Proc. Indian Acad. Sci. (Chem. Sci.), Vol. 99, Nos 1 & 2, August 1987, pp. 9–20.

On the use of generalized Euler transformation in handling divergent series

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Abstract. A parametrized version of the Euler transformation, introduced fairly recently, is employed to study the behaviour of functions, given their formal power-series (alternating) expansions in λ with finite radii of convergence, in the limit $\lambda \to \infty$. The strategy requires only the first few low-order data. Results are tested with quite a few known cases and found remarkably satisfactory. The role of some other methods in this context are briefly discussed.

Keywords. Power series; divergence; perturbation theory; asymptotic expansions; Euler transformation.

1. Introduction

The appearance of divergent series is quite common in various fields of theoretical physics and chemistry (see, for example, Baker 1965; Baker and Gammel 1970) and these days one would *not* agree with Abel's remark (Hardy 1956) that 'it is shameful to base on them any demonstration whatsoever'.

A series representation of some function $F(\lambda)$ of the form

$$F(\lambda) = \sum_i f_i \lambda^i, \tag{1}$$

well outside the radius of convergence (λ_0), is only a formal one. However, in some cases, the asymptotic sum $F_a(\lambda) \approx F(\lambda)$ may be useful for small positive values of $(|\lambda| - \lambda_0)$. But, when $|\lambda| \gg \lambda_0$, $F_a(\lambda)$ makes no sense and we are really in trouble if $F(\lambda)$ is to be evaluated even to a moderate degree of accuracy. To deal with such cases, fortunately, a number of strategies are now available (and are being developed) which offer reliable estimates of $F(\lambda)$, given the coefficients f_i. Of these, the more widely studied ones during the past few years are the Padé approximants (PA) (Baker 1965, 1975; Baker and Gammel 1970; Baker and Graves-Morris 1980), the Borel transformation (Hardy 1956; Zinn Justin 1981; Simon 1982), continued fraction techniques (CFT) (Wall 1948; Jones and Thron 1980; Cizek and Vrscay 1982, 1984; Vrscay 1986), Aitken's Δ^2 method (Baker and Graves-Morris 1980), Wynn's ε-convergence algorithm (Shanks 1955), the method of order-dependent mapping (ODM) (Seznec and Zinn Justin 1979), the functional method (Arteca *et al* 1984) etc.

Most of the above techniques employ a considerably large number of coefficients f_i to obtain $F(\lambda)$ rather accurately. However, often, it so happens that one does not have accurate estimates of f_i in hand except for the very first few, due chiefly to the complicated nature of the problem concerned. Keeping this *practical* difficulty in mind, we have recently developed (Bhattacharyya 1982) a parametrized version of the well-known Euler transformation, to be henceforth called the parametrized Euler transformation (PET). It has been observed that such simple 1- and 2-parameter transforms work very successfully, keeping in mind our limitations regarding the knowledge of $\{f_i\}$. For further applications and employment of some variants of this transformation, we refer to Silverman (1983) and Sangaranarayanan and Rangarajan (1983, 1984). The PET, we have explicitly demonstrated, is closely related to PA, and hence to CFT of specific varieties. Sangaranarayanan and Rangarajan (1983) found a close kinship of PET(1), the 1-parameter transform, to Aitken's Δ^2 method. Thus, PET appears to be a very general scheme by which a poorly-converging series may be accelerated or, more profitably, a divergent series can be successfully handled.

Perturbation theory often offers itself as a logical candidate for applying the various techniques of handling divergent series, not just because accurate results of $F(\lambda)$ are otherwise available, the coefficients f_i are known to considerably large orders, or because of practical importance; the more interesting aspect is that there are some inherent flexibilities in a perturbative approach which *can* be employed for series-acceleration. Stated otherwise, perturbation theory by itself is capable of offering some specific summability schemes. These are actually achieved by scaling. Rather surprisingly, one finds that such *basically perturbative* summability techniques are also versions of the PET (for detail, see Bhattacharyya 1982). This observation again points to the increased generality of the PET. In other cases, however, one has to check whether a particular technique is *superior* to these scaling schemes. Only, PA of specific types have correspondences with scaling of the zero-order Hamiltonian (Wilson *et al* 1977).

