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Statistics of the kinetic energy of heavy, inertial particles in weakly rotating turbulence

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Abstract. We revisit the problem of Lagrangian irreversibility and report new results on the statistics of the kinetic energy of heavy inertial particles suspended in a weakly rotating turbulent flow. We show that the interplay of the strength of rotation and particle inertia leads to a complex asymmetry in the nature of energy losses and gains along the trajectories of such particles.

Keywords. Rotating turbulence; inertial particles; Lagrangian statistics.

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Homogeneous and isotropic turbulent flows are amongst the more well-known examples of dissipative, out-of-equilibrium systems which are characterised by a chaotic, intermittent and irreversible behaviour [1–3]. While the chaotic nature of turbulent flows are all too familiar and tracing the origins of intermittency continues to remain a challenge for theorists, recent measurements – first by Xu *et al.* [4] and subsequently by others [5–7] – have shed fresh light on how (time) irreversibility manifests itself within a Lagrangian framework [8–10]. In a manner which is deceptively simple, this irreversibility manifests itself in an asymmetry in the statistics of the kinetic energy E measured for any fluid particle, along its trajectory, as a function of time: there is a gradual gain but an abrupt loss of kinetic energy along the particle trajectories. This irreversibility Ir is quantified through the skewness of the probability distribution function (PDF) of the (Lagrangian) power $p = \frac{\mathrm{d}E}{\mathrm{d}t}$: $\mathrm{Ir} \equiv \frac{\langle p^3 \rangle}{\langle p^2 \rangle^{3/2}}$. Furthermore, this measure of irreversibility was shown to depend on a suitably defined Reynolds number through simple power laws, namely $\mathrm{Ir} \sim \mathrm{Re}_\lambda^{4/3}$, for three-dimensional (3D) and $\mathrm{Ir} \sim \mathrm{Re}_\lambda^2$, for two-dimensional (2D) turbulence [4]. Subsequently, these ideas were further developed for anisotropic flows resulting from a Coriolis force as well as for trajectories of heavy, inertial particles. Recently, Picardo *et al.* [7] carried out a detailed analysis of where in the flow such asymmetries may arise.

In this brief report, we revisit the problem of weakly rotating turbulent flows – and extend the work of Maity *et al.* [6] – to examine the nature of this asymmetry in the trajectories of heavy, inertial particles. In particular, we show that the degree of irreversibility has a subtle dependence on both the strength of rotation – and hence anisotropy – measured through the Rossby number (Ro) as well as the inertia of the particle measured through its Stokes number St.

Rotating turbulent flows [11–23] are of course common in a variety of systems spanning areas of astrophysics [24–26], geophysics [27], and industrial flows [28]. Rotation, while doing no work, causes an accumulation of energy in modes perpendicular to the plane of rotation [21–31] eventually leading to a quasi-two-dimensionalization of the flow through the generation of columnar vortices parallel to the axis of rotation [11–37] and an emergent inverse cascade of energy in 3D turbulence [38–40]. Recent studies [34, 41] have also investigated the effects of such a Coriolis force on the intermittent nature of the velocity field and the Eulerian energy dissipation rates. Surprisingly, though, Lagrangian studies of such systems are more recent [6–44].

We begin with the three-dimensional Navier–Stokes equation (for the velocity field $\mathbf{u}$), subject to a Coriolis force with a constant angular velocity $\boldsymbol{\Omega} \equiv (0, 0, \boldsymbol{\Omega})$, along the z-axis, and augmented by an incompressibility constraint:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + 2(\boldsymbol{\Omega} \times \mathbf{u}) = -\nabla P' + \nu\nabla^2\mathbf{u} + \mathbf{f}; \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0. \quad (2)$$

The effective pressure P' for such a system has a contribution from the natural pressure term P_0 (in the absence of rotation) as well as that arising from the centrifugal contribution to yield a net pressure $P' = P_0 - \frac{1}{2}|\Omega \times \mathbf{r}|^2$, where $\mathbf{r}$ is the position vector from the axis of rotation (chosen, for convenience, along the z-axis and in the middle of the box). The smallest length and time scales associated with eq. (2) correspond to the Kolmogorov length η and time τ_η scales. The Coriolis force introduces, additionally, a second length scale – the Zeman scale $l_\Omega \sim \sqrt{\frac{\epsilon}{\Omega^3}}$ – where the local fluid turnover time $\tau_l = \epsilon^{-1/3}k^{-2/3}$ (k is the wavenumber) is of the same order as Ω^{-1}. For very strong rotation, the kinetic energy spectrum $E(k)$ scales as $E(k) \sim k^{-2}$ in the wave number range $k \leq 1/l_\Omega$ while the Kolmogorov scaling $E(k) \sim k^{-5/3}$ persists for $k > 1/l_\Omega$ [12–47].

