Proc. Indian Acad. Sci. (Earth Planet. Sci.), Vol. 101, No. 1, March 1992, pp. 77–88.

Application of the Fourier method to the numerical solution of moving boundary problem in heat conduction

A MANGLIK, D V RAMANA, A O GLIKO* and R N SINGH

National Geophysical Research Institute, Hyderabad 500007, India

*Institute of Physics of the Earth, USSR Academy of Science, Moscow, USSR

MS received 30 April 1991; revised 29 November 1991

Abstract. An algorithm for the solution of a nonlinear problem of phase boundary movement and evolution of temperature distribution due to the perturbation in the basal heat flux has been discussed. The reduction of the problem to a system of nonlinear ordinary differential equations with the help of a Fourier series method leads to a stiff system. This stiffness is taken care of by the use of a modified Euler's method. Various cases of basal heat flow variation have been considered to show the performance and stability of the technique for such a nonlinear system. The first case of step-wise function is taken to analyse the performance of the technique, and the study has been extended to other general cases of linear increase, periodic variation, and box and triangular function type variations in the heat flux. In the step-wise case the phase boundary attains a constant position rapidly if the supplied heat flux is sufficiently large. The effect of periodicity in the heat flow is clearly depicted in the phase boundary movement, where the phase boundary oscillates about the mean position at large times. The absence of any constant level in the case of linear increase in heat flux is due to a very large value of heat flux. In the cases of box car and triangular heat flux the boundary starts moving downward after the cessation of excess heat flux but does not immediately return to its original preperturbation state, instead approaches it at large times. This technique may be applied to more general cases of heat flow variation.

Keywords. Fourier series; moving boundary; heat flux; stiffness.

1. Introduction

A problem of the kind in which the position of a boundary is determined as a function of space and time (Crank 1984), after the system is perturbed from its equilibrium state, is termed as the moving boundary problem (MBP). The simplest example of MBP is the classical Stefan's problem of freezing of material initially in liquid state in a half-space under a step-wise change in the surface temperature (Stefan 1889; Turcotte and Schubert 1982). In the Earth system the moving boundaries, with or without phase change, occur in various thermal, hydrological and tectonic processes. The problem of moving boundary with phase change is very complex because of the inherent nonlinearity of the system but is of great importance in the understanding of various phenomena related to the evolution of the Earth. For example, many geophysical and geological phenomena such as thickening and thinning of the lithosphere (Crough 1983; Withjack 1979; Spohn and Schubert 1982; Crough and Thompson 1976; Nakiboglu and Lambeck 1985; Kono and Ogawara 1989) uplift and subsidence of the Earth's surface in response to surface unloading and loading and subsurface phase transition (O'Connell and Wasserburg 1967,1972; Peltier 1974; Mareschal and Gangi 1977; Mareschal 1978), metamorphism (England and Richardson

1977; Thompson 1977, 1981; England 1978; Wells 1980) etc can be explained by the mathematical models which include a moving boundary as a part of the problem.

The mathematical simulation of these complex phenomena through the use of a system of nonlinear equations precludes the possibilities of obtaining a simple analytical solution. Different approximate analytic and numerical schemes have been discussed in literature to solve complex MBP. Some of these methods are the heat balance method (Goodman 1964; Crank 1975), quasi-steadystate approximation method (Lyubov 1980), nonlinear asymptotic approach (Tao 1978; Gliko 1986) and numerical methods (Crank 1984). The last group includes the finite difference, finite element and the Fourier method.

In the Fourier series method the unknown function is expressed in terms of a set of orthogonal functions (Gottlieb and Orszag 1977; Canuto *et al* 1988). The partial differential equation (PDE) is solved using this series expansion and the property of orthogonality of the functions describing the Fourier series. This reduces the PDE into an infinite set of ordinary differential equations (ODE), the solution of which gives the unknown coefficients of the Fourier series. In a transient problem e.g. a heat conduction problem, these unknown coefficients are time-dependent. Therefore the finite difference scheme in time is followed and the system of ODEs is solved for every time step to get the unknown Fourier coefficients (Rakitskiy *et al* 1979; Gliko and Rovensky 1985).

In this paper we discuss the applications of modified Fourier method, originally proposed by Melamed (1958) for the solution of problems related to frost propagation, to lithospheric heat conduction problems. This method has been used by Gliko and Rovensky (1985) to solve the problem of lithospheric thinning, treating it as a MBP, for step-wise and bell-shaped increase in the heat flux. We review the method and apply it to the more general types of heat flow variations. A software has been developed to this effect.

