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The climate attractor

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Abstract. Time series of proxy data representing long-term variation of the terrestrial climate presumably show aperiodic changes, which has given rise to the hypothesis that the dynamics of the earth's climate is governed by a strange attractor. Here a study of such attractors is presented, with emphasis on determination of its dimension and the reported results. Finally, a one dimensional delayed albedo feedback climate model is discussed with the related strange attractor and its dimension.

Keywords. Attractor; dimension; fractal; delay.

1. Introduction

Time series of proxy data representing variations of the earth's climate (Hays *et al* 1976) show aperiodic behaviour. On any scale these plots show irregular fluctuations. The history of the search for glaciation cycles is a very long one, the main interest being in the possibility of climate prediction. However, that dream has remained elusive due to the aperiodic nature of the climatic time series–a result of non-linear instabilities in the terrestrial climate system.

Such phenomena may be conveniently studied by constructing mathematical models of the system. The behaviour of many such models can be understood from the nature of their attractor sets. An attractor set is that region (invariant submanifold) in the phase space where the trajectories of the system converge starting from any initial condition (basins) after the transients have died down. It follows that attractor sets can exist only for dissipative systems, because a shrinking of the volume in phase space for conservative systems is ruled out by Liouville's theorem. If the system develops deterministically then the attractor set generally is a low dimensional smooth topological set, e.g., a point, a limit cycle, or a torus. Then the trajectories do not diverge from each other (i.e., stay at a constant distance) and long term predictability is preserved. For some nonlinear dynamical systems, however, the trajectories diverge and the attractor sets are 'fractal' (Mandelbrot 1977) sets, i.e., sets with non-integer dimensions. These attractors are called 'strange' attractors. Their strangeness arises out of the sensitivity to initial conditions. But, the solutions for conservative systems also show similar sensitivity. Although the region visited by a strange attractor in phase space cannot be specified with anymore accuracy than a cloud of points, the system executes a perpetual wandering motion in that region.

2. Attractor dimension

The dimension of an attractor set (d) can be determined by embedding the attractor into an m dimensional phase space made up of the original time series and its m-1 derivatives (Whitney 1936; Takens 1981), where $m = 2d + 1$. Hence, as long as $n > m$ (where n is the dimension of the full phase space, i.e., the number of original variables) the single variable time series can be used for the attractor dimension computation. One reason why such computations are important is that the attractor dimension, in turn, determines the minimum number of variables needed to describe the dynamic system. Attractors of simple systems are characterized by low dimensional topological sets, which give small integral value for the dimension. Thus, for a point attractor $d = 0$, and for a limit cycle $d = 1$. For a torus, $d = 2$ or higher. A strange attractor, on the other hand, is characterized by fractal dimension, which yields noninteger value for d; this is another reason why determination of the attractor dimension is important. It helps to identify strange attractors relatively easily. The fractal set mentioned above is really a cloud of points in phase space through which the dynamic system is wandering.

In this article by 'attractor dimenion' we will mean a correlation dimension. Before we present this we have to define a correlation function. It is defined as follows

$$C(R) = \frac{1}{N^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} H(R - |\vec{X}_i - \vec{X}_j|) \tag{2.1}$$

where $H(x)$ is the Heaviside function. This correlation function represents the number of data points X_j within a distance R from a reference point X_i, for all values of i. N is the number of the observation points being analysed.

If we consider a hypersphere of radius R in d-dimensional phase space then the correlation function is proportional to R, where d is the correlation dimension. For smooth manifolds d is an integer, while for fractals d is not an integer. This can be utilized for determining the dimension of a regular attractor as well as that of a strange attractor. This is the basis of an algorithm provided by Grassberger and Procaccia (1983) which has been extensively used. It is an advance on the earlier box-counting algorithm (Greenside *et al* 1982) which converged very slowly.