We solve eq. (2) on a 2π triply-periodic cube with $N^3 = 512^3$ collocation points, by using a standard pseudo-spectral method [48] and a second-order Adams–Bashforth scheme to evolve in time. We use a constant energy injection force $\mathbf{f}$ on wavenumbers $k_f \leq 3$ to drive the system to a statistically steady state (see, e.g., Ref. [49] for details). Since rotation leads to an inverse cascade of energy [38–40] and a consequent pile-up at the smallest wavenumbers, we introduce a small frictional force equal to $\alpha\nabla^{-2}\mathbf{u}$ acting on wave numbers $k \leq 10$; we choose $\alpha = 0.005$. Our choice of forcing and coefficient of kinematic viscosity ν yields (in the non-rotating limit) a Taylor-scale based Reynolds number $\mathrm{Re}_\lambda = \frac{u_{\mathrm{rms}}\lambda}{\nu} \approx 100$, where $\mathbf{u}_{\mathrm{rms}}$ denotes the root-mean-square velocity and the Taylor microscale $\lambda = \sqrt{\frac{15\nu}{\epsilon}}u_{\mathrm{rms}}$; the energy dissipation rate is given by ϵ. Finally we choose five different strengths of rotation (including the no rotation case) to yield Rossby numbers Ro – a non-dimensional measure of the strength of the inertial to rotation forces – in the range $\infty \geq \mathrm{Ro} \geq 0.08$. We recall that Rossby number is defined as $\mathrm{Ro} \equiv \frac{u_{\mathrm{rms}}}{2L\Omega}$, where L is the box size and equal to 2π in our case. (We refer the reader to Ref. [6] for more details on the parameters of our numerical simulations.)

We, of course, want to make this flow particle-laden. To do this, we adopt the following strategy. We begin with an initial condition (for the fluid velocity field) and, in the absence of the Coriolis force, allow the flow to reach a statistically steady state because of the combination of viscous damping and the external drive $\mathbf{f}$. At this point we turn on the Coriolis term and wait long enough for the flow to reorganise itself and reach a new statistically steady configuration. It is at this point that we seed our flow, randomly and homogeneously, with $N_0 = 10^6$ particles of radius $a \ll \eta$ and density $\rho_p \gg \rho_f$, where ρ_f is the density of the advecting fluid, and hence a Stokes time $\tau_p = 2\rho_p a^2/9\rho_f \nu$ with an associated Stokes number $\mathrm{St} = \tau_p/\tau_\eta$.

Given this set up and assuming non-interacting particles in a dilute suspension, the dynamics of each particle (decoupled from the others) in terms of its position $\mathbf{x}$ and velocity $\mathbf{v}$ is given by the standard linearised Stokesian drag model, while accounting for the additional forces stemming from the rotation:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{v} \quad (3)$$

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{(\mathbf{v} - \mathbf{u}_\mathrm{p})}{\tau_\mathrm{p}} - 2(\Omega \times \mathbf{v}) - \Omega \times \Omega \times \mathbf{r}, \quad (4)$$

where $\mathbf{r}$ is the distance vector of the particle from the axis of rotation and $\mathbf{u}_\mathrm{p}$ is the fluid velocity interpolated (numerically, through a trilinear interpolation scheme) to the typically off-grid particle position. We choose six different sets of particles, corresponding to different degrees of inertia (including the tracer problem for comparison); this allowed us to obtain results for St = 0, 0.25, 0.63, 0.81, 1.0, and 1.5.