The primary aim of the present communication is to explore whether, by employing the PET, some information about the behaviour of $F(\lambda)$ in the limit $\lambda \to \infty$ can be extracted, if the parent series (1) of $F(\lambda)$ is known to, say, the third order, i.e. the coefficients from f_4 onwards are all unknown. At a first thought, it appears ridiculous. If at all, the sum

$$F(3) = \sum_{i=0}^{3} f_i \lambda^i \tag{2}$$

can at best approximate $F(\lambda)$ only when $\lambda \to 0$, $\lambda_0 > 0$. On the other hand, if $\lambda_0 = 0$, $f(3)$ may satisfy $f(3) \approx F_a(\lambda) \approx F(\lambda)$, again for very small values of $|\lambda|$ and over a small range. But, as we shall see, the coefficients f_i indeed contain reasonable information regarding the large-λ behaviour of $F(\lambda)$. However, there is a condition: the series expansion (1) *must be alternating* in nature. Of course, this is a severe restriction and has to be ensured a priori. Also, it limits the applicability of the scheme we are going to propose. Still, we may remark that there are many such alternating series where some kind of symmetry forces them to be so. It is precisely in these cases where without explicitly calculating the higher-order coefficients ($\geqslant f_4$) we can be certain about the signs of such terms. Anyway, the problem that we have posed here is in no way going to become a trivial one owing to the imposed alternating nature of $F(\lambda)$. Moreover, to our knowledge, *none* of the summability schemes have ever been employed for this purpose with such little resource. Thus, the PET will probably be the first scheme to play a significant role in this context. We shall also pay some attention to building up a connection between the PET and the generalized Euler transformation (GET) of Silverman (1983). A link between the ODM and the PET (2) will again be considered very briefly. Finally, the possibility of profitably using some of the other summability techniques with the information we extract will be indicated.

Our organization will be as follows. In §2, some aspects of the PET will be considered, for convenience. Section 3 will deal with the workability of the scheme. An introduction to the possible use of some other techniques in this particular context will be taken up in §4. We reserve §5 for a discussion on the relation of the PET with GET and ODM. In §6, some problems regarding the workability of the present scheme will be discussed.

2. Aspects of the PET

The Euler transformation (Morse and Feshbach 1953) is basically concerned with a change of variable. Thus, for an alternating series of the form

$$\bar{F}(\lambda) = \sum_{i=0} (-1)^i f_i \lambda^i, \tag{3}$$

a new variable $\lambda_1 = \lambda/(1+\lambda)$ is chosen to obtain

$$\bar{F}(\lambda) = [1/(1+\lambda)] \sum_{i=0} f_i^{(1)} \lambda_1^i, \tag{4}$$

where the new coefficients are related to the old ones. For non-alternating series of the form (1), it is sometimes useful to define a new variable, $\bar{\lambda}_1 = \lambda/(1-\lambda)$, to find

$$F(\lambda) = [1/(1-\lambda)] \sum_{i=0} \bar{f}_i^{(1)} \bar{\lambda}_1^i. \tag{5}$$

In the above cases, we actually applied the transformation repeatedly. But, we could have applied it a finite number of times. For example, if we apply it just once to (3), we obtain

$$\bar{F}(\lambda) = [f_0/(1+\lambda)] + \lambda_1 \sum_{i=0} (-1)^n (f_n - f_{n-1}) \lambda^n. \tag{6}$$

The GET (Morse and Feshbach 1953) involves a known function $G(\lambda)$ of the form

$$G(\lambda) = \sum_{i=0} g_i \lambda^i, \tag{7}$$

and leads, for the series (1), to the transformed expansion

$$F(\lambda) = \sum_{i=0} (-1)^i D_i \frac{\lambda^i}{i!} \frac{d^i}{d\lambda^i} G(\lambda), \tag{8}$$

where $$D_i = \sum_{r=0}^{i} (-1)^r \binom{i}{r} a_r, \; a_r = f_r/g_r. \tag{9}$$

For the series (3), we have taken $G(\lambda) = (1+\lambda)^{-1}$.