3. Computation of attractor dimension

Starting from the original time series (Ruelle 1980) n new time series are constructed by shifts of $\tau, 2\tau, \ldots, (n-1)\tau$ to each original data point $X_0(t)$ of the original time series. In other words, we have

$$\begin{gathered}
X_0(t_1),\ X_0(t_2), \ldots\ldots, X_0(t_{N-1}),\ X_0(t_N) \\
X_0(t_1 + \tau),\ X_0(t_2 + \tau), \ldots\ldots, X_0(t_{N-1} + \tau),\ X_0(t_N + \tau) \\
\vdots \\
X_0(t_1 + (n-1)\tau),\ X_0(t_2 + (n-1)\tau), \ldots, X_0(t_{N-1} + (n-1)\tau), \\
X_0(t_N + (n-1)\tau)
\end{gathered} \tag{3.1}$$

If n is large enough so that $n > m$ then Whitney's embedding theorem guarantees that the d dimensional attractor can be embedded safely into an n dimensional phase space. Thus, the first step is to choose an appropriate shift which is greater than the autocorrelation time of the series. This ensures the linear independence of the phase space axes. This is easily determined by plotting the autocorrelation against delay time and locating the delay where the first zero occurs. Next one computes the values of the correlation function $C(r)$ for different values of r, and a log-log plot of r against $C(r)$ should yield a straight line. Starting from a small value of n it is successively increased and a straight line is plotted for each of them. It is found that with increasing n the slopes of the straight lines reach a limiting value. This limiting value is the attractor dimension (d) and the value of n at which this occurs is the minimum number of equations needed to describe the physical system. However, the success of this procedure depends on the length of the time series and also the data points have to be free from artificial dependencies that are introduced by smoothing. This has been repeatedly emphasized by Grassberger and Procaccia (1983). It is the basis of criticism of the work of Nicolis and Nicolis (1984) on determination of the attractor dimension from the paleoclimatic time series.

Nicolis and Nicolis (1984) applied the above procedure on the deep sea core V28-238 obtained from the equatorial Pacific and obtained $d = 3{\cdot}1$, which implies the presence of a strange attractor in the earth's long term climatic record. However, this interesting result was criticised by Grassberger (1986). According to him the fact that 184 raw data points were interpolated to 500 points (Pestiaux 1984) means that the data points were not uncorrelated (due to smoothing) and even this number is much less than the minimum number of data points needed for accurate and confident determination of the attractor dimension. Grassberger (1986) concluded that from this data set the lower bound of the dimension of the climate attractor, if it exists, cannot be determined and the climate system cannot be described by only four equations. Fraedrich (1986) also found evidence for a low dimensional climatic attractor, but only after he separated the data into summer and winter parts.

Low dimensional attractors have been found in weather time series. Essex *et al* (1987) computed the attractor dimension from a series of daily objectively analysed 500 mb geopotential heights at 12 GMT for about 40 years. They found a dimension of 6·2 which can be embedded in a phase space of 9 dimensions. Tsonis and Elsner (1988) applied it to a series of vertical wind velocities near the ground for 11 hours. They found $d = 7{\cdot}3$ with an embedding dimension of 8. However, this work was criticised by Procaccia (1988) for the same reason which led Grassberger (1986) to criticize Nicolis and Nicolis (1984). Keppenne and Nicolis (1989) studied height data from five European radiosonde stations by three different methods including the present one and found that $6{\cdot}8 < d < 8{\cdot}4$. The corresponding minimum phase space dimension varied between 7 and 9.

4. An example of a strange attractor: One dimensional delayed feedback climate model

So far we have discussed the detection of strange attractors and the determination of their dimensions from the time series of observed data. We have seen that strange attractors have been found in many cases of weather and climate data. Now we will show how strange attractors can even be found in the behaviour of a simple one dimensional non-linear model. In this case the number of variables of the model is known in advance, so the results will provide a confirmation of the theory.

The model will be described briefly; interested readers may refer to earlier work on the topic (Bhattacharya *et al* 1982; Bhattacharya 1991). It is a one-dimensional energy-balance model of Sellers (1969) that has been extended in several ways. The model equation is

$$G(x)\frac{\partial T(x,t)}{\partial t} = R_i[x, T(x,t)] - R_0[T(x,t)] + F\left[x, t, \frac{\partial T(x,t)}{\partial x}, \frac{\partial^2 T(x,t)}{\partial x^2}\right] \tag{4.1}$$

where $x = 2\Phi/\pi$, Φ being the colatitude. $T(x, t)$ is the surface temperature that depends on latitude and time only. $G(x)$ is the latitude dependent heat capacity of the earth's surface.

