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Static deformation of an orthotropic multilayered elastic half-space by two-dimensional surface loads

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Abstract. The transfer matrix approach is used to solve the problem of static deformation of an orthotropic multilayered elastic half-space by two-dimensional surface loads. The general problem is decoupled into two independent problems. The antiplane strain problem and the plane strain problem are considered in detail. Integral expressions for displacements and stresses at any point of the medium due to a normal line load and a shear line load, acting parallel to a symmetry axis, are obtained. In the case of a uniform half-space, closed form analytic expressions for displacements and stresses are derived. The procedure developed is quite easy and convenient for numerical computations.

Keywords. Transfer matrix; static deformation; orthotropic medium; multilayered half-space; two-dimensional surface loads; normal line load; shear line load.

1. Introduction

The behaviour of horizontally layered elastic materials under surface loads is of great interest in engineering, soil mechanics and geophysics. Laminated composite materials are finding increasing applications in engineering. Many earthworks, such as fills or pavements, consist of horizontal layers of materials of different types. Quite often natural deposits in the earth are also horizontally layered. However, the elastic properties of the material at a point of a layer may be different in different directions, i.e., the medium may be anisotropic. Most anisotropic media of interest in seismology have, at least approximately, a horizontal plane of symmetry. The most general system with one plane of symmetry is the monoclinic system. A material with three mutually perpendicular planes of elastic symmetry at a point is said to possess orthotropic or orthorhombic symmetry. This symmetry is exhibited by olivine and orthopyroxenes, the principal rock-forming minerals of the deep crust and upper mantle. Therefore, it is useful to determine the static field due to surface loads acting on the surface of an orthotropic multilayered elastic half-space. It may also find applications in the study of reservoir-induced seismicity.

Garg and Singh (1985) studied the static deformation of an isotropic multilayered half-space by two-dimensional surface loads. Singh (1986), Garg and Singh (1987) and Pan (1989) assumed the multilayered half-space to be transversely isotropic in which there are five elastic constants. Chaudhuri and Bhowal (1989) extended the results of Garg and Singh (1987) by introducing nonhomogeneity. They assumed exponential type variations of elastic parameters with depth. The static deformation of a multilayered semi-infinite medium by surface loads has also been studied by Kuo (1969) and Small and Booker (1984).

The transfer matrix approach is used in the present paper to solve the problem of the static deformation of an orthotropic multilayered semi-infinite elastic medium by two-dimensional surface loads. In an orthotropic material there are nine elastic constants as against five in a transversely isotropic material considered by Garg and Singh (1987). The results for a tetragonal material with six elastic coefficients, for a transversely isotropic material with five elastic coefficients and for a cubic material with three elastic coefficients can be derived as particular cases. We have verified that the results for a transversely isotropic material derived as a special case coincide with the corresponding results of Garg and Singh (1987).

2. Basic equations

In the cartesian coordinates (x_1, x_2, x_3), the equations of equilibrium for zero body forces are

$$\frac{\partial p_{11}}{\partial x_1}+\frac{\partial p_{12}}{\partial x_2}+\frac{\partial p_{13}}{\partial x_3}=0, \tag{1}$$

$$\frac{\partial p_{21}}{\partial x_1}+\frac{\partial p_{22}}{\partial x_2}+\frac{\partial p_{23}}{\partial x_3}=0, \tag{2}$$

$$\frac{\partial p_{31}}{\partial x_1}+\frac{\partial p_{32}}{\partial x_2}+\frac{\partial p_{33}}{\partial x_3}=0, \tag{3}$$

where p_{ij} is the stress tensor. Let (u_1, u_2, u_3) denote the components of the displacement vector. The strain-displacement relations are

$$e_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \quad (i,j=1,2,3). \tag{4}$$

For an orthotropic elastic medium, with coordinate planes coinciding with the planes of symmetry, the stress-strain relations are

$$\begin{bmatrix} p_{11} \\ p_{22} \\ p_{33} \\ p_{23} \\ p_{13} \\ p_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{23} \\ 2e_{13} \\ 2e_{12} \end{bmatrix} \tag{5}$$

A transversely isotropic elastic medium, x_3 axis coinciding with the axis of symmetry, is a special case of an orthotropic elastic medium for which

$$c_{22}=c_{11}, \quad c_{23}=c_{13}, \quad c_{55}=c_{44}, \quad c_{66}=\tfrac{1}{2}(c_{11}-c_{12}) \tag{6}$$

and the number of independent elastic constants reduces from 9 to 5. For an isotropic elastic medium

$$c_{11}=c_{22}=c_{33}=\lambda+2\mu, \quad c_{12}=c_{13}=c_{23}=\lambda, \quad c_{44}=c_{55}=c_{66}=\mu, \tag{7}$$

where λ and μ are the Lamé constants.