As regards the PET, we have considered the 1- and 2-parameter transforms—PET(1) and PET(2), respectively. In PET(1), we have taken $G(\lambda) = (1+k\lambda)^{-1}$ and in PET (2), the auxiliary function has been rendered more flexible by choosing it as $(1+k\lambda)^{-m}$, where k and m are two adjustable parameters. Such transformations may again be applied just once or repeatedly. The new variables in the latter case become $\lambda/(1+k\lambda)$ and $\lambda/(1+k\lambda)^m$, respectively, for PET(1) and PET(2). For a detailed discussion and explicit relations, we refer to the original work (Bhattacharyya 1982).

Here we shall mainly concentrate on PET(2) and apply it just once to a series like (1) to obtain

$$F(\lambda) = [1/(1+k\lambda)^m]\left(f_0+\lambda \sum_{i=0} D_1 f_i \lambda^i\right), \tag{10}$$

where
$$D_1 f_{n-1} = \sum_{r=0}^{n} f_{n-r} \binom{m}{r} k^r . \tag{11}$$

The parameters k and m are fixed by setting

$$D_1 f_0 = 0 = D_1 f_1 , \tag{12}$$

to approximate $F(\lambda)$ by

$$F(\lambda) \approx f_0(1+k\lambda)^{-m}, \tag{13}$$

where

$$k = (f_1^2 - 2f_0 f_2)/f_0 f_1 , \quad m = f_1^2/(2f_0 f_2 - f_1^2). \tag{14}$$

This is termed as the [0, 2] PET(2), i.e. the right side of (13) is designated by [0, 2] PET(2) $F(\lambda)$. If, on the other hand, we split $f(\lambda)$ as

$$f(\lambda) = f_0+\lambda \sum_{i=1} f_i \lambda^{i-1} = f_0+\lambda F_1(\lambda), \tag{15}$$

and construct [0, 2] PET(2)$F_1(\lambda)$ along similar lines, we obtain another approximation for $F(\lambda)$ which we shall denote by

$$F(\lambda) \approx f_0+\lambda[0, 2]\ \mathrm{PET}(2) F_1(\lambda) = [1, 3]\ \mathrm{PET}(2) F(\lambda). \tag{16}$$

For alternating series, the *most* important point that we like to mention here is that either the following inequality,

$$[0, 2]\ \mathrm{PET}(2)F(\lambda) > F(\lambda) > [1, 3]\ \mathrm{PET}(2)F(\lambda), \tag{17}$$

is obeyed or the reverse,

$$[0, 2]\ \mathrm{PET}(2)F(\lambda) < F(\lambda) < [1, 3]\ \mathrm{PET}(2)F(\lambda), \tag{18}$$

holds. We are yet to prove these inequalities, but many cases have been studied and the inequalities, either (17) or (18), are found to be satisfied. In fact, we have noticed that any two successive PET(2) obey either of them. This is an additional qualification of the PET.

3. Applications

Before proceeding further, a brief outline of the scheme that we shall follow deserves mention. So, we first discuss what exactly is to be done and what information is needed.

3.1 *The scheme*

Choose an alternating series for which f_0 to f_3 are known. Be sure that λ_0 is finite. Find from some other source one or two values of $F(\lambda)$, reasonably accurate, when $|\lambda| \gg \lambda_0$. Check the inequalities (17) or (18) after constructing the [0, 2] and [1, 3] PET(2) approximants to $F(\lambda)$ in regions $|\lambda| \gg \lambda_0$. Choose now the closer approximant and infer from its behaviour how $F(\lambda)$ will change with λ in the limit $\lambda \to \infty$. The variation of $F(\lambda)$ with λ in this limit will thus be bounded either above or below by virtue of (17) or (18).

We may remark at this stage that (i) if values of $F(\lambda)$, $|\lambda| \gg \lambda_0$ are not available, one still obtains bounds to variations of $F(\lambda)$ with λ, for large λ, but one of such bounds may not be very useful; (ii) such a procedure is of value if λ_0 is finite; (iii) it is unlikely that at two different λ-values beyond λ_0, two *different* said approximants are closer. Regarding the last point, our experience is that if, e.g., the [0, 2] PET(2) approximant is closer to $F(\lambda)$ for some $|\lambda| > \lambda_0$, it *continues* to remain so. Moreover, it does not change its role—if it is found to be an upper bound for some $|\lambda| > \lambda_0$, the property continues.

