Trajectories of Brownian particles with space-correlated noise[†]

EDOARDO MILOTTI*

Physics Department, University of Trieste, Via Valerio 2, I-34127 Trieste, Italy
Email: milotti@units.it

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Abstract. The Langevin equation used to model Brownian motion includes a stochastic process that is routinely assumed to be a Gaussian white noise. Spatial correlations of the noise are usually ruled out, and the paths traced by the random walkers are statistically independent. In this study, I consider instead noise which is white in time and has a Gaussian correlation in space, and by means of numerical simulation, I show how the spatial correlation determines the time evolution of the spatial separation of random walkers.

Keywords. Stochastic processes; fluctuations; random walks.

1. Introduction

There are physical systems that display spatially correlated randomness,^{1,2} and several properties of Brownian motion in the spatially correlated noise landscapes have already been explored.^{1–5} This is especially important in disordered systems, where spatial disorder amounts to an additional noise term. Both nonlinear effects in transport,² and entropic forces that are relevant in colloidal mixtures⁶ produce time dependent spatial correlations, in contrast with stationary random landscapes.

Here I consider a still unexplored feature of random walks with spatially correlated white noise: the time-dependence of the distance of pairs of random walkers.

2. Theoretical/Computational

2.1 Theory

The Langevin equation provides a standard and intuitive description of Brownian motion

$$m \frac{d\mathbf{v}}{dt} = -\gamma \mathbf{v}(t) + \xi \mathbf{v}(t), \quad (1)$$

where γ is the coefficient of friction, ξ^2 is the spectral density of noise, $\mathbf{v}(t)$ is a vector of statistically independent white Gaussian processes with zero mean and unit variance, and where we assume isotropy (so that both γ and ξ are scalars); the corresponding stochastic process is the well-known Ornstein-Uhlenbeck process.^{7,8} It is

also quite well-known that the two-sided noise spectral density is

$$\xi^2 = 2D\gamma^2 = \frac{2(k_B T)^2}{D}, \quad (2)$$

where D is the diffusion coefficient. Yet another well-known result is that the mean square distance from the origin in n dimensions is $2nDt$.

When we consider two random walkers that start simultaneously from the origin, we can subtract the equations that describe each walker and obtain,

$$m \frac{d}{dt}(\mathbf{v}_1 - \mathbf{v}_2) = -\gamma(\mathbf{v}_1 - \mathbf{v}_2) + \xi(\mathbf{v}_1 - \mathbf{v}_2), \quad (3)$$

which is a new Langevin equation for the interparticle separation $\mathbf{x}_1 - \mathbf{x}_2$, then assuming that $\mathbf{v}_1$ and $\mathbf{v}_2$ are statistically independent, it easy to show that the new noise process $\mathbf{v}_1 - \mathbf{v}_2$ is again a Gaussian white noise process. However, the variance of the process is twice as large, and this means that the mean square distance between these two random walkers is

$$\langle r^2 \rangle = 4nDt \quad (4)$$

in n -dimensional space. In the cases of interest here, $n = 2$ or $n = 3$. Notice also that when one of the random walkers starts with a small offset $\delta\mathbf{x}$, we can define its position as $\mathbf{x}_2 + \delta\mathbf{x}$, where $\mathbf{x}_2$ is the position without offset, and the following relations hold:

$$\langle \mathbf{x}_1 - (\mathbf{x}_2 + \delta\mathbf{x}) \rangle = -\delta\mathbf{x} \quad (5a)$$

$$\langle [\mathbf{x}_1 - (\mathbf{x}_2 + \delta\mathbf{x})]^2 \rangle = \langle (\mathbf{x}_1 - \mathbf{x}_2)^2 \rangle + \delta\mathbf{x}^2 = 4nDt + \delta\mathbf{x}^2 \quad (5b)$$

We see that the growth of the mean square separation is rather trivial in random walks like those described by eq. (1).

*For correspondence

[†] Dedicated to the memory of the late Professor Charusita Chakravarty.