2. Mathematical formulation

The one-dimensional transient heat conduction equation for normalized thermal perturbation is given as:

$$\partial v/\partial t = \partial^2 v/\partial z^2, \quad 0 \leqslant z \leqslant \xi(t), \tag{1}$$

where

$$v(z,t) = u(z,t) - u_0(z),$$

where u is the normalized temperature distribution, $u_0(z)$ the initial temperature profile, $\xi(t)$ the moving boundary position, z the space co-ordinate positive downward, and t the normalized time. u, t and z are normalized with respect to T_m, τ and l_0 respectively, where T_m is the solidus temperature, τ the characteristic time (l_0^2/k; k is thermal diffusivity) and l_0 the initial lithospheric thickness.

The initial and boundary conditions are given by

$$v(0,t) = 0; \quad v(\xi(t),t) = \phi(\xi(t)); \quad v(z,0) = 0$$

$$\left.\frac{\partial v}{\partial z}\right|_{z=\xi(t)} + \left.\frac{\partial u_0}{\partial z}\right|_{z=\xi(t)} - Rq(t) = \alpha\frac{\mathrm{d}\xi}{\mathrm{d}t} + \beta\xi(t)\frac{\mathrm{d}\xi}{\mathrm{d}t}, \tag{2}$$

where q is heat flux normalized with respect to the initial heat flux Q_0 and ξ the normalized moving boundary position with respect to l_0. α, β and R are given as:

$$R = l_0 Q_0 / K T_m; \quad \alpha = \lambda / c T_m; \quad \beta = (\rho_l - \rho_a) g l_0 / \rho_l c T_m, \tag{3}$$

where K is the thermal conductivity of the lithosphere, ρ_l the density of the lithosphere, ρ_a the density of asthenosphere, g the gravity acceleration and λ the latent heat term.

We define a new transformation

$$V(z,t) = v(z,t) - \frac{z}{\xi(t)} \phi(\xi(t)) \tag{4}$$

such that

$$V(z,0) = 0; \quad V(\xi(t),t) = 0 \tag{5}$$

and express this variable in terms of an infinite set of orthogonal functions

$$V(z,t) = \frac{2}{\xi(t)} \sum_{k=1}^{\infty} A_k(t) \sin\left(\frac{\pi k z}{\xi(t)}\right) \tag{6}$$

at any time t.

The coefficients $A_k(t)$ of this Fourier series are unknown and can be obtained from the transformed system of (1) and (2) after the application of the property of orthogonality. The final expression for the coefficients of Fourier series is obtained in the form of first-order ordinary differential equation (ODE):

$$\begin{aligned} \frac{\mathrm{d}A_n(t)}{\mathrm{d}t} = & -\left(\frac{n\pi}{\xi(t)}\right)^2 A_n(t) - \frac{2n}{\xi(t)} \frac{\mathrm{d}\xi}{\mathrm{d}t} \sum_{k=1}^{\infty} A_k(t) P_{nk} \\ & - \frac{(-1)^n}{n\pi} (R-a) \frac{\mathrm{d}\xi}{\mathrm{d}t} + \frac{1}{\xi(t)} \frac{\mathrm{d}\xi}{\mathrm{d}t} A_n(t) \\ & n = 1, 2, \ldots, \infty, \end{aligned} \tag{7}$$

where

$$P_{nk} = \begin{cases} (-1)^{n+k} (k/(n^2 - k^2)) & n \neq k \\ -1/4n & n = k \end{cases}$$

and a is the slope of Clapeyron curve. This ODE system can be solved subject to initial conditions:

$$\xi(0) = 1; \quad A_n(0) = 0; \quad n = 1, 2, \ldots, \infty. \tag{8}$$

The position of the boundary at any time t is obtained from

$$(\alpha + \beta\xi(t)) \frac{\mathrm{d}\xi}{\mathrm{d}t} = \frac{2\pi}{\xi^2(t)} \sum_{k=1}^{\infty} (-1)^k k A_k(t) + \frac{(R-a)}{\xi(t)} (1 - \xi(t)) + R(1 - q(t)). \tag{9}$$

Equations (7) and (9) form a coupled system of first-order ODEs which can be solved iteratively to obtain the position of the boundary and the Fourier coefficients. These values of parameters at any time t can be substituted into (4) and (6) to get the perturbation in the temperature distribution within the domain.