We shall consider a two-dimensional deformation in which the displacement components are independent of x_1 so that $\partial/\partial x_1 \equiv 0$. Then, the general problem is decoupled into two independent problems—plane strain problem ($u_1 = 0$) and the antiplane strain problem ($u_2 = u_3 = 0$). We discuss both the problems separately. In the following, we shall write (x, y, z) for (x_1, x_2, x_3) and (u, v, w) for (u_1, u_2, u_3).

3. Antiplane strain problem

For this problem

$$u = u(y, z), \quad v = w \equiv 0. \tag{8}$$

The non-zero strain and stress components are

$$e_{13} = \tfrac{1}{2}(\partial u/\partial z), \quad e_{12} = \tfrac{1}{2}(\partial u/\partial y), \tag{9}$$

$$p_{12} = c_{66}(\partial u/\partial y), \quad p_{13} = c_{55}(\partial u/\partial z). \tag{10}$$

Equilibrium equations (2) and (3) are identically satisfied and equation (1) becomes

$$\delta_1^2(\partial^2 u/\partial y^2) + (\partial^2 u/\partial z^2) = 0, \tag{11}$$

where

$$\delta_1^2 = c_{66}/c_{55}. \tag{12}$$

A solution of (11) is of the form

$$u = \int_0^\infty [A\exp(-\delta_1 kz) + B\exp(\delta_1 kz)]\begin{pmatrix}\sin ky\\ \cos ky\end{pmatrix}\mathrm{d}k, \tag{13}$$

where A, B are functions of k only. From (10) and (13), we find

$$p_{13} = \delta_2 \int_0^\infty [-A\exp(-\delta_1 kz) + B\exp(\delta_1 kz)]\begin{pmatrix}\sin ky\\ \cos ky\end{pmatrix}k\,\mathrm{d}k, \tag{14}$$

where

$$\delta_2 = (c_{55}c_{66})^{\frac{1}{2}}. \tag{15}$$

We write (13) and (15) in the form

$$u = \int_0^\infty U(z)\begin{pmatrix}\sin ky\\ \cos ky\end{pmatrix}\mathrm{d}k, \tag{16}$$

$$p_{13} = \int_0^\infty T(z)\begin{pmatrix}\sin ky\\ \cos ky\end{pmatrix}k\,\mathrm{d}k. \tag{17}$$

The functions U, T are given by the matrix relation

$$[Y(z)] = [Z(z)][K], \tag{18}$$

where

$$[Y(z)] = [U(z), T(z)]^T, [K] = [A, B]^T \tag{19}$$

and $[\cdots]^T$ denote the transpose of the matrix $[\cdots]$. The matrix $[Z(z)]$ is given below

$$[Z(z)] = \begin{bmatrix} \exp(-\delta_1 kz) & \exp(\delta_1 kz) \\ -\delta_2 \exp(-\delta_1 kz) & \delta_2 \exp(\delta_1 kz) \end{bmatrix}. \tag{20}$$

When the medium is isotropic

$$\delta_1 = 1, \quad \delta_2 = \mu \tag{21}$$

and the matrix $[Z(z)]$ becomes identical with the corresponding matrix given by Garg and Singh (1985).

4. Plane strain problem

In this case

$$v = v(y,z), \quad w = w(y,z), \quad u = 0. \tag{22}$$

The non-zero strain and stress components are

$$e_{22} = \partial v/\partial y, \quad e_{33} = \partial w/\partial z, \tag{23}$$

$$e_{23} = \frac{1}{2}\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right], \tag{24}$$

$$p_{11} = c_{12}e_{22} + c_{13}e_{33}, \tag{25}$$

$$p_{22} = c_{22}e_{22} + c_{23}e_{33}, \tag{26}$$

$$p_{33} = c_{23}e_{22} + c_{33}e_{33}, \tag{27}$$

$$p_{23} = 2c_{44}e_{23}. \tag{28}$$

For the plane strain deformation, the equilibrium equation (1) is identically satisfied and equations (2) and (3) reduce to

$$\frac{\partial p_{22}}{\partial y} + \frac{\partial p_{23}}{\partial z} = 0, \tag{29}$$

$$\frac{\partial p_{23}}{\partial y} + \frac{\partial p_{33}}{\partial z} = 0. \tag{30}$$

Therefore, there exists an Airy stress function $U^*(y,z)$ such that

$$p_{22} = \partial^2 U^*/\partial z^2, \quad p_{23} = -\partial^2 U^*/\partial y \partial z, \quad p_{33} = \partial^2 U^*/\partial y^2. \tag{31}$$