Now, I turn to spatially correlated random walks. Previous investigators have characterized fixed or slowly varying noise landscapes with the space correlation function $g(|\mathbf{x} - \mathbf{x}'|)$, and have considered different types of correlation, like for instance Gaussian or exponential.³

Here I consider a rapidly fluctuating random field, which is Gaussian and white in time and has a Gaussian correlation function in space. In particular, I take a field that is given by a superposition of Gaussian components:

$$\phi(\mathbf{x}, t) = \sum_k \alpha_k \exp\left(-\frac{|\mathbf{x} - \mathbf{x}_k|^2}{2\sigma^2}\right), \quad (6)$$

where the α_k 's are zero-mean independent identically distributed (i.i.d.) random amplitudes, with Gaussian probability density function and such that $\langle \alpha_j \alpha_k \rangle = \sigma_\alpha^2 \delta_{jk}$, and the $\mathbf{x}_k$'s are random positions in space, such that they have uniform distribution, numerical density ρ , are statistically independent and are uncorrelated with the α_k 's. Moreover, the random field is white in time: the α_k 's and the $\mathbf{x}_k$'s are different at each instant and without correlation with the values at different times. With these assumptions at a given time the spatial average at any space position $\mathbf{x}$ does not depend on $\mathbf{x}$ and is

$$\begin{aligned} \overline{\phi(\mathbf{x}, t)} &= \lim_{V \rightarrow \infty} \frac{1}{V} \int_V d^n x \sum_k \alpha_k \exp\left(-\frac{|\mathbf{x} - \mathbf{x}_k|^2}{2\sigma^2}\right) \\ &= (2\pi\sigma^2)^{\frac{n}{2}} \rho \bar{\alpha} = 0, \end{aligned} \quad (7)$$

where $\bar{\alpha}$ is the mean over all the α_k 's in the volume V , and $\bar{\alpha} = 0$, thanks to the assumption that the α_k 's are i.i.d.

A similar calculation yields the spatial correlation function at fixed time:

$$\begin{aligned} \overline{\phi(\mathbf{x}, t) \phi(\mathbf{x} + \mathbf{r}, t)} &= \lim_{V \rightarrow \infty} \frac{1}{V} \int_V d^n x \sum_{j,k} \alpha_j \alpha_k \\ &\quad \exp\left(-\frac{|\mathbf{x} - \mathbf{x}_j|^2 + |\mathbf{x} + \mathbf{r} - \mathbf{x}_k|^2}{2\sigma^2}\right) \\ &= \lim_{V \rightarrow \infty} \frac{1}{V} \sum_{j,k} \alpha_j \alpha_k \int_V d^n x \\ &\quad \exp\left(-\frac{|\mathbf{x} - \mathbf{x}_j|^2 + |\mathbf{x} + \mathbf{r} - \mathbf{x}_k|^2}{2\sigma^2}\right). \end{aligned} \quad (8)$$

The exponent in the exponential in eq. (8) can be rearranged as follows:

$$\begin{aligned} |\mathbf{x} - \mathbf{x}_j|^2 + |\mathbf{x} + \mathbf{r} - \mathbf{x}_k|^2 &= 2 \left[\mathbf{x} + \frac{(\mathbf{r} - \mathbf{x}_j - \mathbf{x}_k)}{2} \right]^2 \\ &\quad + \frac{1}{2} (\mathbf{r} + \mathbf{x}_j - \mathbf{x}_k)^2. \end{aligned} \quad (9)$$

Therefore,

$$\begin{aligned} \overline{\phi(\mathbf{x}, t) \phi(\mathbf{x} + \mathbf{r}, t)} &= \lim_{V \rightarrow \infty} \frac{1}{V} \sum_{j,k} \alpha_j \alpha_k \\ &\quad \exp\left(-\frac{(\mathbf{r} + \mathbf{x}_j - \mathbf{x}_k)^2}{4\sigma^2}\right) \\ &\quad \times \int_V d^n x \exp\left\{-\frac{1}{\sigma^2} \left[\mathbf{x} + \frac{(\mathbf{r} - \mathbf{x}_j - \mathbf{x}_k)}{2} \right]^2\right\} \\ &= (\pi\sigma^2)^{\frac{n}{2}} \lim_{V \rightarrow \infty} \frac{1}{V} \sum_{j,k} \alpha_j \alpha_k \\ &\quad \exp\left(-\frac{(\mathbf{r} + \mathbf{x}_j - \mathbf{x}_k)^2}{4\sigma^2}\right) \\ &= (\pi\sigma^2)^{\frac{n}{2}} \sigma_\alpha^2 \lim_{V \rightarrow \infty} \frac{1}{V} \sum_k \exp\left(-\frac{\mathbf{r}^2}{4\sigma^2}\right) \\ &= (\pi\sigma^2)^{\frac{n}{2}} \sigma_\alpha^2 \rho \exp\left(-\frac{\mathbf{r}^2}{4\sigma^2}\right), \end{aligned} \quad (10)$$