3.2 *Some examples*

(i) $$\ln(1+\lambda) = \lambda(1-\lambda/2+\lambda^2/3-\lambda^3/4+\ldots.) \tag{19}$$

Here, for large λ, one finds the [0, 2] variety closer and that

$$F(\lambda) < [0, 2]\ \mathrm{PET}(2)F(\lambda) = \lambda/(1+\tfrac{5}{6}\lambda)^{3/5}. \tag{20}$$

Thus, we have

$$F(\lambda) < \sim \lambda^{2/5}, \tag{21}$$

which cannot be directly tested, however. Another interesting feature emerges from (20). We find

$$1+\lambda \leqslant e^{\lambda/(1+5\lambda/6)^{3/5}}, \tag{22}$$

or, $$e \geqslant (1+\lambda)^{(1+5\lambda/6)^{3/5}/\lambda}, \tag{23}$$

which holds for any λ. Writing $\lambda = 1/n$ we obtain

$$e \geqslant \left(1+\frac{1}{n}\right)^{n(1+5/6n)^{3/5}}, \tag{24}$$

which may be compared with the standard definition of e

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n, \tag{25}$$

ignoring the limit or, with a recently introduced (Mermin 1984) refined variety†,

$$e \approx \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}. \tag{26}$$

Table 1 shows the results. The bounding nature of (25), $e \geqslant (1+1/n)^n$, follows from the inequality $\ln(1+\lambda) \leqslant \lambda$ for $0 \leqslant \lambda$. On the other hand, (26) offers no such systematism. The advantage of using PET(2) is clear from the table.

(ii)
$$\frac{\tan^{-1}\lambda}{\lambda} = 1-\lambda^2/3+\lambda^4/5-\ldots. \tag{27}$$

In this case, one finds that again the [0, 2] approximant is closer to the function for large λ. Calculation shows

$$\lim_{\lambda\to\infty} \frac{\tan^{-1}\lambda}{\lambda} < \sim \lambda^{-10/13}. \tag{28}$$

The agreement is remarkable if we remember that $\lim_{\lambda\to\infty} \tan^{-1}\lambda = \pi/2$ and hence the left part of (28) varies as λ^{-1}.

(iii) We shall now consider the anharmonic oscillator case for which the Hamiltonian is

$$H = -\nabla^2+x^2+\lambda x^4. \tag{29}$$

The energy eigenvalues $E_n(\lambda)$ for such a system are known to be *formally* represented by alternating power-series expansions in λ. The expansion coefficients are taken from Reid (1967). We have studied this system earlier and found that the [0, 2] PET(2) is closer to $E_n(\lambda)$ for any λ and n (note that here $\lambda_0 = 0$) and obeys

$$E_n(\lambda) > [0,2]\ \mathrm{PET}(2)\ E_n(\lambda). \tag{30}$$

An important point with such eigenvalue functions is that the large-λ behaviour of every $E_n(\lambda)$ is the same: $\lim_{\lambda\to\infty} E_n(\lambda) \sim \lambda^{1/3}$. This, in fact, follows from a coordinate scaling argument applied to H and is originally due to K Symanzik (Simon 1970). The results are presented in table 2. Clearly, the approximate behaviour gradually improves with n, though it is unlikely to reach the exact value. After all, that is too much to expect from such a simple scheme involving really *very limited* information.

Table 1. The value of e.

n	$(1+1/n)^n$	$(1+1/n)^{n+1/2}$	$(1+1/n)^{n(1+5/6n)^{3/5}}$
1	2·0	2·828...	2·7106...
10	2·59...	2·7203...	2·71826...
100	2·704...	2·71830...	2·718281...

†It follows from (24) as a first approximation to modify (25) for finite n.

Table 2. Large-λ behaviour of eigenvalues of H given by (29).