3. Computational details

The coupled system which describes the phase boundary movement is solved using finite difference scheme because of the complexity of the system of ODEs. The modified Euler's method due to Melamed (1958) is used in the analysis. The system is given as:

$$A_n^{m+1} = A_n^m \exp(-(n\pi/\xi^m)^2 h) - \frac{d\xi^m}{dt}\left[\frac{2n}{\xi^m}\sum_{k=1}^{\infty} A_k^m P_{nk} + \frac{(-1)^n}{n\pi}(R-a) - \frac{1}{\xi^m}A_n^m\right] * \left[\left(\frac{n\pi}{\xi^m}\right)^{-2}(1-\exp(-(n\pi/\xi^m)^2 h))\right],$$

$$\xi^{m+1} = \xi^m + h\frac{R^m}{\alpha + \beta\xi^m}, \quad (10)$$

where

$$R^m = \frac{2\pi}{(\xi^m)^2}\sum_{k=1}^{\infty}(-1)^k k A_k^m + \frac{(R-a)}{\xi^m}(1-\xi^m) + R(1-q^m), \quad (11)$$

m denotes the mth time step and h denotes the discretization interval in time.

The perturbation in the temperature distribution or the phase boundary movement is expressed in terms of an infinite set of orthogonal functions but in actual computations only a finite set is considered. Therefore the solution thus obtained differs from the actual solution. The increase in the number of Fourier coefficients or the orthogonal functions reduces the error of misfit but in turn requires a large computation time. The computation time increases with the square of the number of coefficients considered. Therefore an optimization between the error of misfit and computer time is done by considering a sufficiently large number of orthogonal sets and the upper limit in summation is replaced by the finite integer value N.

A software has been developed to compute the phase boundary movement and the temperature distribution due to the increased time-varying basal heat flux. The input parameters to the program are: α the normalized Stefan number, β the coefficient describing the contribution due to density changes, N the total number of orthogonal functions used, q the heat flux, R the parameter describing the nature of initial temperature profile, h the time discretization interval and t_{max} the maximum time up to which the computation is required. The value of R, α and β can be computed from equation (3) if all the parameters describing these three coefficients are adequately known. For the present purpose, these are taken as 1·0, 0·1 and 0 respectively in all the computations with a view to understand the performance of the technique. The parameters required for the computation of R, α and β are taken from Turcotte and Schubert (1982; pp. 172). These values give α equal to 0·25 but only a value of 0·1 for this coefficient is used to simulate partial melting problem. β equal to 0 gives a case of no density changes between the two phases.

4. Results and discussion

The truncation of Fourier series in actual computation results in the deviation of the computed solution from the true solution. In figures 1a and 1b the effect of increase in the number of Fourier coefficients on the reduction of the error is analysed to

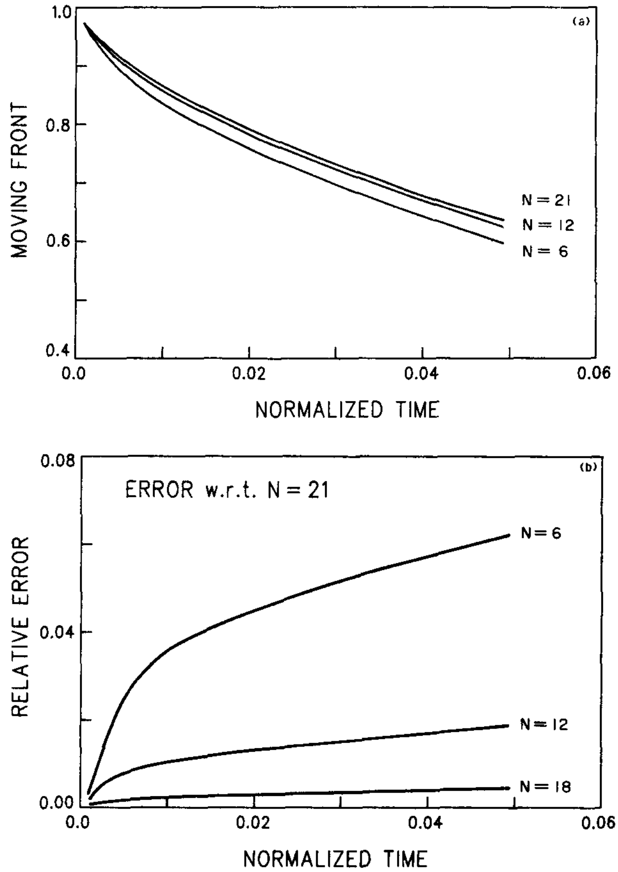

Figure 1. (**a**) Time-varying moving boundary position computed for three different values of total number of the Fourier coefficients N. These values are 6, 12 and 21. The positions of moving boundary show a converging trend with the increase in the number of Fourier coefficients; and (**b**) The relative error in the solution, incurred due to the truncation of the Fourier series. The relative error is computed w.r.t. $N = 21$ to show the convergence of the solution in the case of a complex nonlinear problem.

