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Laguerre unitary ensemble to Gaussian unitary ensemble crossover: Eigenvalue statistics

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Abstract. We consider an ensemble of complex random matrices interpolating between Wishart–Laguerre and Wigner–Gaussian ensembles, and use the Dyson's Brownian motion approach to obtain the corresponding eigenvalue statistics. The crossover parameter (τ) in this case serves as a positive-definiteness violation parameter. The joint probability density of eigenvalues of this random matrix model evolves from that of Laguerre unitary ensemble (LUE) to Gaussian unitary ensemble (GUE) as τ is varied from 0 to ∞. It exhibits a biorthogonal structure and hence eigenvalue correlation functions of all orders follow using a generalization of Andréief's integration formula.

Keywords. LUE–GUE crossover; sum of random matrices; Dyson's Brownian motion; eigenvalue statistics; biorthogonal structure.

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1. Introduction

Random matrix theory constitutes an immensely powerful multivariate statistics tool to deal with a plethora of scientific problems in fields ranging from mesoscopic physics to communication theory [1–5]. In physics, the classification scheme for random matrix ensembles based on symmetries is due to the seminal works of Wigner and Dyson [6, 7]. This scheme is based on the time-reversal and spin-rotation symmetries exhibited by the system under consideration. The matrices to model the corresponding Hamiltonian are accordingly real-symmetric, complex-Hermitian or self-dual quaternion [1–6]. Dyson famously referred to this classification scheme as the three-fold way [6]. Similar invariant ensembles of unitary matrices were introduced by Dyson which are referred to as the circular ensembles [8]. This classification turned out to be of immense importance and has been particularly very successful in describing the universal aspects of eigenspectra. However, there are many systems where some additional symmetries or constraints result in the corresponding universal properties which cannot be described by Dyson's three-fold scheme. Therefore, an extension of this classification to a ten-fold way has been worked out by Altland, Zirnbauer and their co-workers [9–12]. At a fundamental level, this ten-fold way is related to Cartan's classification for symmetric spaces [12, 13]. In addition to the invariant ('pure') ensembles, one can have situations when a symmetry is 'partially' broken [14–18], for example, a weak magnetic field can lead to a 'partial' breaking of the time-reversal symmetry [19–23]. To model such systems, one needs intermediate or crossover ensembles which can interpolate from one symmetry class to other as a certain parameter is varied. Dyson introduced the Brownian motion model of random matrices to investigate such ensembles and characterized them by a fictitious-time parameter, which acts as the symmetry breaking parameter [24]. Interestingly, pure as well as crossover ensembles also find applications in problems where time-reversal and rotation symmetries do not have any direct meaning; for example in multiple-antenna communication [25] and statistics of zeros of Riemann zeta and *L*-functions [26, 27].

An exact solution for the crossovers in Gaussian ensembles was provided by Pandey and Mehta [28, 29] and for the circular ensembles by Pandey and Shukla [30]. These results were extended to other classical random matrix ensembles by Forrester and co-workers [3, 31, 32]. These classical crossover ensembles have also been studied in Refs [33, 34]. These symmetry transitions along with similar others have been studied using random matrix techniques and have found applications in a variety of problems [35–43].

The crossover ensembles can be categorized within ensembles of 'composite' matrices where the members of the ensembles are themselves composed of two or more matrices from certain well-defined distributions. For instance, sum and product of random matrices taken from some distributions constitute such composite ensembles. These more general ensembles have attracted a great deal of attention in recent times due to their fascinating mathematical properties [44–70] and applications in several problems where they are more suited to model the relevant 'operators'. An example is the field of multiple-channel communication theory where sum, product and quotient of Wishart matrices have been used to work out various performance metrics of communication systems under consideration [64–70]. Similarly, ensembles comprising sum of Wigner and Wishart real matrices have been used to model the Hessian matrices arising in supergravity [71–73] and machine learning models in neural networks [74, 75].