Using (31), we note that the equilibrium equations (29) and (30) are identically satisfied. The non-zero compatibility equation is

$$\frac{\partial^2 e_{22}}{\partial z^2} + \frac{\partial^2 e_{33}}{\partial y^2} = 2\frac{\partial^2 e_{23}}{\partial y \partial z}. \tag{32}$$

From equations (26)–(28), (31) and (32), we find

$$\delta_3 \frac{\partial^4 U^*}{\partial y^4} + \delta_6 \frac{\partial^4 U^*}{\partial y^2 \partial z^2} + \delta_5 \frac{\partial^4 U^*}{\partial z^4} = 0, \tag{33}$$

where

$$\delta_3 = c_{22}/c_{44}, \quad \delta_4 = c_{23}/c_{44}, \quad \delta_5 = c_{33}/c_{44}, \quad \delta_6 = \delta_3\delta_5 - \delta_4^2 - 2\delta_4. \tag{34}$$

Let α and β be given by the relations

$$\alpha^2 + \beta^2 = \delta_6/\delta_5, \quad \alpha^2\beta^2 = \delta_3/\delta_5. \tag{35}$$

Then (33) is factorized as

$$\left(\alpha^2 \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\beta^2 \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) U^* = 0. \tag{36}$$

In the case of an isotropic medium

$$\alpha = \beta = 1 \tag{37}$$

and U^* becomes biharmonic.

A solution of (36) is of the type (assuming $\alpha \neq \beta$)

$$U^* = \int_0^\infty [A\exp(-\alpha kz) + B\exp(\alpha kz) + C\exp(-\beta kz) + D\exp(\beta kz)] \times \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} \mathrm{d}k. \tag{38}$$

Corresponding to the Airy stress function (38), the stress can be obtained from (31) and then the displacements can be obtained by integrating the stress-displacement relations (25)–(28). Following Singh and Garg (1985) and Garg and Singh (1987), we write

$$v = \int_0^\infty V(z) \begin{pmatrix} -\cos ky \\ \sin ky \end{pmatrix} k\,\mathrm{d}k, \tag{39}$$

$$w = \int_0^\infty W(z) \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} k\,\mathrm{d}k, \tag{40}$$

$$p_{23} = \int_0^\infty S(z) \begin{pmatrix} \cos ky \\ -\sin ky \end{pmatrix} k^2\,\mathrm{d}k, \tag{41}$$

$$p_{33} = \int_0^\infty N(z) \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} k^2\,\mathrm{d}k. \tag{42}$$

The functions V, W, S, N are given by the matrix relation

$$[Y(z)] = [Z(z)][K], \tag{43}$$

where

$$[Y(z)] = [V, W, S, N]^T, \quad [K] = [A, B, C, D]^T. \tag{44}$$

The matrix $[Z(z)]$ is given below

$$[Z(z)]=\begin{bmatrix} r_1 e^{-\theta} & r_1 e^{\theta} & r_2 e^{-\phi} & r_2 e^{\phi} \\ s_1 e^{-\theta} & -s_1 e^{\theta} & s_2 e^{-\phi} & -s_2 e^{\phi} \\ \alpha e^{-\theta} & -\alpha e^{\theta} & \beta e^{-\phi} & -\beta e^{\phi} \\ -e^{-\theta} & -e^{\theta} & -e^{-\phi} & -e^{\phi} \end{bmatrix}, \tag{45}$$

where

$$r_1=\frac{c_{33}\alpha^2+c_{23}}{c_{33}c_{22}-c_{23}^2},\quad r_2=\frac{c_{33}\beta^2+c_{23}}{c_{33}c_{22}-c_{23}^2},$$

$$s_1=\frac{c_{23}\alpha+(c_{22}/\alpha)}{c_{33}c_{22}-c_{23}^2},\quad s_2=\frac{c_{23}\beta+(c_{22}/\beta)}{c_{33}c_{22}-c_{23}^2}, \tag{46}$$

$$\theta=\alpha kz,\quad \phi=\beta kz. \tag{47}$$

5. Deformation of a multilayered half-space

We consider a semi-infinite elastic medium made up of $p-1$ parallel homogeneous orthotropic elastic layers lying over a homogeneous orthotropic elastic half-space. The layers are assumed to be in welded contact implying the continuity of the displacements and stresses across the interfaces. The layers are numbered serially, the layer at the top being layer 1 and the half-space, layer p. The origin of the cartesian coordinate system (x, y, z) is taken at the boundary of the semi-infinite medium and the z-axis is drawn into the medium. The nth layer is bounded by the interfaces $z=z_{n-1}$ and $z=z_n$ and is of thickness d_n where $d_n=z_n-z_{n-1}$. Clearly $z_0=0$ and $z_{p-1}=H$, where H denotes the depth of the last interface [figure 1].