select an optimum value for the number of Fourier coefficients. In figure 1a, the error reduces drastically for N greater than 12. The convergence of the moving boundary position for increasing number of Fourier coefficients is shown in figure 1a which suggests that the solution of MBP converges to the actual solution. In the absence of the true analytical solutions for such complex MBP, this serves the purpose of the reliability estimates of the results obtained. The error of misfit is computed for different values of N with respect to the N equal to 21 (figure 1b). The relative error is about 6·2% for N equal to 6 at the time of 0·048 units where as for N equal to 12 and 18 it is 1·8% and 0·4% respectively. This step has been followed in the computation of the moving boundary position for different types of heat flux functions in order to get a reliable, convergent solution. The successive computations are carried out for N equal to 21.

Although a general function of heat flow variation can be incorporated in the computation of moving boundary position and temperature distribution, we consider only a limited number of cases, relevant for geophysical problems, of heat flow variation in the present study and analyse the nature of phase boundary movement.

4.1 *Case I*

In this case a step-wise function is considered for the additional heat flux. The function is defined as:

$$q(t) = q_0 H(t), \tag{12}$$

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geqslant 0 \end{cases}$$

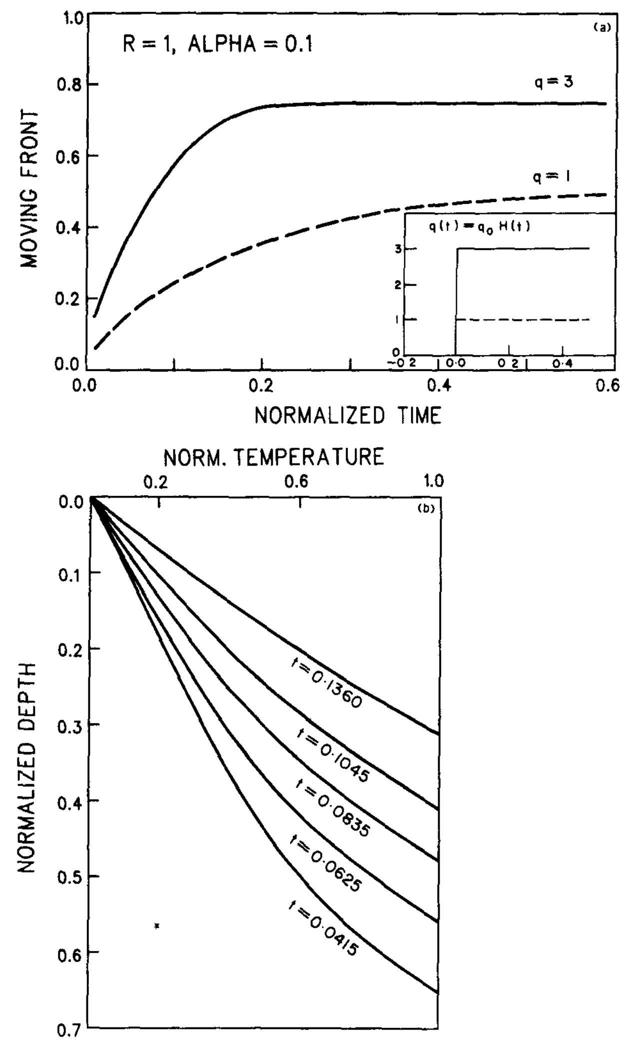

Figure 2. (**a**) Time-dependent moving phase boundary position due to the step-wise increase in the basal heat flux. The supplied additional heat flux is shown in the inset. The moving front position is computed for two values, 1 and 3, of additional heat flux q_0. The boundary approaches a constant position. (**b**) Temperature distribution in the domain at different times for additional heat flux q_0 equal to 3. The computations for only one value of heat flux are shown in the paper in order to optimize the space.