2. Crossover ensembles: Dyson's Brownian motion approach

Consider an ensemble of parameter (τ)-dependent $n\times n$-dimensional matrices

$$\mathbf{A}(\tau) = \mathbf{A}_0 e^{-\tau/2} + (1-e^{-\tau})^{1/2}\mathbf{G}. \tag{1}$$

In this equation $\mathbf{A}_0$ is an 'initial' $n\times n$-dimensional random matrix from some well-defined distribution (to be decided later) and $\mathbf{G}$ is an $n\times n$-dimensional complex Ginibre matrix governed by the probability density

$$\mathcal{P}_G(\mathbf{G}) \propto \exp\left(-\frac{1}{2v^2}\operatorname{tr}\mathbf{G}\mathbf{G}^\dagger\right), \tag{2}$$

where 'tr' represents the trace and v^2 is the variance of both real and imaginary parts of the zero-mean complex Gaussian random variables constituting the matrix $\mathbf{G}$. The corresponding probability measure is given by $P_G(\mathbf{G})d[\mathbf{G}]$, where $d[\mathbf{G}]$ represents the product of differentials of all independent variables occurring in $\mathbf{G}$. Clearly, from the definition of $\mathbf{A}(\tau)$, we see that for $\tau = 0$ we obtain $\mathbf{A}_0$. On the other hand, for $\tau\to\infty$ we obtain a matrix $\mathbf{G}$ from the complex Ginibre ensemble. We now consider the ensemble of Hermitian matrices

$$\begin{aligned}\mathbf{H}(\tau) &= \mathbf{A}(\tau) + \mathbf{A}(\tau)^\dagger\\ &= \mathbf{H}_0 e^{-\tau/2} + (1-e^{-\tau})^{1/2}\mathbf{H}_\infty,\end{aligned} \tag{3}$$

where we have defined

$$\mathbf{H}_0 = \mathbf{A}_0 + \mathbf{A}_0^\dagger, \quad \mathbf{H}_\infty = \mathbf{G} + \mathbf{G}^\dagger. \tag{4}$$

The random matrix $\mathbf{H}_\infty$, by construction, belongs to the Gaussian unitary ensemble (GUE) of random matrices and has the associated probability density

$$\mathcal{P}_{H_\infty}(\mathbf{H}_\infty) \propto \exp\left(-\frac{1}{8v^2}\operatorname{tr}\mathbf{H}_\infty^2\right). \tag{5}$$

With the choice $v^2 = 1/8$, the corresponding joint probability density of eigenvalues $\{\lambda\}(\equiv \lambda_1,\dots,\lambda_n)$ is

$$P(\{\lambda\};\infty) = \frac{2^{n(n-1)/2}}{\pi^{n/2}\prod_{i=1}^n i!}\Delta^2(\{\lambda\})\prod_{j=1}^n e^{-\lambda_j^2}, \tag{6}$$

where $\Delta(\{\lambda\}) = \det[\lambda_j^{k-1}]_{j,k} = \prod_{j>i}(\lambda_j-\lambda_i)$ is the Vandermonde determinant (det).

Given the matrix model of eq. (1), one can use Dyson's Brownian motion approach to arrive at the joint probability density of eigenvalues of $\mathbf{H}(\tau)$ appearing in eq. (3). The usual application of this approach is in solving the GOE–GUE and GSE–GUE crossover problems, where GOE and GSE denote the Gaussian orthogonal ensemble and Gaussian symplectic ensemble, respectively. Physically, these crossover ensembles model a system undergoing gradual time-reversal symmetry breaking. Here we use this formalism to obtain the eigenvalue statistics for an ensemble interpolating between LUE and GUE, where LUE stands for Laguerre unitary ensemble. This relates to the recent investigations pertaining to the sum of Wishart–Laguerre and Wigner–Gaussian matrices [49] where the matrix integral approach was implemented.