where the integral corresponds to the normalisation of an n -dimensional Gaussian, and can also be evaluated taking the formulas for volume and surface of the n -dimensional hypersphere given, *e.g.*, in ref.⁹ Thus, we see from eq. (10) that this model has a spatial correlation function with a Gaussian structure, *i.e.*, it is a realization of random field with a Gaussian correlation function of the type described in ref.³

It is also possible to evaluate the ensemble averages $\langle \phi(\mathbf{x}, t) \rangle$ and $\langle |\phi(\mathbf{x}, t)|^2 \rangle$. As a result of the statistical properties of the α_k 's and of the $\mathbf{x}_k$'s, the ensemble average of the field vanishes

$$\langle \phi(\mathbf{x}, t) \rangle = \sum_k \langle \alpha_k \rangle \left\langle \exp\left(-\frac{|\mathbf{x} - \mathbf{x}_k|^2}{2\sigma^2}\right) \right\rangle = 0, \quad (11)$$

and the mean square value is,

$$\begin{aligned} \langle |\phi(\mathbf{x}, t)|^2 \rangle &= \sum_{j,k} \langle \alpha_j \alpha_k \rangle \left\langle \exp\left(-\frac{|\mathbf{x} - \mathbf{x}_j|^2}{2\sigma^2}\right) \right. \\ &\quad \left. \exp\left(-\frac{|\mathbf{x} - \mathbf{x}_k|^2}{2\sigma^2}\right) \right\rangle \\ &= \sigma_\alpha^2 \sum_k \left\langle \exp\left(-\frac{|\mathbf{x} - \mathbf{x}_k|^2}{\sigma^2}\right) \right\rangle \\ &= \rho \sigma_\alpha^2 \int_0^\infty \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} r^{n-1} \exp\left(-\frac{r^2}{\sigma^2}\right) dr \\ &= \pi^{\frac{n}{2}} \sigma^n \rho \sigma_\alpha^2 \end{aligned} \quad (12)$$

Figure 1. 3D map (left) and contour plot (right) of a 2D random field with 400 $\mathbf{x}_k$'s, with each coordinate in the range $(-5, +5)$ and related amplitudes α_k 's, so that $\rho = 4$. Other parameters are $\sigma = 0.5$, $\sigma_\alpha = 1$

If the field represents a potential energy, then the corresponding stochastic forces in the Langevin equation are,

$$\mathbf{F} = -\nabla\phi = \sum_k \alpha_k \frac{\mathbf{x} - \mathbf{x}_k}{\sigma^2} \exp\left(-\frac{|\mathbf{x} - \mathbf{x}_k|^2}{2\sigma^2}\right) \quad (13)$$

and therefore $\langle \mathbf{F} \rangle = 0$, and

$$\begin{aligned} \langle |\mathbf{F}|^2 \rangle &= \frac{1}{\sigma^4} \sum_{j,k} \left\langle \alpha_j \alpha_k (\mathbf{x} - \mathbf{x}_j) \cdot (\mathbf{x} - \mathbf{x}_k) \right. \\ &\quad \left. \exp\left(-\frac{|\mathbf{x} - \mathbf{x}_j|^2}{2\sigma^2} - \frac{|\mathbf{x} - \mathbf{x}_k|^2}{2\sigma^2}\right) \right\rangle \\ &= \frac{\sigma_\alpha^2}{\sigma^4} \sum_k \left\langle |\mathbf{x} - \mathbf{x}_k|^2 \exp\left(-\frac{|\mathbf{x} - \mathbf{x}_k|^2}{\sigma^2}\right) \right\rangle \\ &= \frac{\sigma_\alpha^2 \rho}{\sigma^4} \int_0^\infty \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} r^{n+1} \exp\left(-\frac{r^2}{\sigma^2}\right) dr \\ &= n\pi^{\frac{n}{2}} \sigma^{n-2} \sigma_\alpha^2 \rho = n \frac{\langle |\phi(\mathbf{x}, t)|^2 \rangle}{\sigma^2} \quad (14) \end{aligned}$$