Our particles evolve through a simultaneous solution of eqs (1)–(4) and their trajectories stored at every time step. The equations of motion (eqs (3) and (4)) for the particle trajectories are solved by using a fourth-order Runge–Kutta scheme with the same time-step as that used to solve for the Eulerian fluid velocity field; this time-step is chosen to be smaller than not only the Komogorov time-scale of the flow but the smallest Stokes time-scale (corresponding to St = 0.25) used in our simulations. Furthermore, since the particles are typically not found on the regular cubic grid used to solve for the fluid velocity field, we use a trilinear interpolation scheme to obtain the fluid velocity at the particle positions.

We have checked, by computing the PDFs of the particle velocities in the planes perpendicular $\mathbf{v}_\perp$ and parallel $\mathbf{v}_\|$ to the axis of rotation, that the latter display a stronger anisotropy, with decreasing Ro, consistent with the experimental observations of Castello and Clercx [44]. This anisotropy is due to the quasi-two-dimensionalisation of the flow which restricts, preferentially, the motion to planes perpendicular to the axis of rotation.

A natural consequence of this is the ratio of the mean kinetic energies of the particles perpendicular ($E_\perp$) and parallel ($E_\|$) to the axis of rotation. In the absence of rotation (Ro = ∞), this ratio is 2 (because of isotropy). With decreasing Ro, this ratio obtains values larger than 2 and has a weak dependence on the Stokes numbers. In figure 1a we show a plot of this ratio versus St for different values of Ro.

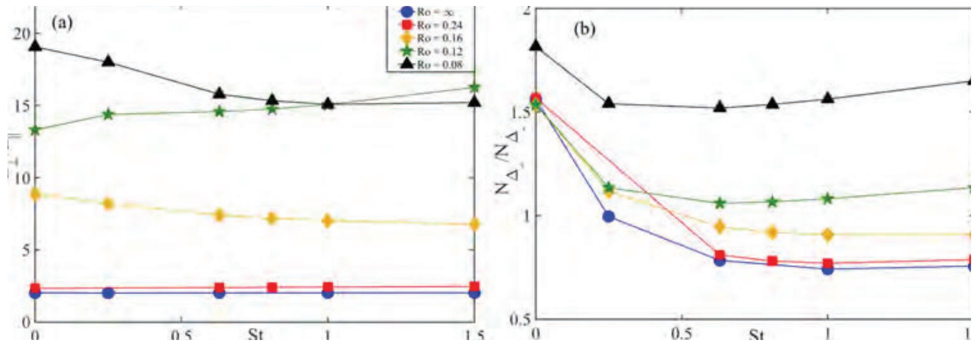

Figure 1. (**a**) Plots of the ratio of (**a**) the kinetic energies in the planes perpendicular and parallel to the axis of rotation and (**b**) the number of particles in vortical (N_{Δ_+}) to those in straining region (N_{Δ_-}) versus the Stokes number St. The different curves, in both the figures, correspond to different values of the Rossby number as given in the legend in panel (**a**).

The most significant effect of the Coriolis force, however, is on the geometry and the reorganisation of the vortical and straining regions of the flow. A common and useful method for identifying such regions is the so-called Δ-criterion [50, 51] which allows us to differentiate between regions of the flow based on the sign of the discriminant Δ of the characteristic equation for the velocity gradient matrix ∇u. This discriminant $\Delta = \frac{27R^2}{4} + Q^3$, where Q and R are the two invariants of velocity gradient tensor, determines the nature of eigenvalues of the velocity gradient matrix: vortical regions are identified with local values of $\Delta \geq 0$ whereas straining regions are associated with $\Delta < 0$. We have checked by calculating the PDFs of Δ, for the fluid velocity field, and seen that these distributions have a preferential bias for $\Delta > 0$ as Ro decreases suggesting the formation of stronger rotation-induced vortices.

