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Multiplicative noise-induced intermittency in maps

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Abstract. Intermittency arising due to parametric noise has been observed across a variety of physical systems. A simple and generic approach for characterizing the onset of such intermittency in one-dimensional maps due to aperiodic fluctuations of a system parameter is presented. Two mutually complementary conditions for the onset of intermittency, with respect to the evolution during the two distinct phases constituting intermittent response are derived. We illustrate these ideas with numerical results from a noisy logistic map, with intermittency about a noise-free transcritical bifurcation and an aeroelastic system with intermittency about a noise-free Hopf bifurcation.

Keywords. Intermittency; multiplicative noise; 1-d maps.

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1. Introduction

Noise is known to significantly affect the stability properties of systems in multiple ways, like shift in a transition or bifurcation point, noise-induced intermittency (NII) and stochastic resonance. Also, the effect of noise on a bifurcation scenario depends on whether the type of noise is multiplicative or additive. The former refers to a case where the effect of noise on system behaviour depends on the system variables [1]. This generally occurs when a parameter in system-governing equations is considered to be fluctuating. This in turn means that the stability properties of the system equilibria, which are dependant on the parameter, are time varying. On the other hand, additive noise appears as an additional term in system equations and disrupt the movement of trajectory in phase space. This study focusses on multiplicative noise-induced intermittency (mNII).

NII has been observed and studied in a variety of physical systems. Noise-induced transition, i.e., the noise-induced advancement or delay of bifurcation has been observed in a wide range of systems – chemical, biological, geophysical and aeroelastic – and for both additive and multiplicative noises since 1950s [1]. NII, however, was observed as a distinct phenomenon only in the 1980s [2, 3]. NII could be observed in systems which undergo transition between two regimes – for example rest and oscillatory – when the bifurcation parameter fluctuates. This parametric noise is called 'multiplicative' noise or 'state-dependent noise' since the effect of noise depends on the state variables [1] and the corresponding intermittent behaviour is called the 'multiplicative noise-induced intermittency' (mNII). The mNII has been used to describe phenomena in diverse physical systems – geophysical [4], biological systems [5] etc. Thus understanding the origin of NII and its universal characteristic features is of interest to several areas of science.

The role of the fluctuation time scales has previously been found to be significant in noise-induced transitions [6]. Lefever *et al.* [6] found that a bifurcation could be delayed or advanced in the presence of noise depending on the difference in time scales between the system and noise. Thus understanding the interplay of the noise and system time scales holds the key to understanding noise-induced transitions and mNII. This paper studies this interplay for mNII.

2. Dynamics of a noisy map

Consider the logistic map

$$x_n = a_n x_n(1 - x_n), \tag{1}$$

where a_n which fluctuates randomly over time, is expressed as

$$a_n = a_m + \sigma\xi_n, \tag{2}$$

where ξ_n is mean zero independent noise increment that follows probability density function $p(\xi)$ and a_m and σ

are constants. For the noise-free case, the logistic map has a transcritical bifurcation at $a_m = 1$ and attractors $x^* = 0$ for $a_m \le 1$ and $x^* = 1 - 1/a_m$ for $a_m > 1$. In the presence of multiplicative noise, the logistic map is known to exhibit intermittency. The response alternates between a phase of quiescent, nearly zero, response called 'laminar' phase and another phase of oscillations away from zero called 'burst' phase. Heagy *et al.* [7] characterized the intermittent behaviour in a logistic map through a study of its linearized version $x_n \approx a_n x_n$. In fact, this linear model serves as a simplification of most one-dimensional (1-d) maps. The linearization was justified on the grounds that the response $x \to 0$ during the laminar phase. Thus, the response can be equivalently written in a logarithmic form as

$$\ln(x_n) = \ln(x_0) + \sum_{k=0}^{n-1} a_k. \tag{3}$$

Heagy *et al.* [7] assumed that for a laminar phase existing between $k = 0$ and $k = n$, $\ln(x_n) - \ln(x_0) = 0$. This led to the onset condition for mNII being,

$$\mathrm{E}[\ln(a)] = 0, \tag{4}$$

where $\mathrm{E}[\cdot]$ is the ensemble expectation operator. The condition was verified numerically in a logistic map driven by a_n for two cases, with a_n following a uniform distribution and a tent map.

However, such a condition confines the discussion to laminar phases alone and does not provide insight into the burst phases. In fact, similarities have been noted between the features of laminar and burst phases in the evolution of an intermittent response [8]. A more generic onset condition can be derived by relaxing the linearity assumption. For a generic system with an attractor χ, the rate of evolution of an orbit can be defined as

$$r_n = \ln\left(\left|\frac{x_n - \chi}{x_{n-1} - \chi}\right|\right). \tag{5}$$