We take $\mathbf{A}_0$ to be a positive-definite Hermitian matrix belonging to the complex Wishart–Laguerre ensemble, i.e., LUE, with the corresponding probability density

$$\mathcal{P}_{A_0}(\mathbf{A}_0) \propto \det(\mathbf{A}_0)^\alpha \exp(-2\operatorname{tr}\mathbf{A}_0);\quad \alpha > -1. \tag{7}$$

The random matrix $\mathbf{A}_0$ can be constructed using an $n\times m$-dimensional complex Ginibre matrix $\widetilde{\mathbf{G}}$ as $\mathbf{A}_0 = \widetilde{\mathbf{G}}\widetilde{\mathbf{G}}^\dagger$, where we assume $n \le m$. The corresponding probability density is chosen to be $\mathcal{P}_{\tilde{G}}(\widetilde{\mathbf{G}}) \propto \exp(-2\operatorname{tr}\widetilde{\mathbf{G}}\widetilde{\mathbf{G}}^\dagger)$. In this case, the parameter α can be identified as the rectangularity parameter $m-n$. Now, evidently, $\mathbf{H}_0 = \mathbf{A}_0 + \mathbf{A}_0^\dagger = 2\mathbf{A}_0$ also belongs to the LUE and the associated probability density is

$$\mathcal{P}_{H_0}(\mathbf{H}_0) \propto \det(\mathbf{H}_0)^\alpha \exp(-\operatorname{tr}\mathbf{H}_0);\quad \alpha > -1. \tag{8}$$

The corresponding joint probability density of eigenvalues is

$$P(\{\lambda\};0) = \frac{1}{\prod_{i=1}^n i!\Gamma(i+\alpha)}\Delta^2(\{\lambda\})\prod_{j=1}^n \lambda_j^\alpha e^{-\lambda_j}\Theta(\lambda_j), \tag{9}$$

where $\Theta(x)$ is the Heaviside theta function.

With the above choice of $\mathbf{A}_0$, we can see that for $\tau = 0$, $\mathbf{H}(\tau)$ belongs to the ensemble of positive-definite Hermitian matrices described by LUE, and the corresponding eigenvalues are restricted to the positive real domain, i.e., $(0, \infty)$. As τ is gradually increased, the positive-definiteness of $\mathbf{H}(\tau)$ is violated and the eigenvalues start to enter the negative real side. Eventually for $\tau \to \infty$, $\mathbf{H}(\tau)$ becomes a Hermitian random matrix described by GUE and the eigenvalues cover the entire real domain, i.e., $(-\infty, \infty)$. Therefore, τ can be interpreted as a parameter which destroys the positive definiteness of the 'operator' $\mathbf{H}(\tau)$. Consequently, this matrix ensemble is of potential use in investigating 'violation of positive-definiteness' of random Hermitian operators in linear algebra and quantum mechanics. In the corresponding real-matrix ensemble, if $\mathbf{H}(\tau)$ is used to model a Hessian matrix pertaining to a real scalar field describing a random landscape, then positive definiteness would indicate local minima and increasing τ would destroy these. In Refs [71–73], the authors have used the sum of Wishart–Laguerre and Wigner–Gaussian random real matrices to model the Hessian matrix associated with the landscape of de Sitter solutions of string theory and thereby examined the metastability of the critical points. Similarly, such a matrix model is of potential use in the study of disordered systems where one is interested in counting the number of local minima of a random scalar field [76]. Moreover, very recently, such a composite ensemble of real matrices has also been used to replicate the Hessian matrix associated with the loss function of machine learning models [74, 75]. The complex ensemble considered here can be of relevance in investigating complex Hessian matrices associated with complex scalar fields, see for example [77].