R_i, the absorbed part of the incoming solar radiation, is represented as

$$R_i = \mu Q(x)\{1 - \alpha[x, T(x,t)]\} \tag{4.2}$$

where $Q(x)$ is the meridional distribution of the incoming solar radiation, α is the surface albedo and μ is a normalizing constant. $\mu = 1$ corresponds to present day solar radiation.

R_0, the outgoing longwave radiation from the earth's surface, is represented as

$$R_0 = c[T(x,t)]\cdot\sigma\cdot[T(x,t)]^4 \tag{4.3}$$

where σ is the Stefan-Boltzmann constant and c is the emissivity coefficient.

The meridional heat transport along the earth's surface (F) is given by a linear diffusive approximation

$$F = \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}[\sin\phi\cdot k(\phi)]\frac{\partial T}{\partial\phi} \tag{4.4}$$

where $k(X)$ is a latitude dependent constant. This term models both atmospheric and oceanic sensible heat fluxes, as well as the atmospheric latent heat fluxes.

The model has several new features, two of which affect the results greatly. These are now being described in some detail.

The first is the introduction of a time delay in the albedo. This implies that whenever snow or ice is present at the surface, the albedo was determined not only by the current temperature but also by the earlier temperatures. The weights given to the earlier temperatures decrease exponentially as the time recedes to the past, with the maximum weight at the delay time $t = \tau$. There is a weight cut off at $t = 2\tau$. When snow or ice is not there, the surface albedo is of course determined by the current surface temperature. Physically, this reflects the thermal inertia in growth and decay of continental ice sheets.

The second feature is the addition of a jump in the albedo-temperature curve. Physically, this represented the fact that near the edge of the ice cap there is intense baroclinic activity due to a strong temperature gradient which is reflected by increased cloudiness. This has been observationally supported by several authors (Goody 1980; Schwerdtfeger and Kachelhoffer 1973). Consequently, the albedo increase from the equator to pole is not monotonic.

The other minor features involve the way the global surface heat capacity was computed, and the heat transport was determined by only one temperature distribution, that of the present climate. The details are given in Bhattacharya *et al* (1982).

A time independent (steady state) version of the model yielded multiple solutions through Hopf bifurcations; for the current climate ($\mu = 1$) five solutions were obtained against three in the original Sellers model. These are being denoted as $T_1(x)$, $T_2(x),\ldots, T_5(x)$. It was shown in Bhattacharya *et al* (1982) that both the warmest $(T_1(x))$ and coldest $(T_5(x))$ solutions are linearly stable; the former corresponds to the present climate, while the latter corresponds to a 'deep freeze' state. The third solution $(T_3(x))$ is also linearly stable, while the two other intermediate solutions are linearly unstable.

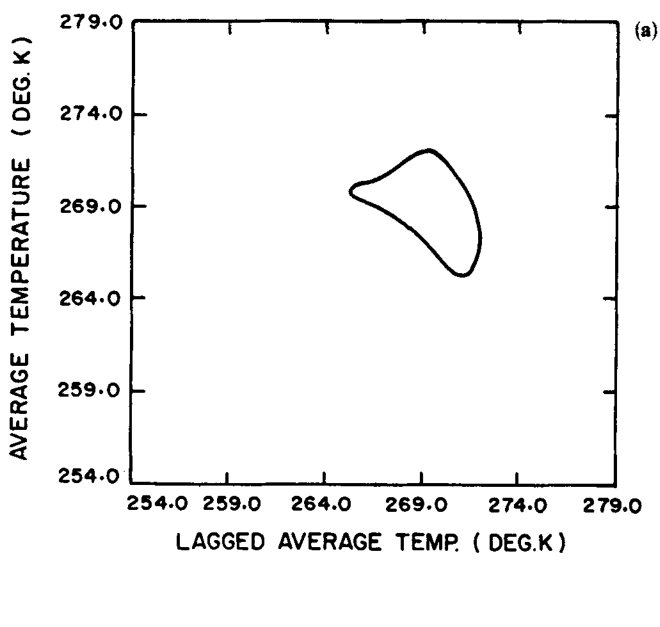
Solutions of the time dependent version of the model showed an interesting behaviour. Warm initial perturbations to the solution representing the present climate only yielded either decaying or damped oscillations. 'Cold' perturbations, but not too cold, i.e.,

