{r+1}/br!)\Gamma((r+1)/b)$$

  and

  $$\Psi(-t)=1+n\sum((-ita/n)^{r+1}/br!)\Gamma((r+1)/b).$$

  These are conjugates.
- $|\Psi(t)|\leq\int|e^{ith}|\mathrm{d}F\leq 1=\Psi(0)$.

  $\Psi(t)$ when approximated to first two terms becomes

  $$\Psi(t)=1+n(ita/nb)\Gamma(1/b).$$

Therefore

$$|\Psi(t)|=\sqrt{(1+[ta\Gamma(1/b)]^2/b^2)};$$

$$\Psi(0)=1\int_{-\infty}^{\infty}|e^{itx}|\mathrm{d}F=1$$

i.e. When 't' is sufficiently small,

$$|\Psi(t)|=\int_{-\infty}^{\infty}|e^{itx}|\mathrm{d}F=\Psi(0).$$

Hence the conditions are satisfied. Therefore $\Psi(t)$ represents a characteristic function and its density function is approximated to be

$$\mathrm{d}F=(b/na)(h/na)^{b-1}\exp-(h/na)^b\mathrm{d}h$$

which is the sampling model for redefined significant wave heights.

- Another commonly adopted testing procedure for the validation of the sampling model for mean is that the average computed from the sampling model and the parent model for the population should be the same,

  $$\phi(it)=1+\sum((ita)^{r+1}/br!)\Gamma((r+1)/b)$$
  $$=1+ita\Gamma(1/b)/b0!+(ita)^2\Gamma(2/b)/b1!+\cdots$$

Put $it=\theta$.

Then $[\mathrm{d}\phi(\theta)/\mathrm{d}\theta]_{\theta=0}$ gives the mean of the parent Weibull model for wave height population, i.e.

$$\phi(\theta)=1+\theta a\Gamma(1/b)/b0!+(\theta a)^2\Gamma(2/b)/b1!+\cdots$$

$$\mathrm{d}\phi(\theta)/\mathrm{d}\theta=a\Gamma(1/b)/b0!+2\theta a^2\Gamma(2/b)/b1!+\cdots$$

$$[\mathrm{d}\phi(\theta)/\mathrm{d}\theta]_{\theta=0}=a\Gamma(1+1/b).$$

The mean of the sampling model for redefined significant wave heights can be derived in the same way. The characteristic function of the sampling model as given above is

$$\Psi(it) = 1 + n\sum((ita/n)^{r+1}/br!)\Gamma((r+1)/b).$$

Putting $it = \theta$

$$\Psi(\theta) = 1 + n\sum((a\theta/n)^{r+1}/br!)\Gamma((r+1)/b)$$

$$[\mathrm{d}\Psi(\theta)/\mathrm{d}\theta]_{\theta=0} = na\Gamma(1/b)/nb = a\Gamma(1+1/b)$$

which is the mean of the sample model and is equal to the population mean and hence an important result in the distribution theory is established.

The mean for the sampling model for redefined significant wave heights is also derived by considering the expectation of the model, i.e.

$$E(h/n) = \int_0^\infty (h/n)f(h)\mathrm{d}h,$$

$$\text{where } h = h_1 + h_2 + h_3 + \cdots + h_n$$

$$= (1/n)na\Gamma(1+1/b) = a\Gamma(1+1/b),$$

the interval being '$1/n$' instead of unity; and hence this distribution is that of the mean i.e. expectation of '$1/n$' of the sum (i.e. h) is equal to the population mean and hence it is evident that this represents the model for redefined significant wave heights.

$$\therefore \mathrm{d}F = (b/na)(h/na)^{b-1}\exp-(h/na)^b\mathrm{d}h$$

is the density function of the redefined significant wave height.

3. Results and discussions

The parametric relations derived for redefined significant wave height parameters from the model are given below.

- Mean redefined significant wave height ($\bar{H}_{rs}$).

The mean of the distribution is

$$\bar{H}_{rs} = a\Gamma(1+b^{-1}).$$

- Most probable maximum redefined significant wave height ($H_{rs(\mathrm{MPM})}$).

The most probable maximum redefined significant wave height is the mode of the distribution $f(H_{rs(\max)})$. This is

$$H_{rs(\mathrm{MPM})} = a(1-1/mb)^{1/b}$$

where 'm' is the sample size.

- Maximum redefined significant wave height ($H_{rs(\max)}$).

$$H_{rs(\max)} = (a/b)[m\Gamma(1/b)/1! - m(m-1)\Gamma(1/b)/2!2^{1/b} + \cdots + (-1)^{r+1}\Gamma(1/b)/m^{1/b}].$$

- Average of the highest one-third redefined significant wave height ($H_{rs(1/3)}$).

Average of the highest one-third wave heights of the total redefined significant wave heights was found to be

$$H_{rs(1/3)} = a(\log 3)^{1/b} + (a/b)3I_{(\log 3)}(1/b)$$

where $I_{(x)}(P) = \int_x^\infty \mathrm{e}^{-t}t^{p-1}\mathrm{d}t$, is the incomplete gamma function.

Similarly the average of the one-tenth highest wave heights of the total redefined significant wave heights is derived as

$$H_{rs(1/10)} = a(\log 10)^{1/b} + (a/b)10I_{(\log 10)}(1/b).$$

- Extreme redefined significant wave height ($H_{rs(\mathrm{extreme})}$).

An extreme redefined significant wave height to occur in a given period of time 't' is derived to be

$$H_{rs(\mathrm{extreme})} = a\{-\log[1-(1-1/t)^{1/n}]\}^{1/b},$$

where 'n' is the time of observation.

- Analysis of return periods.

The return period of an extreme redefined significant wave height is derived as

$$t = \{1-(1-\exp-(h_{rs(\mathrm{extreme})}/a)^b)^n\}^{-1}$$

where 'n' is the time of observation.

- Probability of realising an extreme redefined significant wave height $H_{rs(\mathrm{extreme})}$ less than a designated duration (m) in a given period of time 't' is found to be

$$P = 1 - \{1-(1-\exp-(h_{rs(\mathrm{extreme})}/a)^b)^n\}^M$$

where

$$M = m/t; m < t.$$

n = time of observation.

Earlier models considered significant wave height as average of individual wave heights during a constant time interval, the number of waves considered being unknown and variable. The present model makes use of a redefined significant wave height which is average of a constant number of individual wave heights, yielding the model statistically more reliable. Since the characteristic function of the mean qualifies all the tests for a function to be a characteristic function, the model will explain the long-term distribution pattern of redefined significant wave heights effectively and hence the parametric relations derived from this model will more effectively predict the various redefined significant wave height parameters.

The redefined significant wave heights are computed for 3 sets of recorded wave data at 0900, 1200 and 1500 hrs for Valiathura, Kerala coast (6 days recording from 24th to 29th June 1984). The Weibull model simulates the actual distribution of the redefined significant wave heights averaged as above in 100.00% cases using χ^2-test at 0.05 level of significance suggesting that the model could be used for simulating redefined significant wave height distributions and the various redefined wave height parameters obtained using the parametric relations derived from the model will be sufficiently accurate for ocean engineering activities.

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