As indicated above, Dyson's Brownian motion approach can be used to derive the joint probability density of the interpolating ensemble [30–34]. Using the notation of Refs [33, 34], we have

$$P(\{\lambda\};\tau) = \int_0^\infty d\mu_1 \cdots \int_0^\infty d\mu_n P(\{\lambda\};\tau|\{\mu\};0)P(\{\mu\};0). \tag{10}$$

Here $P(\{\lambda\};\tau|\{\mu\};0)$ is the conditional joint probability density (many-body Green's function) associated with probability of finding the eigenvalues $\{\lambda\}$ at 'time' τ when the initial eigenvalues are $\{\mu\}$. To construct $P(\{\lambda\};\tau|\{\mu\};0)$ one derives the Fokker–Planck equation for the joint probability density of eigenvalues, $P(\{\lambda\};\tau)$, using the Brownian motion approach. The Fokker–Planck equation is then mapped to a Schrödinger equation in imaginary time $i\tau$ for n independent fermions. The corresponding Hamiltonian is of Calogero–Sutherland type, in which the two-body term drops out if transition to unitary ensemble is considered. The conditional density can then be obtained using eigenfunctions of this Hamiltonian, and reads for the present case as

$$P(\{\lambda\};\tau|\{\mu\};0) = \frac{1}{n!}e^{n(n-1)\tau/4}\frac{\Delta(\{\lambda\})}{\Delta(\{\mu\})}\prod_{i=1}^{n}\frac{e^{-\lambda_i^2/2}}{e^{-\mu_i^2/2}} \times \det\left[K(\lambda_j,\mu_k;\tau)\right], \tag{11}$$

where the indices j, k inside the determinant run from 1 to n, and the kernel $K(\lambda,\mu;\tau)$ is given by

$$K(\lambda,\mu;\tau) = \sum_{j=0}^{\infty} e^{-j\tau/2}\phi_j(\lambda)\phi_j(\mu)$$
$$= \frac{1}{[\pi(1-e^{-\tau})]^{1/2}} \times \exp\left[-\left(\frac{1+e^{-\tau}}{1-e^{-\tau}}\right)\frac{(\lambda^2+\mu^2)}{2} + \left(\frac{2e^{-\tau/2}}{1-e^{-\tau}}\right)\lambda\mu\right]. \tag{12}$$

Here $\phi_j(x) = (2^j\pi^{1/2}j!)^{-1/2}e^{-x^2/2}H_j(x)$ are the harmonic oscillator wave functions with $H_j(x)$ being the Hermite polynomials. For $\tau = 0$, completeness yields $K(\lambda,\mu;0) = \delta(\lambda-\mu)$, where $\delta(x)$ is the Dirac delta function. This may also be seen by taking the limit $\tau \to 0$ in the Gaussians. Consequently, the multidimensional integral in eq. (10) gives rise to the result $(1/N!)[P(\{\lambda\};0)$ + permutations of eigenvalues]. Since we are considering joint probability density of unordered eigenvalues, this is effectively the same as $P(\{\lambda\};0)$, as it must. We should add that the result (10) can also be obtained using the celebrated Harish–Chandra–Itzykson–Zuber unitary group integral [78, 79].

3. Joint probability density of eigenvalues: Biorthogonal structure

We use eq. (9) in (10) and, after some rearrangement of terms, find that the evaluation of joint probability density of eigenvalues of the crossover ensemble involves integrals over the product of two determinants:

$$P(\{\lambda\};\tau) = \frac{e^{n(n-1)\tau/4}}{n![\pi(1-e^{-\tau})]^{n/2}}\Delta(\{\lambda\})\prod_{i=1}^{n}\frac{e^{-\lambda_i^2/(1-e^{-\tau})}}{i!\Gamma(i+\alpha)} \times \int_0^\infty d\mu_1\cdots\int_0^\infty d\mu_n \det\left[\mathcal{S}_j(\mu_k;\tau)\right] \times \det\left[\mathcal{K}_k(\mu_j;\tau)\right]. \tag{13}$$