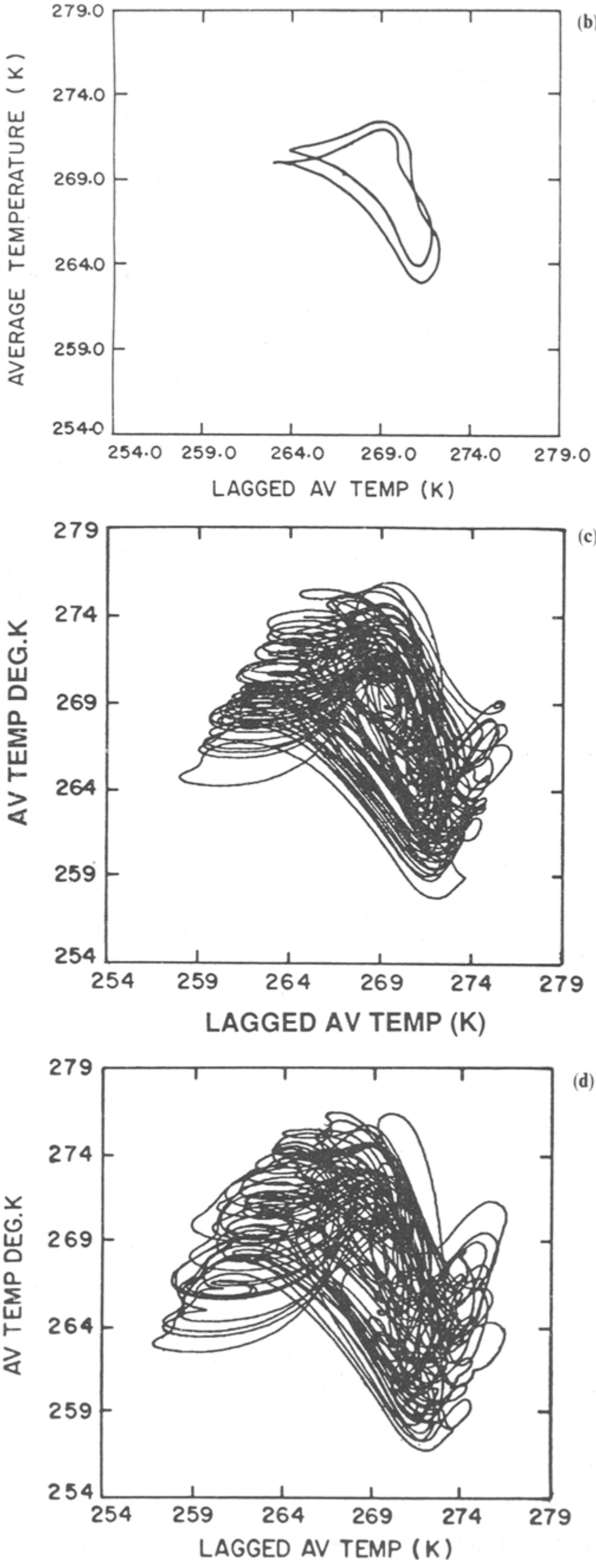
$$T_1(x) - 10\,\mathrm{K} < T(x,0) < T_1(x) - 7{\cdot}5\,\mathrm{K} \tag{4·5}$$

gave sustained small-amplitude oscillations for certain range of delay τ. For $\tau = 100$ Yr, simple periodic oscillations obtain. For $\tau = 165$ Yr, quasi-periodic oscillations, and for $\tau > 300$ Yr chaotic oscillations obtain, as shown in figures 1(a)–(e) of Bhattacharya (1991). The amplitudes were found to depend on the width of the normal distribution (used to represent the dependence of the surface albedo on the delay) in an interesting way.