The forces, eq. (13), replace the usual term $\xi \mathbf{v}(t)$ in eq. (1). Since the mean square value, eq. (14), coincides with the variance, which is the sum of the n equal variances due to the individual noise components, we find that in the present context the value of ξ^2 is

$$\xi^2 = \pi^{\frac{n}{2}} \sigma^{n-2} \sigma_\alpha^2 \rho = \frac{\langle |\phi(\mathbf{x}, t)|^2 \rangle}{\sigma^2} \quad (15)$$

When we use the usual relation with the diffusion coefficient and the friction term

$$\xi^2 = 2D\gamma^2 = 2k_B T \gamma \quad (16)$$

we can find the diffusion coefficient

$$D = \frac{\xi^2}{2\gamma^2} = \frac{\pi^{\frac{n}{2}} \sigma^{n-2} \sigma_\alpha^2 \rho}{2\gamma^2} \quad (17)$$

associated with this stochastic background, and the constraint

$$\pi^{\frac{n}{2}} \sigma^{n-2} \sigma_\alpha^2 \rho = 2k_B T \gamma \quad (18)$$

Figure 2. Representation of a 3D random field with 512 $\mathbf{x}_k$'s, with each coordinate in the range $(-5, +5)$. The dots show the positions of the centres $\mathbf{x}_k$'s, while the spheres have the same centres and radius equal to the amplitude α_k . In this case $\rho = 0.512$, while the other parameters are $\sigma = 0.5$, $\sigma_\alpha = 1$.

which relates the dimensionality of space n with the parameters σ , σ_α , and ρ . Notice also that dimension $n = 2$ is unique in that neither ξ^2 , nor D depend on σ .

2.2 Numerical simulation

Here, I use a standard update scheme¹⁰ for the Ornstein-Uhlenbeck process described by the Langevin eq. (1):

$$x(t + \Delta t) = x(t) - \lambda x(t)\Delta t + \xi(\Delta t)^{\frac{1}{2}}s \quad (19)$$

where s is a sample drawn from a Gaussian distribution $N(0, 1)$ (zero mean, unit standard deviation). More

accurate update schemes are available, however in this first computational study I have decided to balance speed and accuracy by choosing a conventional scheme and a small time-step instead. For a discussion of an alternate integration scheme, see ref.¹⁰

The simulation of space-correlated noise is computationally intensive, as it requires the construction of a random field $\phi(\mathbf{x}, t)$ by means of the extraction of a large number of space points (the $\mathbf{x}_k$'s) and amplitudes (the α_k 's) at each time step. Figure 1 shows a snapshot of one simulated 2D random field. Figure 2 shows a

Figure 3. Paths of two 2D random walkers (marked in red and blue) in a background of space-correlated noise. The simulated random fields have on average 400 $\mathbf{x}_k$'s (the actual number is a Poisson variate) so that here $\rho = 4$. Other parameters that define the random fields are $\sigma = 0.5$, $\sigma_\alpha = 1$. Moreover, here as in the other simulation runs $\lambda = 10$, $\Delta t = 0.00001$. In this case the initial separation of the random walkers is 0.3. Left panel: complete display of the generated paths (8192 steps). Right panel: only the final part of the walks (steps 6000 to 8192); the final positions are marked by the dots.