The phase boundary movement and the evolution of temperature profile is computed for two values of q (1, 3) and Stefan's number α equal to 0·1. The boundary moves slowly and attains a constant level at a depth of 0·56 in response to q equal to 1 (figure 2a). This signifies that the thinning of 44% is possible for this amount of additional heat flux. For q equal to 3 the phase boundary rapidly moves upward and attains a constant level at a depth of 0·24 in approximately 0·25 units of time (figure 2a). This amounts to thermal thinning of 76% in a short-time interval. The temperature evolution for q equal to 3 shown in figure 2b suggests that the temperature distribution has become steady-state in a very short time and there is a considerable increase in the surface heat flow.

4.2 *Case II*

As a second case we consider a linear increase in the heat flow with time and compute the nature of phase boundary movement. The heat flow function for this case is defined as:

$$q(t) = mt, \tag{13}$$

where m is the gradient of increase in heat flow.

The additional heat flow function is considered for the two values of gradient m (35, 54) as shown in figure 3 (inset). These two values are chosen arbitrarily but with the objective that the required basal heat flow should reach the high heat flux value in short time, thus departing slightly from the step-wise increase in the heat flux. For example, the heat flux in this case is constrained to reach a value in the range from 1·5 to 3·0 in a short time of 0·05 units. In this case only the phase boundary movement is analysed. The moving boundary position for these two values of gradient is shown in figure 3. The boundary does not reach any constant level as seen in the previous case of step-wise function because of the considerable increase in the heat flux values.

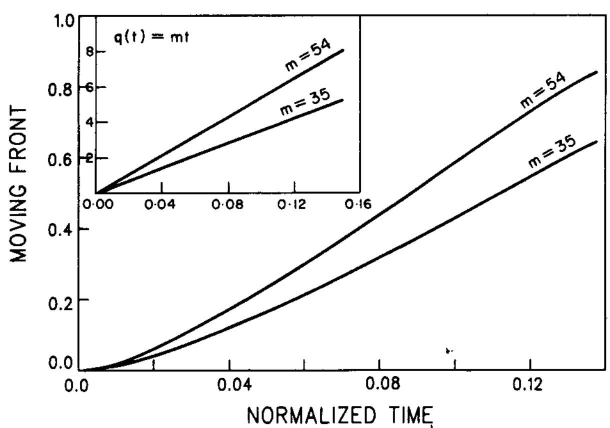

Figure 3. The moving phase boundary position in response to the linearly increasing additional heat flux (shown in the inset). The computations are carried out for two values, 35 and 54, of the heat flux gradient. In this case, the front does not reach any constant position because of the continuous increase in the heat flux.

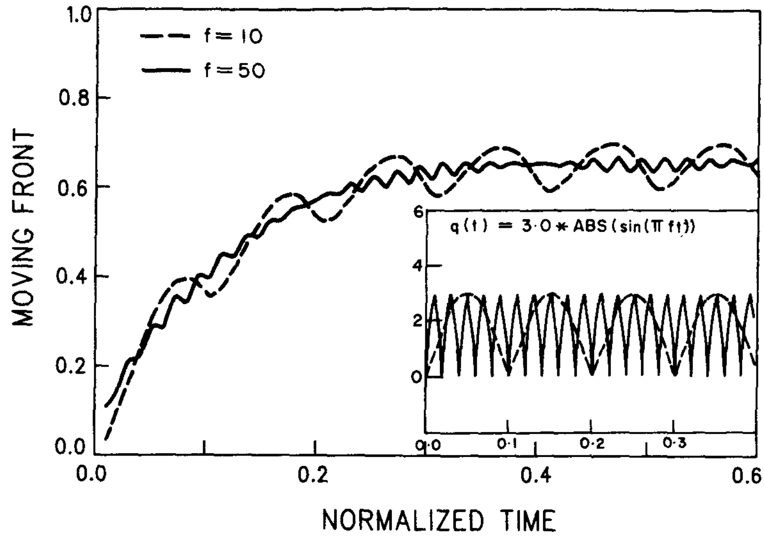

Figure 4. The moving phase boundary position due to the supplied heat flux given by $q(t) = q_0 \mathrm{abs}(\sin(\pi f t))$. The input function is shown in the inset. The value of q_0 is taken as 3 and computations are performed for f equal to 10 and 50. The oscillations in the moving front are superimposed on the mean position.

4.3 *Case III*

In this case a periodic function for additional heat flux is considered which is defined as:

$$q(t) = q_0 \mathrm{Abs}(\sin(\pi f t)) \tag{14}$$

where q_0 is equal to 3 in the present case.