In this paper, however, the main interest lies in the nature of the attractor set of this model and the way it changes with change in the delay τ. Figures 1(a)–(e) demonstrate rather clearly the limit cycle, the two-torus and then the sudden transition to the strange attractor as shown by the irregular motion in the phase plane diagrams as well as filling up of the well-defined region. The phrase 'period three implies chaos' is supported in this case because only a slight change in τ from 165 Yr to 170 Yr is enough to push the system from a motion of two in commensurate frequencies to chaotic motions. Using a zero dimensional model which is in many ways similar to ours, Andersson and Lundberg (1988) obtained similar plots (figures 4(a)–(b) of their paper); one of these figures show a three-period quasi periodic oscillation.

We also applied the Grassberger and Procaccia's (1983) algorithm for computing




(b)
AVERAGE TEMPERATURE (K)
279.0
274.0
269.0
264.0
259.0
254.0
254.0 259.0 264.0 269.0 274.0 279.0
LAGGED AV TEMP (K)
(c)
AV TEMP DEG.K
279
274
269
264
259
254
254 259 264 269 274 279
LAGGED AV TEMP (K)
(d)
AV TEMP DEG.K
279
274
269
264
259
254
254 259 264 269 274 279
LAGGED AV TEMP (K)

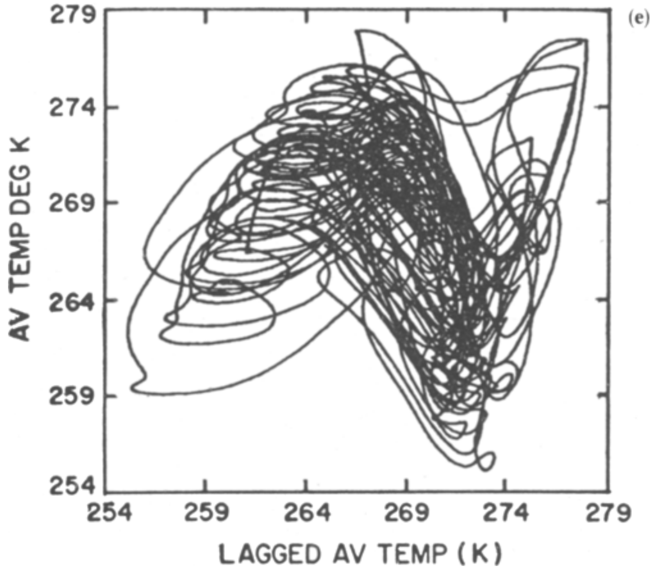


Figure 1. Phase plots for sections of the stable solutions. Values of the time delay τ for the five cases shown are: (**a**) 100 years (**b**) 165 years (**c**) 300 years (**d**) 500 years (**e**) 5000 years. Abscissa denotes the lagged hemispherically averaged temperature $T(t-\tau)$, and the ordinate denotes the hemispherically averaged temperature $T(t)$.

the correlation dimensions of the attractor of our model. In the strange attractor region, these dimensions are all fractal dimensions, with embedding dimension of 2; this is understandable, since only two variables (temperature and latitude) are needed to describe our model.

5. Discussion

There are two important issues on which the success of the procedure described previously on computation of attractor dimension depends. They are discussed below.

The first has already been mentioned briefly. This has to do with the fact that confidence in this kind of computation depends on the length N of the datasets as well as the accuracy of the uncorrelated data. The requirement $N > 42^{[D]}$ has been given by Smith (1988), where $[D]$ is the integer part of D, for an accuracy of 5 per cent. This was the basis of the criticism of the work of Nicolis and Nicolis (1984) by Grassberger (1986), as well as that of the work of Tsonis and Elsner (1988) by Procaccia (1988). Subsequent authors seem to have kept this in mind while reporting on dimension computations of attractors.

The other relates to the fact that it is very easy to mistake random fractal sequences for the time series representing chaotic dynamics (Tsonis and Elsner 1992). Random fractal sequences arise when coloured (correlated) noise is added to periodic signals. In this case the predictability may decay exponentially, just as in the case of chaos. The above mentioned authors have shown that by measuring the scaling properties of the prediction error as a function of time, it is possible to distinguish between these two processes. They tested the procedure on a time series of the Southern

Oscillation Index and found it to be indeed chaotic, thus confirming the results of Vallis (1986) and Hense (1987); according to the latter author the fractal dimension is around five.

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