In this study, of course the main focus is on the Lagrangian trajectories of heavy particles. To understand this, it is useful to measure Δ along individual trajectories. A natural question that arises is, how do particles sample positive or negative values of Δ, which for convenience, we denote as Δ_+ and Δ_-, respectively, as a function of the Stokes and Rossby numbers. In figure 1b, we show plots of the ratio of particles in vortical (N_{Δ_+}) to those in the straining (N_{Δ_-}) regions for various values of St and Ro. These results suggest that the majority of tracers (St = 0) are confined to the vortical regions as a consequence of the prevalence of weak vortical structures in the flow. As St number is increased, the ratio $\frac{N_{\Delta_+}}{N_{\Delta_-}}$ falls below 1 because these heavy particles are more likely to be ejected from vortical structures and hence sample straining regions preferentially. For a given Stokes number, though, this ratio increases with decreasing Ro as weaker and extended vortical structures proliferate the flow and hence unavoidably sampled by the heavy particles.

All of this brings us to the central result of this work, namely the statistics of the kinetic energy E of an inertial particle, along its trajectory, in a weakly rotating turbulent flow. We recall that a useful way to make sense of this is through the measurement of the Lagrangian power $p = \frac{\mathrm{d}E}{\mathrm{d}t}$ and thence the irreversibility $\mathrm{Ir} = \frac{\langle p^3 \rangle}{\langle p^2 \rangle^{3/2}}$. We take a further step in this direction by conditioning our statistics based on whether the particle is in a vortical or straining region.

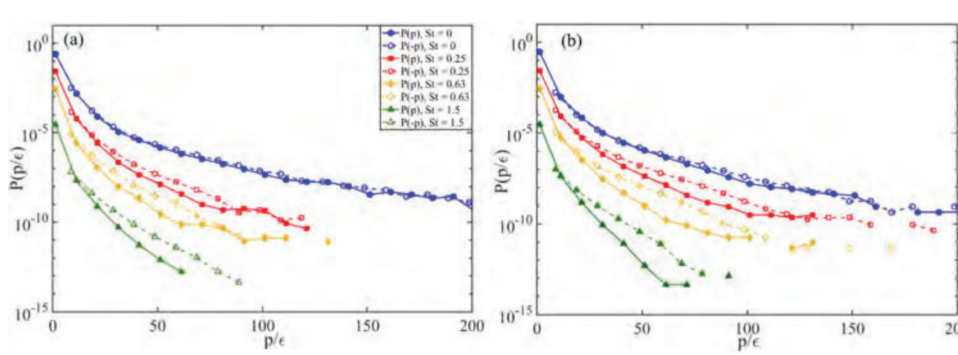

Figure 2. Representative plots of the PDFs of the Lagrangian power (normalised by the energy dissipation rate ϵ), for Ro = 0.08, conditioned on whether the particles are sampling (**a**) vortical ($\Delta \geq 0$) or (**b**) straining ($\Delta < 0$) regions. The left tails of each PDF (for negative p) are reflected for easy comparison with the right tail. The PDFs for different values of the Stokes numbers (shown in the legend in panel (**a**)) are shifted by factors of 10 for clarity.

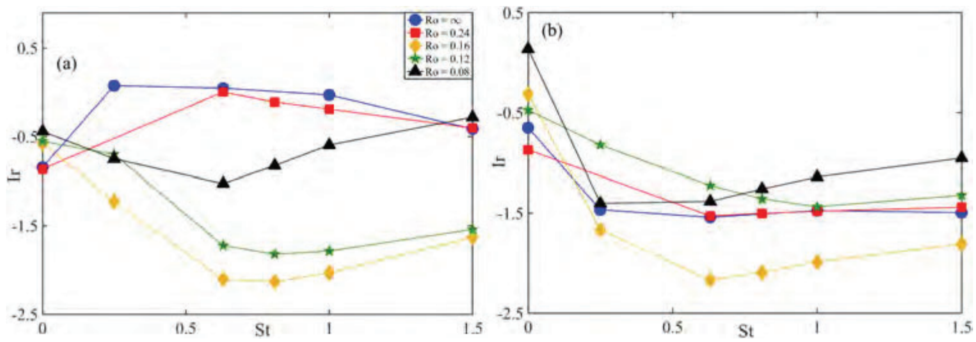

Figure 3. Plots of the Lagrangian irreversibility Ir as a function of the Stokes number St, for different values of Ro (see the legend in panel (**a**)), conditioned on whether the particles are in (**a**) vortical or (**b**) straining regions.