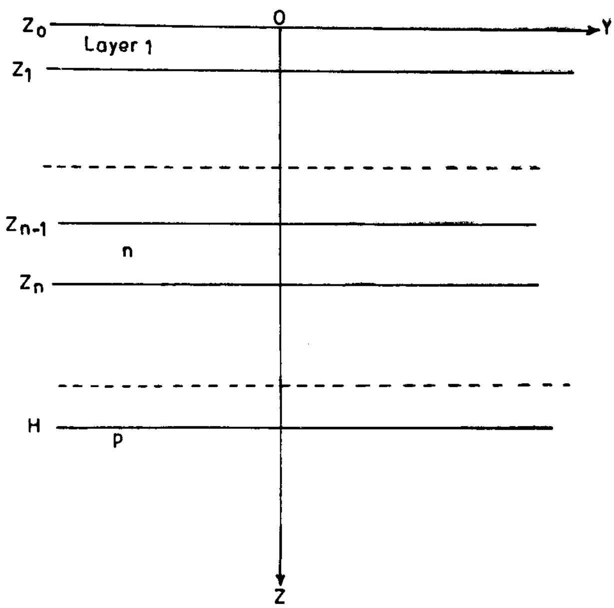

Figure 1. Multilayered half-space.

5.1 *Antiplane strain deformation*

Introducing the subscript n to the quantities related to the nth-layer, (18) becomes

$$[Y_n(z)] = [Z_n(z)][K_n], \tag{48}$$

where the matrix $[Z_n(z)]$ is obtained from the matrix $[Z(z)]$, given in (20) on replacing δ_1 and δ_2, respectively, by the corresponding elastic constants of the nth layer.

It has been shown by Singh (1970) and Singh and Garg (1985) that the deformation fields at the boundaries of the consecutive layers satisfy the relation

$$[Y_{n-1}(z_{n-1})] = [a_n][Y_n(z_n)], \tag{49}$$

where the transfer matrix $[a_n]$ is

$$\begin{bmatrix} \mathrm{ch}(\delta_1 kd) & -\delta_2^{-1}\,\mathrm{sh}(\delta_1 kd) \\ -\delta_2\,\mathrm{sh}(\delta_1 kd) & \mathrm{ch}(\delta_1 kd) \end{bmatrix} \tag{50}$$

with ch = cosh and sh = sinh.

For the p-th layer, $B_p = 0$, otherwise $U_p(z) \to \infty$ as $z \to \infty$. Making a repeated use of (48) and (49), we find

$$[U_1(0),\, T_1(0)]^T = [F][A_p, 0]^T, \tag{51}$$

where

$$[F] = [a_1][a_2][a_3]\cdots[a_{p-1}][Z_p(H)]. \tag{52}$$

When the surface load is prescribed, the boundary condition is of the type

$$p_{13} = f(y) \text{ at } z = 0. \tag{53a}$$

We write [see, (17)]

$$f(y) = \int_0^\infty \bar{f}(k)\begin{pmatrix}\sin ky \\ \cos ky\end{pmatrix} k\,\mathrm{d}k, \quad \bar{f}(k) = \frac{2}{\pi k}\int_0^\infty f(y)\begin{pmatrix}\sin ky \\ \cos ky\end{pmatrix}\mathrm{d}y. \tag{53b}$$

This determines the value of A_p:

$$A_p = \bar{f}(k)/F_{21}. \tag{54}$$

The deformation field at any point z of the nth layer can be obtained from the relation

$$[U_n(z),\, T_n(z)]^T = [G(z)][A_p, 0]^T, \tag{55}$$

where

$$[G(z)] = [a_n(z_n - z)][a_{n+1}]\cdots[a_{p-1}][Z_p(H)]. \tag{56}$$

From the relations (16), (17), (54) and (55), the stress and the displacement at any point of the nth layer caused by the surface load acting on the boundary is given below in the integral form:

$$u = \int_0^\infty \left(\frac{G_{11}}{F_{21}}\right)\bar{f}(k)\begin{pmatrix}\sin ky \\ \cos ky\end{pmatrix}\mathrm{d}k, \tag{57}$$

$$p_{13} = \int_0^\infty \left(\frac{G_{21}}{F_{21}}\right)\bar{f}(k)\begin{pmatrix}\sin ky \\ \cos ky\end{pmatrix}k\,\mathrm{d}k. \tag{58}$$