Two values of f (10, 50) are considered to simulate the situation of low-frequency and high-frequency responses. The input function is shown in the inset of figure 4. The oscillatory nature of moving front is clear in figure 4. The boundary oscillates in response to the periodic heat flux but attains a constant level depending on the value of q_0, the amplitude of the periodic function. These oscillations reduce with the increase in frequency.

4.4 *Case IV*

A more general function of periodic heat flow variation is considered in this case, where the oscillatory behaviour of the heat flux is superimposed on a constant increase in the heat flux. This function is given as:

$$q(t) = QH(t) + \bar{q}\sin(\pi f t). \tag{15}$$

The heat flux is suddenly increased to a high value Q which oscillates as a Sine function. Q and $\bar{q}$ are taken as 3. The behaviour of oscillatory moving boundary is shown in figure 5 for two values 10 and 50 of f. The oscillations for f equal to 10 are very large but for $f = 50$ these are small. The boundary reaches a mean position with oscillations around it in a short time. This boundary oscillates between 0·55 and 0·83 for $f = 10$, and 0·69 and 0·79 for $f = 50$ at large times. The mean values of oscillations

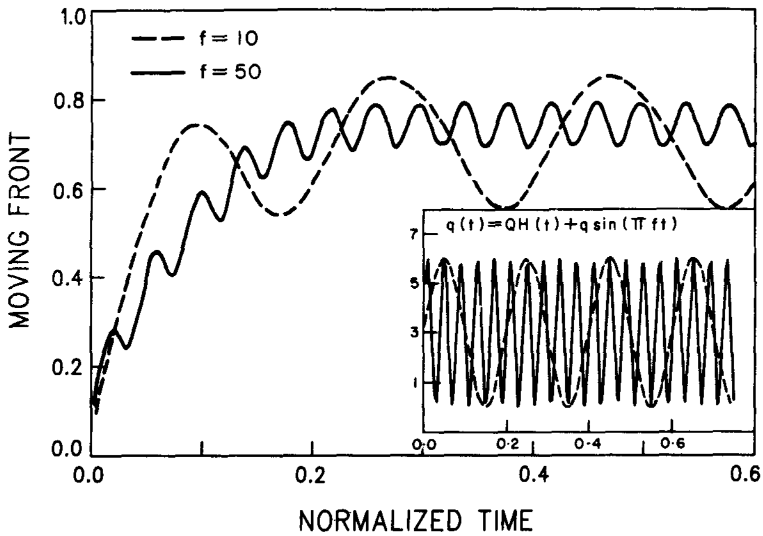

Figure 5. The moving phase boundary position due to the supplied periodic additional heat flux given by the function $q(t) = QH(t) + \bar{q}\sin(\pi ft)$. The nature of this function is shown in the inset. The computations are done for Q and $\bar{q}$ equal to 3, and f equal to 10 and 50 to simulate the low and high-frequency perturbations. The moving boundary oscillates about the mean position. The results of figure 2 are a special case of this function.

in this case are 0·69 and 0·74 respectively whereas in the case of step-wise function (figure 2a) the boundary attains a mean position of 0·76 for the corresponding heat flow.

4.5 *Case V*

In this case we consider a situation where heat flux increases suddenly to some large value for a finite time interval t_0. The heat flux variation is given as:

$$q(t) = q_0(H(t) - H(t - t_0)), \tag{16}$$

where $H(t)$ is given in equation (12).

The moving boundary position is computed for two values 1 and 3 of additional heat flux q_0. The basal heat flux is supplied for a short time (inset of figure 6). The boundary first moves upward in response to the heat flux and then starts moving downward after the cessation of the excess basal heat flux. The results are computed up to 0·6 units of time (figure 6). The boundary in both the cases of q_0 does not reach the initial boundary position but approaches this value at large times.

4.6 *Case VI*

This case presents the nature of the phase boundary movement in response to a triangular type of heat flux variation. The heat flux is given as:

$$q(t) = \begin{cases} mt & 0 < t \leqslant t_0 \\ -mt & t_0 < t \leqslant 2t_0 \end{cases}, \tag{17}$$

where m is the gradient of the heat flux.