$n =$	0	1	2	3	5	7	10	20
$E_n(\lambda) >$	$\lambda^{0.1765}$	$\lambda^{0.1852}$	$\lambda^{0.1983}$	$\lambda^{0.2033}$	$\lambda^{0.2067}$	$\lambda^{0.2079}$	$\lambda^{0.2086}$	$\lambda^{0.2091}$

(iv) Here we choose the hydrogenic s-state Hamiltonian perturbed by a radial electric field

$$H = -\frac{1}{2}\nabla^2 - \frac{1}{r} + \lambda r, \tag{31}$$

and consider the ground state eigenvalue $E_0(\lambda)$ for which the expansion coefficients are available (Austin 1980). A scaling argument in this case dictates that the eigenvalues $E_n(\lambda)$ should behave as $\sim \lambda^{2/3}$ in the large-λ regime. For this system, calculation shows that the [1, 3] approximant is closer to $E_0(\lambda)$, $\lambda > \lambda_0 = 0$, and obeys

$$E_0(\lambda) < [1, 3]\ \mathrm{PET}(2)\ E_0(\lambda). \tag{32}$$

Also,

$$\lim_{\lambda\to\infty}\ [1, 3]\ \mathrm{PET}(2)\ E_0(\lambda) \sim \lambda^{0.875}, \tag{33}$$

so that we get the result

$$\lim_{\lambda\to\infty}\ E_0(\lambda) < \sim \lambda^{0.875}. \tag{34}$$

Needless to mention, we could have also taken advantage of other n-values to obtain, hopefully, still better estimates.

(v) As the last example, we take the function

$$\int_0^\infty e^{-t}\,\mathrm{d}t/(1+\lambda t) = 1 - 1!\,\lambda + 2!\,\lambda^2 - 3!\,\lambda^3 + \ldots, \tag{35}$$

with the associated expansion ($\lambda_0 = 0$) which is again asymptotic in nature. The value of the function at $\lambda = 1$ is ≈ 0.5963 and it may be checked that the [0, 2] PET(2) approximant is closer to this value satisfying

$$F(\lambda) < [0, 2]\ \mathrm{PET}(2)\ F(\lambda). \tag{36}$$

Further, the right side of (36) in the large-λ limit becomes $\sim \lambda^{-1/3}$ so that we have

$$\lim_{\lambda\to\infty}\ F(\lambda) < \sim \lambda^{-1/3}. \tag{37}$$

Actually, however, $\lim_{\lambda\to\infty} F(\lambda) \sim (\ln \lambda)/\lambda$. The result (37) that we have just obtained here is not very good if we remember the result of our first example: $\ln(\lambda+1) \approx \ln\lambda < \sim \lambda^{2/5}(\lambda \to \infty)$. Coupling this with the exact behaviour, we find

$$\lim_{\lambda\to\infty} F(\lambda) \sim \frac{\ln \lambda}{\lambda} < \lambda^{-0\cdot 6}, \tag{38}$$

which immediately points to the rather poor performance of (37).

3.3 *Discussion*

Our observations are collected in table 3. It demonstrates convincingly the power of the PET in extracting the behaviour of $\lim_{\lambda\to\infty} F(\lambda)$ from just the first few terms. $E(\lambda)$ in this table refers to any eigenvalue of the Hamiltonian defined by (29) and $E_0(\lambda)$ signifies the ground state eigenenergy corresponding to the Hamiltonian (31). Let us note that the worst estimate is obtained for (35). Indeed, the goodness of the predicted behaviour depends on the *tightness* of the bounds. In certain situations, to avoid looseness of the bounds, we are free to take the derivative series. The point is, one should then have at least one or two accurate data for the derivative for λ-values well beyond λ_0.

4. Role of a few other techniques

As we have mentioned earlier, so far no summability technique has been used for the purpose in hand. But it is easy to see that if the large-λ behaviour of $F(\lambda)$

Table 3. Large-λ behaviour of some functions.