It is obvious that $r_n < 0$ if the nth iterate evolves towards the attractor χ, and $r_n > 0$ if the iterate evolves away from it. For the logistic map, the attractor $\chi = 0$ and the rate reduces to,

$$r_n = \ln\left(\left|\frac{x_n}{x_{n-1}}\right|\right). \tag{6}$$

Thus, one can rewrite the response in a logarithmic form as

$$\ln(x_n) = r_n + \ln(x_{n-1}). \tag{7}$$

Following such an expression, the relationship between the state variable T iterations apart can be given as

$$\ln(x_n) = \ln(x_{n-T}) + \sum_{k=n-T+1}^{n} r_k. \tag{8}$$

Defining the evolution rate between x_{n-T} and x_n as

$$\Gamma(n, T) = \ln\left[\left|\frac{x_n}{x_{n-T}}\right|\right], \tag{9}$$

it follows that

$$\Gamma(n, T) = \sum_{k=n-T+1}^{n} r_k. \tag{10}$$

In fact, $\Gamma(n, T) < 0$ when there is a net evolution towards χ_n during the T iterations. Likewise, $\Gamma(n, T) > 0$ when there is a net evolution away from χ_n.

3. Onset condition for mNII

For a logistic map, the attractor $\chi = 0$ for $a \le a_c = 1$ and the evolution of the trajectories towards it leads to the state variables approaching zero. In such cases where $\chi = 0$, the laminar phase $\mathcal{L}$ can be defined as a phase when $x_n, x_{n+1}, \ldots, x_{n+\ell-1} \le \alpha$, with $x_{n-1} > \alpha$ and $x_{n+\ell} > \alpha$. Similarly, the burst phase $\mathcal{B}$ can be defined as a phase when $x_n, x_{n+1}, \ldots, x_{n+\ell-1} > \alpha$, with $x_{n-1} \le \alpha$ and $x_{n+\ell} \le \alpha$. Following the expression presented in eq. (10), the total evolution of the trajectories in a $\mathcal{L}$ phase is given by

$$\Gamma(\mathcal{L}) = \ln \frac{|x_{n+\ell} - \chi_{n+\ell}|}{|x_n - \chi_n|} = \sum_{k=n}^{n+\ell-1} r_k, \tag{11}$$

while the total evolution of the trajectories in a $\mathcal{B}$ phase is given by

$$\Gamma(\mathcal{B}) = \ln \frac{|x_{n+b} - \chi_{n+b}|}{|x_n - \chi_n|} = \sum_{k=n}^{n+b-1} r_k. \tag{12}$$

Here, $\ell, b \in \aleph$.

An expression for the mean evolution in the $\mathcal{L}$ phase of the given length ℓ is obtained by taking expectations on both sides of eq. (11) and using the law of iterated expectations, it can be expressed as

$$\mathrm{E}[\Gamma(\mathcal{L})] = \mathrm{E}[\ell]\mathrm{E}[r | x \le \alpha], \tag{13}$$

Here, since $x \le \alpha$ always within a laminar phase, the equation is seen to involve conditional expectations.

Also, $\Gamma(\mathcal{L})$ is the evolution between x_n and $x_{n+\ell}$, necessarily $\Gamma(\mathcal{L}) \geq 0$ for a laminar phase. Essentially, this indicates that a run of $x_k : x_k \leq \alpha$ should end when x_k exceeds α. This ensures that a laminar phase is followed by a burst phase $\mathcal{B}$. Here, mNII is being interpreted as the irregular and aperiodic growth and decay of the trajectories. Similarly, it is necessary that $\Gamma(\mathcal{B}) \leq 0$ during a burst phase for the phase to end and be followed by a laminar phase. Thus, during intermittency, $\Gamma(\mathcal{L}) \geq 0$ and $\Gamma(\mathcal{B}) < 0$. This follows from the characteristic of mNII that bursts and laminar phases alternate each other. An orbit which completely converges onto $\chi = 0$ also has $x \leq \alpha$. However, this is not really a laminar phase, since no bursts follow it and consequently it is not a part of a mNII response. One can define similar relation for the burst phase of the given length b as

$$\mathrm{E}[\Gamma(\mathcal{B})] = \mathrm{E}[b]\mathrm{E}[r|x > \alpha]. \tag{14}$$

Using the condition that $\Gamma(\mathcal{L}) \geq 0$ and $\Gamma(\mathcal{B}) < 0$, and the fact that ℓ and b are non-negative by definition, eqs (13) and (14) yield the following conditions to be satisfied during intermittency:

$$\mathrm{E}[r|x \leq \alpha] \geq 0, \quad \mathrm{E}[r|x > \alpha] < 0. \tag{15}$$

Both these conditions are expected to hold during an intermittent regime. Further, if the pre-intermittency response is treated as one never-ending laminar phase with response perpetually decreasing below the threshold α, the evolution during such a laminar phase (see eq. (10)), $\Gamma(n, T) < 0$ necessarily. Thus, the onset of intermittency can be defined as the point in the a_m, σ, τ_c plane when $\Gamma(\mathcal{L})$ vanishes. In other words, the onset condition for intermittency is,

$$\mathrm{E}[r|x \leq \alpha] = 0, \quad \mathrm{E}[r|x > \alpha] < 0. \tag{16}$$

Thus, the onset of mNII can be mathematically defined as when $\mathrm{E}[r|x \leq \alpha]$ undergoes a change from a negative value to zero as the bifurcation parameter is changed.