Figure 4. Left panel: histogram of the squared distance between the two random walkers in 1000 realizations of the walks at timestep $j = 8192$. Here, the initial separation is 0.5, the other parameters are $\rho = 4$, $\sigma = 0.5$, $\sigma_\alpha = 1$, $\lambda = 10$, $\Delta t = 0.00001$. Right panel: median of the squared distance between random walkers vs. timestep. I use the median instead of the mean because of the very high skewness of the distribution in the right panel. Instead of increasing, as predicted by the standard theory of Brownian motion, eq. (20), where the mean coincides with the median, here we see that the median decreases as time increases. The behaviour shown here is typical and is qualitatively the same for a wide range of initial separations.

Figure 5. Upper panel: paths traced by 10 different random walkers that start at positions scattered in the square with vertices ± 2 (50000 timesteps). The different colours correspond to different walkers: the range of colours is compressed by the convergence of several paths. Lower panel: the last 10000 steps show that the paths converge to just two configurations, which can be expected to merge in the long time.

representation of a simulated 3D random field. Here, as well as in the following figures and simulations, all units are arbitrary.

3. Results and Discussion

In the spatially-correlated noise scenario, the random background is the same for all random walkers, therefore two random walks can be different only if they start from different origins, and the simulation runs have been carried out with different separations between the origins.

Random walkers that are widely separated at start are likely to follow different paths in the space-correlated noise landscape. On the contrary, random walkers that are close at start can be expected to be drawn to converging paths. Simulation runs show that this trend is extremely strong, and the different random walkers are very strongly drawn to very close trajectories, as shown in Figure 3.

The behaviour displayed by the 2D random walkers in Figure 3 is replicated in varying degrees for different initial separations, but it does not go away even for an initial separation 1 (which, in this case, is twice the width of the Gaussian components of the random field, $\sigma = 0.5$).

This behaviour deviates starkly from the standard behaviour predicted by eqs. (4) and (5). From the simulation parameters, we can compute the diffusion coefficient, see eq. (16), and in this case, we find $D \approx 0.063$. Since the timestep interval in the simulations is $\Delta t = 0.00001$, the predicted mean square separation as a function of timestep j in the 2D case is

$$\langle (\mathbf{x}_1 - \mathbf{x}_2)^2 \rangle + \delta \mathbf{x}^2 = 8Dj\Delta t + \delta \mathbf{x}^2 \approx 5 \cdot 10^{-6} j + \delta \mathbf{x}^2, \quad (20)$$

i.e., for the largest timestep $j = 8192$,

$$\sqrt{\langle (\mathbf{x}_1 - \mathbf{x}_2)^2 \rangle + \delta \mathbf{x}^2} \approx \sqrt{0.04 + \delta \mathbf{x}^2}. \quad (21)$$

This must be compared with the behaviour of $\langle (\mathbf{x}_1 - \mathbf{x}_2)^2 \rangle + \delta \mathbf{x}^2$ observed in the simulation runs, see Figure 4. With space-correlated noise, median separation decreases, instead of increasing as predicted by eq. (20), and this does not depend on the initial separation of the random walkers.

This decrease of the mean square distance is especially striking when we consider more than two random walkers. Figure 5 shows the paths traced by 10 different random walkers, again with parameters $\rho = 4$, $\sigma = 0.5$, $\sigma_\alpha = 1$, $\lambda = 10$, $\Delta t = 0.00001$. The simulation shows that eventually, after 50000 timesteps, the paths converge

to just two different configurations which we can expect to merge in the long time.

4. Conclusions

The theory and the numerical results presented in this paper show that the spatially-correlated noise considered here acts as a powerful entropic force which pushes different walkers to follow very close paths, at least in 2D. This kind of behaviour is seemingly unnatural, and in these simulations it appears already with a noise that has a rather short-range correlation. So, we can expect that the problem worsens with longer-range correlations and it is natural to wonder what kind of change in this scheme could make more physical sense. There are some leads that I plan to follow: a rather obvious one is that the noise in the Langevin equation could have an uncorrelated component. Another possibility is that in 3D the overall behaviour does differ from 2D, and there are a couple hints that it could be so. The first one is that random walks in 3D are quite different from those in 2D: Pólya's theorem guarantees that 2D walks always intersect, while this does not happen in 3 and higher dimensions.¹⁰ The second one comes from equations (15) and (17), which show that dimension 2 is unique in making the noise variance and the diffusion coefficient independent from σ .

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