	$\lim_{\lambda\to\infty} F(\lambda)$	
$F(\lambda)$	Predicted	Exact
$\ln(1+\lambda)$	$< \lambda^{0\cdot 4}$	—
$\frac{\tan^{-1}\lambda}{\lambda}$	$< \lambda^{-0\cdot 77}$	λ^{-1}
$E(\lambda)$	$> \lambda^{0\cdot 209}$	$\lambda^{0\cdot 333}$
$E_0(\lambda)$	$< \lambda^{0\cdot 875}$	$\lambda^{0\cdot 667}$
$\int_0^\infty \frac{e^{-t}dt}{1+\lambda t}$	$< \lambda^{-0\cdot 33}$	$\frac{\ln\lambda}{\lambda} < \lambda^{-0\cdot 6}$

is known beforehand, we can profitably employ suitable summability schemes. For example, if $\lim_{\lambda\to\infty} F(\lambda) \sim \lambda^m (m : \text{integer})$ we may advocate the use of $[(n+m)/n]$ PA$(n = 0, 1, 2, \ldots)$ for $m > 0$. Similarly, if m were a negative integer, $[n/(n+m)]$ PA could be tried at first. On the other hand, if $\lim_{\lambda\to\infty} F(\lambda) \sim$ constant, we can try $[n/n]$ PA. Keeping in mind the close correspondence between the PA and the CFT, we refrain from a discussion on the possible impact of a knowledge of $F(\lambda)$ (in the large-λ regime) on CFT. Rather, we note that such information is efficiently taken into account in schemes like the ODM or the functional method, leading thus to very accurate estimates of $f(\lambda)$ for $|\lambda| \gg \lambda_0$.

It is now quite apparent that, in the aforesaid issue, potential use of the information we obtain by employing the PET(2) can be made. Thus, our results will *at least* help us in selecting certain other summability techniques to be employed for accurate data. Such a selection is likely to be advantageous, though not always; at least we can say that it will *not* be disadvantageous as compared to some *randomly* selected scheme. For example, Leinaas and Osnes (1980) used $[(n+1)/n]$ PA and coupled Borel-Pade approximants on their way to studying a model intruder problem, guided by the linear λ-dependence of $E(\lambda)$ for large $|\lambda|$ and found encouraging results for negative values of λ. For positive λ, however, the results were not found to be *quite* satisfactory. Likewise, Baker (1965) discussed a few cases where $[n/n]$ PA show an *oscillatory* behaviour, though the functions concerned obey $\lim_{\lambda\to\infty} F(\lambda) \sim$ constant. Keeping aside such pathological cases, we may hope that the predicted large-λ behaviour will guide us properly towards betterment.

5. The PET, GET and ODM

The GET put forward by Silverman (1983) rests on a change of variable from λ to μ, where

$$\mu = \lambda/(1-\sigma\lambda), \tag{39}$$

and the parent series $F(\lambda)$ is rewritten as

$$F(\lambda) = \left(\frac{1}{1-\sigma\lambda}\right)^m \sum_{j=0} a_j \mu^j, \; a_j = a_j(\sigma, m). \tag{40}$$

Another transformation that has been used corresponds to just a scaling of the new variable μ:

$$\nu = (1-\sigma)\mu, \tag{41}$$

and this need not be considered for the present purpose. We note first that this is actually a two-parameter transform. But, whereas in our case the variable was of the form (if applied repeatedly),

$$\lambda_{km} = \lambda/(1+k\lambda)^m, \tag{42}$$

here the variable μ is of the same form as our PET(1) case. In other words, the transformation leading to (40) is actually a *mixed* transformation, as we have checked. First, the PET(2) is applied *just once* by choosing $(1-\sigma\lambda)^{m-1}$ as the parametrized function and then, to the new series that has emerged, the PET(1) is applied *repeatedly* by choosing $(1-\sigma\lambda)$ as the parametrized function [see (21)–(24) in Bhattacharyya 1982]. We fail to recognize how 'in several fundamental aspects' this particular GET differs from earlier work, though the author claims that it is so.