We have introduced here

$$\mathcal{S}_j(\mu_k;\tau) = \mu_k^{\alpha+j-1} e^{-\mu_k},$$
$$\mathcal{K}_k(\mu_j;\tau) = \exp\left[-\left(\frac{e^{-\tau}}{1-e^{-\tau}}\right)\mu_j^2 + \left(\frac{2e^{-\tau/2}}{1-e^{-\tau}}\right)\lambda_k\mu_j\right]. \quad (14)$$

The above multidimensional integral can be performed using Andréief's integration formula [80] and, after some rearrangement, results in the following expression for the joint probability density of eigenvalues:

$$P(\{\lambda\};\tau) = \pi^{-n/2} e^{n(n+\alpha)\tau/2}(1-e^{-\tau})^{n(n+2\alpha-1)/4} \times \Delta(\{\lambda\}) \prod_{i=1}^{n} \frac{e^{-\lambda_i^2/(1-e^{-\tau})}}{i!\Gamma(i+\alpha)} \det[\mathcal{F}_j(\lambda_k;\tau)]. \quad (15)$$

Here, the kernel $\mathcal{F}_j(\lambda_k;\tau)$ is given by the integral

$$\mathcal{F}_j(\lambda_k;\tau) = \left(\frac{e^{-\tau}}{1-e^{-\tau}}\right)^{(j+\alpha)/2} \int_0^\infty d\mu \mathcal{S}_j(\mu;\tau)\mathcal{K}_k(\mu;\tau), \quad (16)$$

and can be evaluated in terms of the confluent hypergeometric function of the first kind (Kummer's function) as

$$\mathcal{F}_j(\lambda_k;\tau) = \frac{1}{2}\Gamma\left(\frac{j+\alpha}{2}\right) \times {}_1F_1\left[\frac{j+\alpha}{2},\frac{1}{2};\left(\frac{\lambda_k}{(1-e^{-\tau})^{1/2}} - \frac{(1-e^{-\tau})^{1/2}}{2e^{-\tau/2}}\right)^2\right] + \left(\frac{\lambda_k}{(1-e^{-\tau})^{1/2}} - \frac{(1-e^{-\tau})^{1/2}}{2e^{-\tau/2}}\right)\Gamma\left(\frac{j+\alpha+1}{2}\right) \times {}_1F_1\left[\frac{j+\alpha+1}{2},\frac{3}{2};\left(\frac{\lambda_k}{(1-e^{-\tau})^{1/2}} - \frac{(1-e^{-\tau})^{1/2}}{2e^{-\tau/2}}\right)^2\right]. \quad (17)$$

We note that with the aid of Andréief's formula [80], eq. (15) may also be written as

$$P(\{\lambda\};\tau) = \frac{\Delta(\{\lambda\})}{n!\det[c_{j,k}]} \prod_{i=1}^{n} e^{-\lambda_i^2/(1-e^{-\tau})} \det[\mathcal{F}_j(\lambda_k;\tau)], \quad (18)$$

where the entries $c_{j,k}$ are given by

$$c_{j,k} = \int_{-\infty}^{\infty} d\lambda e^{-\lambda^2/(1-e^{-\tau})}\mathcal{F}_j(\lambda;\tau)\lambda^{k-1} = \pi^{1/2}\frac{e^{-(\alpha+j+k-1)\tau/2}}{(1-e^{-\tau})^{(\alpha+j-1)/2}}\Gamma(\alpha+j+k-1) \times {}_2F_2\left(\frac{1-k}{2},\frac{2-k}{2};\frac{2-\alpha-j-k}{2},\frac{3-\alpha-j-k}{2};\frac{1-e^{-\tau}}{4e^{-\tau}}\right). \quad (19)$$

Comparison of eqs (15) and (18) leads to the following very interesting identity for a determinant involving hypergeometric functions ${}_2F_2$:

$$\det[c_{j,k}] = \frac{\pi^{n/2}\prod_{i=1}^{n} i!\Gamma(i+\alpha)}{n!e^{n(n+\alpha)\tau/2}(1-e^{-\tau})^{n(n+2\alpha-1)/4}}. \quad (20)$$