5.2 Plane strain deformation

When the surface load is prescribed, the boundary conditions are of the form

$$p_{23} = g(y), \quad p_{33} = h(y) \quad \text{at } z = 0. \tag{59}$$

We write [see, (41) and (42)]

$$g(y) = \int_0^\infty \bar{g}(k) \begin{pmatrix} \cos ky \\ -\sin ky \end{pmatrix} k^2 \, \mathrm{d}k, \quad \bar{g}(k) = \frac{2}{\pi k^2} \int_0^\infty g(y) \begin{pmatrix} \cos ky \\ -\sin ky \end{pmatrix} \mathrm{d}y, \tag{60}$$

$$h(y) = \int_0^\infty \bar{h}(k) \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} k^2 \, \mathrm{d}k, \quad \bar{h}(k) = \frac{2}{\pi k^2} \int_0^\infty h(y) \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} \mathrm{d}y. \tag{61}$$

Proceeding as in the case of the antiplane strain deformation, the expressions for the displacements and stresses at any point of the nth layer are:

$$v = \int_0^\infty [G_{11}(F_{43}\bar{g} - F_{33}\bar{h}) + G_{13}(F_{31}\bar{h} - F_{41}\bar{g})]\Omega^{-1} \begin{pmatrix} -\cos ky \\ \sin ky \end{pmatrix} k \, \mathrm{d}k, \tag{62}$$

$$w = \int_0^\infty [G_{21}(F_{43}\bar{g} - F_{33}\bar{h}) + G_{23}(F_{31}\bar{h} - F_{41}\bar{g})]\Omega^{-1} \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} k \, \mathrm{d}k, \tag{63}$$

$$p_{23} = \int_0^\infty [G_{31}(F_{43}\bar{g} - F_{33}\bar{h}) + G_{33}(F_{31}\bar{h} - F_{41}\bar{g})]\Omega^{-1} \begin{pmatrix} \cos ky \\ -\sin ky \end{pmatrix} k^2 \, \mathrm{d}k, \tag{64}$$

$$p_{33} = \int_0^\infty [G_{41}(F_{43}\bar{g} - F_{33}\bar{h}) + G_{43}(F_{31}\bar{h} - F_{41}\bar{g})]\Omega^{-1} \begin{pmatrix} \sin ky \\ \cos ky \end{pmatrix} k^2 \, \mathrm{d}k, \tag{65}$$

where

$$\Omega = F_{31}F_{43} - F_{33}F_{41}. \tag{66}$$

The transfer matrix $[a_n]$ for the plane strain problem is given in Appendix I.

6. Specified surface loads

In this section, we consider a few particular cases in which the surface loads are specified.

6.1 Antiplane strain problem

Let R be the shear line load per unit length in the positive direction of the x-axis. If the line load passes through the origin, the boundary condition is

$$p_{13} = -R\delta(y), \tag{67}$$

where

$$\delta(y) = \frac{1}{\pi} \int_0^\infty \cos ky \, \mathrm{d}k \tag{68}$$

is the Dirac delta function. From (53), (67) and (68), we find that

$$f(y) = -R\delta(y), \quad \bar{f}(k) = -\mathrm{R}/\pi k \tag{69}$$

and that we must choose the lower solution, $\cos ky$, in the expression (13) for u and the corresponding solution in all the succeeding equations related to u. Substituting this value of $\bar{f}(k)$ in (57) and (58), we obtain

$$u = \frac{-R}{\pi}\int_0^\infty \left(\frac{G_{11}}{F_{21}}\right)\cos ky k^{-1}\,\mathrm{d}k, \tag{70}$$

$$p_{13} = \frac{-R}{\pi}\int_0^\infty \left(\frac{G_{21}}{F_{21}}\right)\cos ky\,\mathrm{d}k. \tag{71}$$

6.2 *Plane strain problem*

In this case, we consider the particular cases of a normal line load and a shear line load.

Normal line load: Let a normal line load P per unit length be acting in the positive z-direction [figure 2]. Then the boundary conditions are

$$p_{23} = 0, \quad p_{33} = -P\delta(y). \tag{72}$$

From (59)–(61) and (72), we find

$$\bar{g}(k) = 0, \quad \bar{h}(k) = -P/\pi k^2 \tag{73}$$

and that we must choose the lower solution, $\cos ky$, in the expression (38) for U^* and the corresponding solution in all the succeeding equations related to U^*. From (73) and (62)–(65), we find

$$v = \frac{-P}{\pi}\int_0^\infty [G_{13}F_{31} - G_{11}F_{33}]k^{-1}\Omega^{-1}\sin ky\,\mathrm{d}k, \tag{74}$$