Repeated application of the PET(2) leads ultimately to

$$F(\lambda) = [1/(1+k\lambda)^m] \sum_{i=0} D_i f_0 \, \lambda_{km}^i, \tag{43}$$

where λ_{km} is given by (42), and

$$D_p f_{n-1} = \sum_{r=0}^{n} D_{p-1} f_{n-r} \binom{m}{r} k^r. \tag{44}$$

Rewriting (43) as

$$F(\lambda)\,(1+k\lambda)^m = \sum_{i=0} b_i \, \lambda_{km}^i, \tag{45}$$

and viewing the left hand side as some series in λ, we may say that the PET(2) has actually transformed some $\sum_i c_i \lambda^i$ to $\sum_i b_i \lambda_{km}^i$. Here $\{c_i\}$ and $\{b_i\}$ are naturally related through k and m. Let us now go back from $\sum_i b_i \lambda_{km}^i$ to $\sum_i c_i \lambda^i$. This will be achieved by putting the relation (42) in the former series for some chosen k and m. If, instead, we like to go back to a scaled-λ series, (42) is to be accordingly modified and then employed in $\sum_i b_i \lambda_{km}^i$. This modified version of (42) will read as

$$\lambda_{km} = \alpha\lambda/(1+k\alpha\lambda)^m. \tag{46}$$

Choosing $k = \alpha^{-1}$, the method of ODM follows immediately. Thus, we recognize ODM as the *scaled inverse* of the PET(2). This completes our discussion on the relationship of the PET(2) with the GET and the ODM.

6. Some unsolved problems

The first problem that we wish to mention is concerned with the bracketing nature of the two consecutive $[p, p+2]$ PET(2) approximants. We have found that this is obeyed for alternating series by choosing a variety of examples. But, a rigorous proof is necessary.

Secondly, we remark that here we are working with power-series expansions for which our scheme leads to either $F(\lambda) \sim \lambda^{\alpha}$- or $F(\lambda) \sim \lambda^{-\alpha}$-type of behaviour ($\alpha$: some positive number) as $\lambda \to \infty$. Hence, functions which either have some kind of exponential dependencies*, or obey $\lim_{\lambda \to \infty} F\lambda) \sim$ constant (finite, nonzero), are unlikely to conform to our scheme. We should, in such unknown situations, be prepared for a poor performance of the PET. Indeed, this is why we have chosen the function $\tan^{-1}\lambda/\lambda$, instead of just $\tan^{-1}\lambda$, as one of the examples. Suitable ways of tackling such situations are wanted.

The third problem is to enquire if there exists a *systematic* way to improve our results. We are not sure whether 3- or 4-parameter transforms, or still higher ones, will also bracket functions defined formally by alternating power-series expansions with increasing tightness. But some such scheme should be constructed.

7. Conclusions

Our endeavour has been to obtain information about the large-λ behaviour of functions for which formal power-series expansions are supplied to a very low order, provided that such expansions are alternating in nature. For convenience, one or two values of the functions well beyond λ_0, the actual radii of convergence, are to be supplied. With such limited resource, we have found that reasonably approximate behaviour *can* be predicted.

It may be argued that mathematical rigour has been relegated to second place throughout. This is true and we may emphasize here that work on divergent series handling techniques *are mostly* empirical in nature. Moreover, if the available resource is very limited, as we require here, one probably cannot be rigorous at the same time. This is precisely why we have stressed on the simplicity of this scheme.

Finally, we have summarized a few unsolved problems. We hope that solutions to such problems will decide the status of PET in the near future.

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Note added in proof

Prof. S. K. Rangarajan has recently noted (private communication) that

$$[0/1]PA(\mathrm{d}\ln F(\lambda)/\mathrm{d}\lambda) = \mathrm{d}\ln([0,2]\,\mathrm{PET}(2)\,F(\lambda))/\mathrm{d}\lambda.$$

This identity, once again, points to the increased generality of the PET. As $F(\lambda) \sim \lambda^{\alpha}(\alpha$: any real number) for large λ, one must take the $\mathrm{d}\ln F(\lambda)/\mathrm{d}\lambda$ series to construct the *natural* $[n/n+1]$ PA. The identity nicely involves the *first* members of both the said PA and the PET. More about the importance of such a relation will be discussed elsewhere. The author is thankful to Prof. Rangarajan for his interest in the work and to INSA for partial financial support.