We begin by showing representative plots (for Ro = 0.12) of the PDFs of the (normalised) Lagrangian power $P(p/\epsilon)$ (with the negative tail, corresponding to energy losses, reflected for easy comparison) in figure 2 conditioned on (a) $\Delta \geq 0$ and (b) $\Delta < 0$ for different values of the Stokes number. Such plots clearly reveal a greater asymmetry in the tails of the distribution for $\Delta < 0$ and becoming more pronounced for higher values of St. In straining region ($\Delta < 0$), large energy losses are always more probable than large energy gains. This disparity grows with the inertia of the particle consistently for all values of Ro. Vortical regions ($\Delta \geq 0$) also show a largely similar behaviour except for $0.2 \lesssim \text{St} \lesssim 0.6$ when the probability of large energy gain is marginally higher. However, for Ro < 0.2, the behavior of the PDFs are qualitatively similar for straining and vortical regions.

This asymmetry is easily quantified by calculating the irreversibility Ir, conditioned likewise on the sign of Δ. In figure 3, we show plots of Ir versus St for different values of Ro for (a) $\Delta \geq 0$ and (b) $\Delta < 0$ leading to an overall non-monotonic behaviour of Ir versus St (for Ro $\neq \infty$) as shown in figure 4a. This non-monotonocity arises because of the interplay of the effects of rotation and particle inertia; we note that our results for the non-rotating case are consistent with those reported earlier by Bhatnagar *et al.* [51] who pointed out this monotonic decrease in irreversibility as a function of particle inertia.

We end by making a final observation on the tails of the PDFs shown in figure 2. One way to do this is to devise a measure for the proliferation and abundance of extreme events – large energy gains or losses – from the PDFs of the Lagrangian power. This led us to first calculate the standard deviation σ of such PDFs and then count the total events which contributed to values of p beyond 3σ either for energy gains (Σ_+) or losses (Σ_-). In figure 4b we plot the logarithm of the ratio of these two numbers as a function of St for different values of Ro. In the non-rotating case, the interpretation is fairly straightforward: the number of extreme energy loss events clearly dominates those with energy gains, becoming even more pronounced with increasing St. This is already reflected in panel (a) and also in the work of Bhatnagar *et al.* [51]. Rotation, however, has a moderating effect on these extreme events. For tracers (St = 0), this ratio is close to 1, while decreasing for finite Stokes numbers. The non-monotonic behaviour (with respect to St) observed in Ir is however less pronounced in this measurement which is a more direct way of looking at extreme value statistics from a Lagrangian point of view.

In summary, in this paper we report new results on the statistics of the kinetic energy of heavy inertial particles

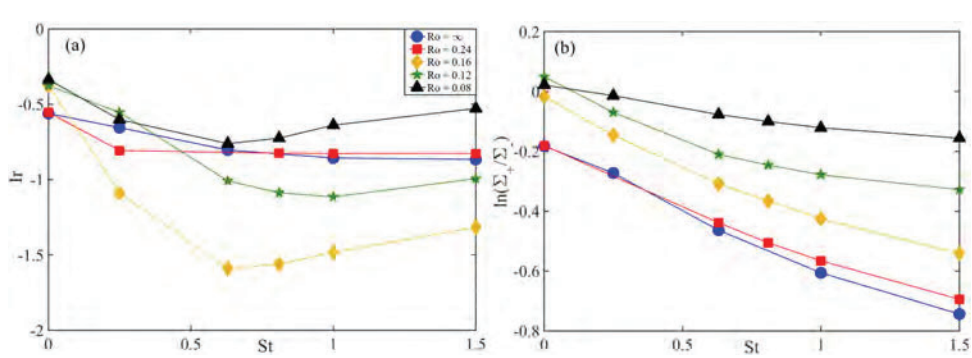

Figure 4. Plots of the (**a**) Lagrangian irreversibility Ir and (**b**) the logarithm of the ratio of large energy gains to losses versus the Stokes number St for different values of Ro.

in weakly rotating turbulent flows. Our results suggest that the interplay of the Coriolis force and the Stokesian drag results in a non-trivial sampling by the particles of different regions of the flow. This results in a complex, and often non-monotonic, behaviour of extreme energy gains and losses along such trajectories in the Stokes–Rossby number phase space. A more detailed investigation in understanding this behaviour is left for future work.

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