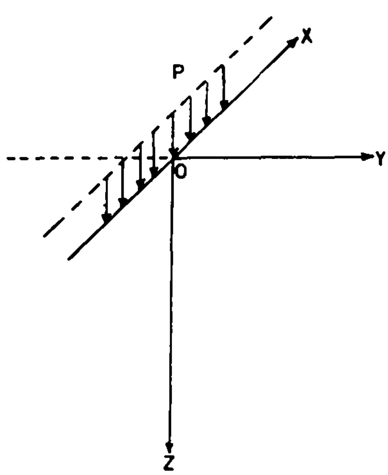

Figure 2. Normal line load P per unit length acting on the boundary of a semi-infinite medium.

$$w = \frac{-P}{\pi}\int_0^\infty [G_{23}F_{31} - G_{21}F_{33}]k^{-1}\Omega^{-1}\cos ky\,dk, \tag{75}$$

$$p_{23} = \frac{P}{\pi}\int_0^\infty [G_{33}F_{31} - G_{31}F_{33}]\Omega^{-1}\sin ky\,dk, \tag{76}$$

$$p_{33} = \frac{-P}{\pi}\int_0^\infty [G_{43}F_{31} - G_{41}F_{33}]\Omega^{-1}\cos ky\,dk. \tag{77}$$

Tangential line load: We assume that a shear line load Q per unit length is acting in the positive y-direction [figure 3]. Then, the boundary conditions are

$$p_{23} = -Q\delta(y), \quad p_{33} = 0 \quad \text{at } z = 0. \tag{78}$$

From (59)–(61) and (78), we obtain

$$\bar{g}(k) = -Q/\pi k^2, \quad \bar{h}(k) = 0 \tag{79}$$

and that the upper solution, $\sin ky$, in the expression (38) for U^* must be taken. As before, we obtain the following integral expressions for the displacements and stresses caused by a shear line load:

$$v = \frac{Q}{\pi}\int_0^\infty [G_{11}F_{43} - G_{13}F_{41}]k^{-1}\Omega^{-1}\cos ky\,dk, \tag{80}$$

$$w = \frac{-Q}{\pi}\int_0^\infty [G_{21}F_{43} - G_{23}F_{41}]k^{-1}\Omega^{-1}\sin ky\,dk, \tag{81}$$

$$p_{23} = \frac{-Q}{\pi}\int_0^\infty [G_{31}F_{43} - G_{33}F_{41}]\Omega^{-1}\cos ky\,dk, \tag{82}$$

$$p_{33} = \frac{-Q}{\pi}\int_0^\infty [G_{41}F_{43} - G_{43}F_{41}]\Omega^{-1}\sin ky\,dk. \tag{83}$$

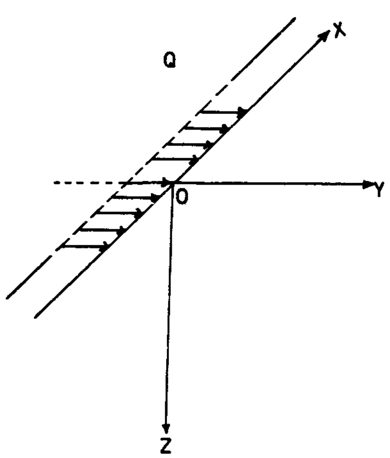

Figure 3. Tangential line load Q per unit length acting on the boundary of a semi-infinite medium.

7. Uniform half-space

In the previous section, we have obtained the integral expressions for the displacements and the stresses at an arbitrary point of the medium caused by the surface line loads acting on the boundary of an orthotropic elastic multilayered half-space. These integrals can be computed numerically by using the method given by Jovanovich *et al* (1974a, b). In the case of an orthotropic elastic uniform half-space ($p = 1$) the integrals giving the stresses can be evaluated anlytically. For a half-space

$$[F] = [Z(0)], \quad [G] = [Z(z)] \tag{84}$$

7.1 *Antiplane strain problem*

$$[F] = \begin{bmatrix} 1 & 1 \\ -\delta_2 & \delta_2 \end{bmatrix}, \quad [G] = \begin{bmatrix} \exp(-\delta_1 kz) & \exp(\delta_1 kz) \\ -\delta_2 \exp(-\delta_1 kz) & \delta_2 \exp(\delta_1 kz) \end{bmatrix}. \tag{85}$$

From (10), (70), (71) and (85), we obtain, using the integrals given in Appendix II,

$$u = -(R/2\pi\delta_2)\log(y^2 + \delta_1^2 z^2), \tag{86a}$$

$$p_{13} = -\frac{R\delta_1}{\pi}\left[\frac{z}{y^2 + \delta_1^2 z^2}\right], \tag{86b}$$

$$p_{12} = -\frac{R\delta_1}{\pi}\left[\frac{y}{y^2 + \delta_1^2 z^2}\right]. \tag{86c}$$