In ref. [49], eigenvalue statistics of weighted sum of a Wishart–Laguerre matrix (from LUE) and a Wigner–Gaussian matrix (from GUE) have been derived using a matrix-integral approach. Using the present notation for matrices, it reads $\mathbf{H}(\tau) = a\mathbf{H}_\infty + b\mathbf{H}_0$. Comparing this with eq. (3) we identify $a = (1-e^{-\tau})^{1/2}$ and $b = e^{-\tau/2}$. Expression (15) or (18) for the joint probability density of eigenvalues of $\mathbf{H}(\tau)$ then agrees completely with the result in [49] once we consider the transformation of the index j in these equations as $j \to n-j+1$ and use $\alpha = m-n$. It should be noted that this new 'j' also runs from 1 to n.

We observe that expressions (15) and (18) comprise the product of two determinants that involve the eigenvalues, namely the Vandermonde determinant and the one containing $\mathcal{F}_j(\lambda_k;\tau)$. Therefore, we identify them to exhibit Borodin's biorthogonal structure [81]. The eigenvalue correlation functions for these ensembles can be expressed in terms of certain biorthogonal polynomials. Another approach, that we implement in the next section, is to use a generalization of Andréief's integration formula [49, 82].

4. Eigenvalue correlation function

The rth order correlation function of eigenvalues is defined as [1, 3],

$$R_r(\lambda_1,\ldots,\lambda_r;\tau) = \frac{n!}{(n-r)!}\int_{-\infty}^{\infty} d\lambda_{r+1}\cdots\int_{-\infty}^{\infty} d\lambda_n P(\{\lambda\};\tau). \quad (21)$$

Using a generalization of Andréief's integration formula [49, 82], this may be expressed in terms of a determinant involving a $(n+r)$-dimensional matrix:

$$R_r(\lambda_1,\ldots,\lambda_r;\tau) = (-1)^r n!\pi^{-n/2}e^{n(n+\alpha)\tau/2}(1-e^{-\tau})^{n(n+2\alpha-1)/4} \times \frac{\prod_{i=1}^{r} e^{-\lambda_i^2/(1-e^{-\tau})}}{\prod_{l=1}^{n} l!\Gamma(l+\alpha)} \det\begin{bmatrix} [0]_{\substack{j=1,\ldots,r\\k=1,\ldots,r}} & [\lambda_j^{k-1}]_{\substack{j=1,\ldots,r\\k=1,\ldots,n}} \\ [\mathcal{F}_j(\lambda_k;\tau)]_{\substack{j=1,\ldots,n\\k=1,\ldots,r}} & [c_{j,k}]_{\substack{j=1,\ldots,n\\k=1,\ldots,n}} \end{bmatrix}. \quad (22)$$