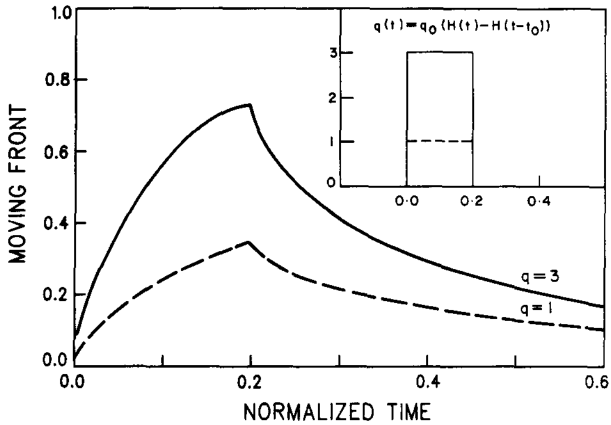

Figure 6. Phase boundary movement in response to the box-car type heat flux function. The excess heat flux is supplied for 0·2 units of time. This input `unction is shown in the inset. The phase boundary moves downward after the excess hea flux is withdrawn and asymptotically approaches the initial unperturbed position.

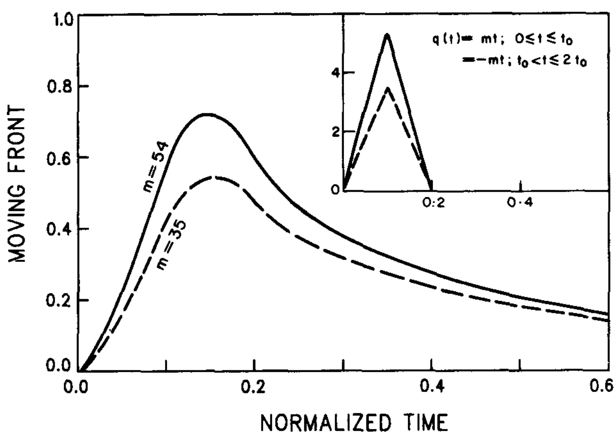

Figure 7. Phase boundary movement due to a triangular excess heat flux function. The excess heat flux is supplied for 0·2 units of time (inset). The moving front is asymptotic to the initial unperturbed position at large times.

The basal heat flux function is shown in the inset of figure 7. For $m = 35$ the boundary moves upward, to a level of 0·54 and then decreases. This boundary does not come back to the original unperturbed position but approaches it at large times. For $m = 54$ the 72% of thinning is obtained. The behaviour of the moving boundary after the excess heat flux is removed is the same as in the earlier case.

5. Conclusions.

The nonlinearity involved in the system which defines the phase boundary motion in response to the thermal and/or pressure variations at the base of the domain restricts the simulation of many physical processes which involve a moving boundary.

In the geophysical context, the problem of lithospheric thickening and thinning, metamorphism, solidification of igneous intrusives, etc. can be modelled using moving boundary formulation. The simple models of step-wise heat flux variation have been used earlier to explain some of these processes. In the present work the Fourier series method has been used to solve the problem of phase boundary motion for several general cases of basal heat flux variation.

The assumption of sudden constant increase in the heat flux reduces the complexity of the problem but at the same time takes the solution away from the real situation. The problem of lithospheric expansion and uplift can be illustrated as an example. Here, the mechanism mainly discussed for the expansion is the thinning at the base and subsequent uplift of the lithosphere in response to the mantle plume. In a step-wise function the moving plate is assumed to be encountering the sudden increase in the basal heat flux during its motion over a hot jet of mantle plume. In fact the variation will be of a more general type depending on the velocity of the plate motion and the uprising mantle plume. For excessively high plate velocity the assumption of step-wise heat flux may be valid but in the case of slow to moderate plate motion the heat flux will increase with time as the plate approaches the plume and then decrease when the plate will move away from the plume. Therefore, a general heat flux function which may be defined in terms of the velocity of plate, should be used instead of a step-wise function. We have attempted this problem to show the efficiency of the technique in solving different general types of heat flux variation problems. The models of box-car and triangular functions have been considered to imitate the solution of plate motion over a mantle plume.

The computations of periodic heat flux variation are discussed to highlight the nonlinearity of the system. The moving boundary calculations for step-wise and oscillatory functions indicate that the mean boundary position at large times in the case of oscillatory function may depart from that of the constant boundary position of step-wise case. For very high frequency of oscillations the mean boundary position at large times approaches the constant boundary position of step-wise function.

Acknowledgements

This work was carried out under the project ILTP-B2·7 (Geological and Geophysical modelling). The authors would like to express their thanks to Prof V K Gaur, the area co-ordinator of the project and Prof D Guptasarma for encouragement. The authors would also like to thank the Department of Science and Technology, New Delhi and the USSR Academy of Science, USSR for their support.