In the case of an isotropic elastic half-space $\delta_1 = 1$, $\delta_2 = \mu$ and (86a–c) reduce to

$$u = \frac{-R}{2\pi\mu}\log(y^2 + z^2), \quad p_{13} = \frac{-R}{\pi}\left(\frac{z}{y^2 + z^2}\right), \quad p_{12} = \frac{-R}{\pi}\left(\frac{y}{y^2 + z^2}\right). \tag{87}$$

7.2 *Plane strain problem*

Here

$$[F] = [Z(0)] = \begin{bmatrix} r_1 & r_1 & r_2 & r_2 \\ s_1 & -s_1 & s_2 & -s_2 \\ \alpha & -\alpha & \beta & -\beta \\ -1 & -1 & -1 & -1 \end{bmatrix}, \quad \Omega = \beta - \alpha \tag{88}$$

and $[G] = [Z(z)]$, where $[Z(z)]$ is given in (45).

7.2a *Normal line load*: From (45), (74)–(77) and (88), we find

$$v = \frac{P}{\pi(\beta - \alpha)}\left[\beta r_1 \tan^{-1}\left(\frac{y}{\alpha z}\right) - \alpha r_2 \tan^{-1}\left(\frac{y}{\beta z}\right)\right], \tag{89}$$

$$w = \frac{P}{2\pi(\beta - \alpha)}[\alpha s_2 \log(y^2 + \beta^2 z^2) - \beta s_1 \log(y^2 + \alpha^2 z^2)], \tag{90}$$

$$p_{23}=\frac{\alpha\beta P}{\pi(\beta-\alpha)}\left[\frac{y}{y^2+\beta^2z^2}-\frac{y}{y^2+\alpha^2z^2}\right], \tag{91}$$

$$p_{33}=\frac{\alpha\beta P}{\pi(\beta-\alpha)}\left[\frac{z}{y^2+\beta^2z^2}-\frac{z}{y^2+\alpha^2z^2}\right]. \tag{92}$$

When the half-space is isotropic, $\alpha=\beta=1$ and the stresses given in (91) and (92) reduce to

$$p_{23}=\frac{-2P}{\pi}\left[\frac{yz^2}{(y^2+z^2)^2}\right],\quad p_{33}=\frac{-2P}{\pi}\left[\frac{z^3}{(y^2+z^2)^2}\right] \tag{93}$$

which are identical with the corresponding results of Sneddon (1951).

7.2b *Tangential line load*: From (45), (80)–(83) and (88), we obtain

$$v=\frac{Q}{2\pi(\beta-\alpha)}[r_1\log(y^2+\alpha^2z^2)-r_2\log(y^2+\beta^2z^2)], \tag{94}$$

$$w=\frac{Q}{\pi(\beta-\alpha)}\left[s_1\tan^{-1}\left(\frac{y}{\alpha z}\right)-s_2\tan^{-1}\left(\frac{y}{\beta z}\right)\right], \tag{95}$$

$$p_{23}=\frac{Q}{\pi(\beta-\alpha)}\left[\frac{\alpha^2z}{y^2+\alpha^2z^2}-\frac{\beta^2z}{y^2+\beta^2z^2}\right], \tag{96}$$

$$p_{33}=\frac{Q}{\pi(\beta-\alpha)}\left[\frac{y}{y^2+\beta^2z^2}-\frac{y}{y^2+\alpha^2z^2}\right]. \tag{97}$$

For an isotropic uniform half-space, stresses given in (96) and (97) become

$$p_{23}=\frac{-2Q}{\pi}\left[\frac{y^2z}{(y^2+z^2)^2}\right],\quad p_{33}=\frac{-2Q}{\pi}\left[\frac{yz^2}{(y^2+z^2)^2}\right]. \tag{98}$$

These stresses coincide with the corresponding results given by Garg and Singh (1985).

8. Conclusions

We have solved the problem of the static deformation of an orthotropic multilayered elastic half-space by two-dimensional surface loads. The results for a tetragonal medium can be found by putting

$$c_{22}=c_{11},\quad c_{23}=c_{13},\quad c_{55}=c_{44}.$$

The results for a transversely isotropic medium can be obtained by taking

$$c_{22}=c_{11},\quad c_{23}=c_{13},\quad c_{55}=c_{44},\quad c_{66}=\tfrac{1}{2}(c_{11}-c_{12}).$$

The results for a cubic material can be obtained on taking

$$c_{22}=c_{33}=c_{11},\quad c_{12}=c_{13}=c_{23},\quad c_{44}=c_{55}=c_{66}.$$

While the problem of a transversely isotropic material has been discussed by Garg and Singh (1987), the solutions of the problems for tetragonal and cubic materials have not been reported in the literature.