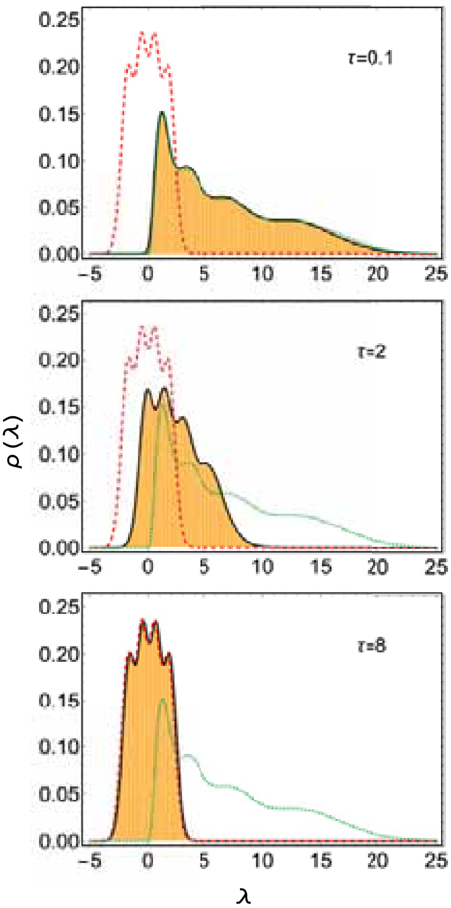

Figure 1. Marginal probability density of a generic eigenvalue for the LUE–GUE crossover ensemble. The solid black lines are the analytical results for the crossover and histograms are from numerical simulation based on the matrix model given in eq. (3). For comparison, the densities for the extreme cases, $\tau = 0$ and $\tau \to \infty$, are also shown using dotted green and dashed red curves, respectively.

Any observable of interest, which depends on the eigenvalues, can then be obtained using the correlation function or the joint probability density in general. For instance, average (first moment) of observables which are linear statistic of the eigenvalues can be calculated with the aid of the level density $R_1(\lambda;\tau)$ only.

In figure 1, we plot the marginal probability density $\rho(\lambda;\tau) = R_1(\lambda;\tau)/n$ of a generic eigenvalue for three τ values. The numerical simulation results based on the matrix model (3) are also shown using histograms, along with the analytic curves. We find an excellent agreement in all cases. For comparison we also show the densities for the extreme cases $\tau = 0$ (LUE) and $\tau \to \infty$ (GUE). These are given, respectively, by

$$\rho(\lambda;0) = \frac{\Gamma(n)}{\Gamma(m)}\lambda^{\alpha}e^{-\lambda} \times \left[L_{n-1}^{(\alpha)}(\lambda)L_{n}^{(\alpha+1)}(\lambda) - L_{n}^{(\alpha)}(\lambda)L_{n-1}^{(\alpha+1)}(\lambda)\right], \quad (23)$$

and

$$\rho(\lambda;\infty) = \frac{e^{-\lambda^2}}{2^n\pi^{1/2}n!}[H_n(\lambda)H_n(\lambda) - H_{n-1}(\lambda)H_{n+1}(\lambda)]. \quad (24)$$

In these equations, $L_j^{(k)}(\lambda)$ and $H_j(\lambda)$ are the associated-Laguerre and Hermite polynomials, respectively. In the figure, we examine the behavior of the eigenvalue density. For $\tau = 0$, the eigenvalue spectrum is located completely in the positive real domain, owing to the positive definiteness of $\mathbf{H}(0)$. As τ is increased, some eigenvalues begin to drift to the negative real domain, and eventually for $\tau \to \infty$ the spectrum is symmetrically distributed about the origin. Thus, we observe the change in the eigenvalue density from that of LUE to GUE as τ is increased from 0 towards large real values.

5. Summary and outlook

In this work, we considered an ensemble of random matrices which exhibits a crossover from the LUE to GUE as parameter τ is varied from 0 to ∞. This matrix ensemble is of use in modeling random Hermitian operators, with τ acting as a positive-definiteness violating parameter. With the aid of Dyson's Brownian motion approach, we derived exact result for the joint probability density of the eigenvalue. This joint density exhibits a biorthogonal structure, and therefore exact expression for an arbitrary order eigenvalue correlation function follows using a generalization of Andréief's integration formula. We verified the corresponding one-eigenvalue density result using Monte-Carlo simulations. This crossover ensemble is directly related to an ensemble comprising weighted sum of complex Wishart–Laguerre and Wigner–Gaussian matrices, which has been recently investigated using a matrix integral method [49].

The present approach, or more generally the group-integral-based approach, can also be used to solve some other crossover problems involving the classical random matrix ensembles. Moreover, besides the sum of two complex random matrices, one can also consider the sum of a real matrix and a complex matrix from the classical invariant ensembles. In this case the joint probability density of eigenvalues will involve the product of a Pfaffian and a determinant. We plan to explore some of these problems in a future work.

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