References

Canuto C, Hussains M Y, Quarteroni A and Zang T A 1988 *Spectral methods in fluid dynamics* (New York: Springer Verlag)

Crank J 1975 *The mathematics of diffusion* (Oxford: Clarendon Press), pp. 310

Crank J 1984 *Free and moving boundary problems* (Oxford: Clarendon Press), pp. 425

Crough S T 1983 Hotspot swells; *Annu. Rev. Earth Planet. Sci.* **11** 165–193

Crough S T and Thompson G A 1976 Numerical and approximate solutions for lithospheric thickening and thinning; *Earth Planet. Sci. Lett.* **31** 397–402

England P C 1978 Some thermal considerations of the Alpine metamorphism—past, present and future; *Tectonophysics* **46** 21–40

England P C and Richardson S W 1977 The influence of erosion upon the mineral facies of rocks from different metamorphic environments; *J. Geol. Soc. London* **134** 201–213

Gliko A O and Rovensky O N 1985 A numerical study of the process of lithospheric thinning under conditions of large heat flow; *Izv. Akad. Nauk. USSR* **21** 416–419

Gliko A O 1986 Asymptotic solution to the problem of thermal thinning of the lithosphere; *Izv. Akad. Nauk. USSR* **22** 257–265

Goodman T R 1964 A heat balance approach to the solution of phase change problems; In *Advances in heat transfer* (ed.) T F D Irvike and J P Martnett (New York: Academic Press) pp. 52–122

Gottlieb D and Orszag S A 1977 A numerical analysis of spectral methods: Theorie and applications; In *CBMS-NSF regional conference series in Applied Mathematics* No. 104 (SIAM, Philadelphia)

Kono Y and Ogawara H 1989 Evolution of the Hawaiian-Emperor swell-thinning of the oceanic lithosphere; *Tectonophysics* **159** 325–335

Lyubov A B 1980 *A theory of crystallization* (Moscow: Nauka) (in Russian)

Mareschal J C 1978 Dynamic behaviour of a phase boundary under non-uniform surface loads: *Geophys. J. R. Astron. Soc.* **54** 703–710

Mareschal J C and Gangi A F 1977 A linear approximation to the solution of a one dimensional Stefan problem and its geophysical implications; *Geophys. J. R. Astron. Soc.* **49** 443–458

Melamed V G 1958 Reducing Stefan's problem to a system of ordinary differential equation; *Izv. Akad. Nauk. USSR, Ser. Geofis.*, **7** 843–869

Nakiboglu S M and Lambeck K 1985 Thermal response of a moving lithosphere over a mantle heat source; *J. Geophys. Res.* **90** 2985–2994

O'Connell R J and Wasserburg G J 1967 Dynamics of a phase boundary to changes in pressure; *Rev. Geophys.* **5** 329–410

O'Connell R J and Wasserburg G J 1972 Dynamics of subsidence and uplift of a sedimentary basin underlain by a phase change boundary; *Rev. Geophys.* **10** 335–368

Peltier W R 1974 The impulse response of a Maxwell earth; *Rev. Geophys.* **12** 469–670

Rakitskiy Yu V, Ustinov S M and Chernorntskiy I G 1979 *Numerical methods for solving a stiff system of differential equations* (Moscow: Nauka) (in Russian)

Spohn T and Schubert G 1982 Convective thinning of the lithosphere: A mechanism for the initiation of continental rifting; *J. Geophys. Res.* **87** 4669–4681

Stefan J 1889 Über einige Probleme der Theorie der Wämeleitung S-B Ween; *Akad. Mat. Natur.* **98** 473–484

Tao L N 1978 The Stefan problem with arbitrary initial and boundary conditions; *Q. J. Appl. Math.* **36** 223–233

Thompson A B 1981 The P-T plane viewed by geophysicists and petrologists; *TERRA Cognita* **1** 11–20

Thompson P H 1977 Metamorphic P-T distributions and the geothermal gradients calculated from geophysical data; *Geology* **5** 520–522

Turcotte D L and Schubert G 1982 *Geodynamics* (New York: John Wiley) pp. 450

Wells P R A 1980 Thermal models for the magmatic accretion and subsequent metamorphism of continental crust; *Earth Planet. Sci. Lett.* **46** 253–265

Withjack M 1979 A convective heat transfer model for lithospheric thining and crustal uplift; *J. Geophys. Res.* **84** 3008–3022