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Appendix I: Transfer matrix for the plane strain problem

From (45), we write

$$[Z(z)] = [Z(0)][X(z)], \tag{A.1}$$

where

$$[X(z)] = \begin{bmatrix} e^{-\theta} & 0 & 0 & 0 \\ 0 & e^{\theta} & 0 & 0 \\ 0 & 0 & e^{-\phi} & 0 \\ 0 & 0 & 0 & e^{\phi} \end{bmatrix}, \quad [Z(0)] = \begin{bmatrix} r_1 & r_1 & r_2 & r_2 \\ s_1 & -s_1 & s_2 & -s_2 \\ \alpha & -\alpha & \beta & -\beta \\ -1 & -1 & -1 & -1 \end{bmatrix}. \tag{A.2}$$

We find

$$[Z(0)]^{-1} = \begin{bmatrix} -\Omega_1 & \beta\Omega_2 & -s_2\Omega_2 & -r_2\Omega_1 \\ -\Omega_1 & -\beta\Omega_2 & s_2\Omega_2 & -r_2\Omega_1 \\ \Omega_1 & -\alpha\Omega_2 & s_1\Omega_2 & r_1\Omega_1 \\ \Omega_1 & \alpha\Omega_2 & -s_1\Omega_2 & r_1\Omega_1 \end{bmatrix} \tag{A.3}$$

where

$$\Omega_1 = [2(r_2 - r_1)]^{-1}, \quad \Omega_2 = [2(\beta s_1 - \alpha s_2)]^{-1}. \tag{A.4}$$

Following Singh (1970), we find that the transfer matrix $[a_n]$ is given by

$$[a_n] = [Z_n(-d_n)][Z_n(0)]^{-1}. \tag{A.5}$$

The elements of the matrix $[a_n]$ are (omitting the subscript n)

$(11) = 2(-r_1 \operatorname{ch}\theta + r_2 \operatorname{ch}\phi)\Omega_1$, $(12) = 2(\beta r_1 \operatorname{sh}\theta - r_2\alpha \operatorname{sh}\phi)\Omega_2$,

$(13) = 2(-r_1 s_2 \operatorname{sh}\theta + r_2 s_1 \operatorname{sh}\phi)\Omega_2$, $(14) = 2(-\operatorname{ch}\theta + \operatorname{ch}\phi)r_1 r_2\Omega_1$,

$(21) = 2(-s_1 \operatorname{sh}\theta + s_2 \operatorname{sh}\phi)\Omega_1$, $(22) = 2(s_1\beta \operatorname{ch}\theta - s_2\alpha \operatorname{ch}\phi)\Omega_2$,

$(23) = 2s_1 s_2(-\operatorname{ch}\theta + \operatorname{ch}\phi)\Omega_2$, $(24) = 2(-s_1 r_2 \operatorname{sh}\theta + s_2 r_1 \operatorname{sh}\phi)\Omega_1$,

$(31) = 2(-\alpha \operatorname{sh}\theta + \beta \operatorname{sh}\phi)\Omega_1$, $(32) = 2\alpha\beta(\operatorname{ch}\theta - \operatorname{ch}\phi)\Omega_2$,

$(33) = 2(-\alpha s_2 \operatorname{ch}\theta + \beta s_1 \operatorname{ch}\phi)\Omega_2$, $(34) = 2(-\alpha r_2 \operatorname{sh}\theta + \beta r_1 \operatorname{sh}\phi)\Omega_1$,

$(41) = 2(\operatorname{ch}\theta - \operatorname{ch}\phi)\Omega_1$, $(42) = 2(-\beta \operatorname{sh}\theta + \alpha \operatorname{sh}\phi)\Omega_2$,

$(43) = 2(s_2 \operatorname{sh}\theta - s_1 \operatorname{sh}\phi)\Omega_2$, $(44) = 2(r_2 \operatorname{ch}\theta - r_1 \operatorname{ch}\phi)\Omega_1$.

In (A.2) $\theta = \alpha kz$, $\phi = \beta kz$ while in (A.5) $\theta = \alpha kd$ and $\phi = \beta kd$.

Appendix II ($z > 0$): Integrals used

(1) $$\int_0^\infty k^{-1} \exp(-kz) \cos ky \, dk = -\frac{1}{2} \log(y^2 + z^2),$$

(2) $$\int_0^\infty k^{-1} \exp(-kz) \sin ky \, dk = \tan^{-1}(y/z),$$

(3) $$\int_0^\infty \exp(-kz) \cos ky \, dk = z/(y^2 + z^2),$$

(4) $$\int_0^\infty \exp(-kz) \sin ky \, dk = y/(y^2 + z^2).$$

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