Assuming small x_n, the condition for the onset of intermittency can be expressed, through a Taylor series expansion of $x_{n=1} = f(x_n)$ and retaining only the linear terms, as

$$\mathrm{E}[\{\ln f'(0)\}] = 0. \tag{17}$$

For a noisy logistic map $x_{n+1} = f(\chi_n) = a_n x_n(1 - x_n)$, with $\chi = 0 \forall a_n$, the above condition for the onset of intermittency can be shown to imply that $\mathrm{E}[\ln(a_n)] = 0$; this condition is consistent with the results available in [7].

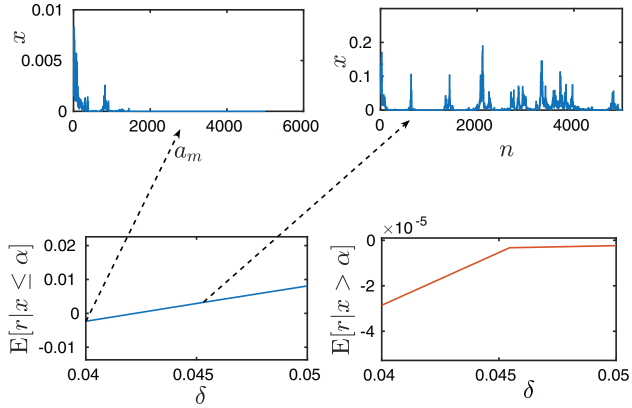

Figure 1. Variation of $\mathrm{E}[r|x \leq \alpha]$ and $\mathrm{E}[r|x > \alpha]$ with a change in $\delta = a_m - a_c$, where $a_c = 1$. The upper panels show iterate histories of x and indicate that intermittency occurs when $\mathrm{E}[r|x \leq \alpha] = 0$ and $\mathrm{E}[r|x > \alpha] < 0$. Intermittency is noticed for $\delta = 0.045$ and beyond.

Equation (16) does not make assumptions on linearity of expressions. In fact, it permits to characterize the onset of intermittency directly from response, without requiring information of the governing equation of the system. Thus, such a condition can be considered to apply for systems with an attractor $\chi = 0$. The following section verifies this result for a logistic map as well as an aeroelastic oscillator.

4. Numerical verification

4.1 *Logistic map – uniformly distributed noise*

Consider the logistic map,

$$x_{n+1} = a_n x_n(1 - x_n), \tag{18}$$

For the generation of uncorrelated noisy parameter, a_n can be generated simply as a random sequence from a pseudorandom number generator. On the other hand, the generation of a_n with a non-trivial correlation length is not straightforward, with a_n being a discrete-time process. In this study, a correlated noise is simulated varying a_n once in every τ_{int} time steps. Following [3, 7], the parametric noise is considered in this case to follow a uniform distribution. Here, imposing the condition $a_n \in [0, 3]$ is necessary to avoid more complicated intermittent patterns involving segments of periodic and chaotic dynamics. Thus, the system has an attractor $\chi = 0$ when $a_n \leq 1$ and another attractor $\psi = 1 - 1/a_n$ for $a_n \in (1, 3)$. Figure 1 shows that eq. (16) holds for a logistic map. Here $\mathrm{E}[r|x \leq \alpha]$ and $\mathrm{E}[r|x > \alpha]$ are calculated numerically.

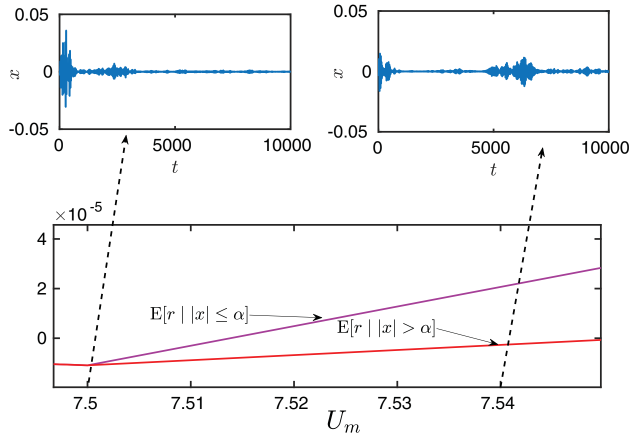

Figure 2. Variation of $E[r||x| \leq \alpha]$ and $E[r||x| > \alpha]$ with a change in U_m, where $a_c = 1$. The upper panels show iterate histories of x and indicate that intermittency occurs when $E[r||x| \leq \alpha] = 0$ and $E[r||x| > \alpha] < 0$.

4.2 *Aeroelastic oscillator*

While simplified analytical results assuming linearization are available for simple dynamical models, these often are difficult to obtain for more sophisticated systems. Further, in many experimental results, modelling the stochastic dynamics appropriately could be a challenge. Equation (16) enables characterization of the onset of intermittency in such cases simply from the acquired response. To demonstrate this utility, the onset conditions are verified for an engineering system such as an aeroelastic oscillator. A two-dimensional airfoil is considered in the presence of fluctuating air inflow. The governing equations for the model and the system parameters are available in [9]. In the noise-free scenario, the system exhibits a Hopf bifurcation. The system exhibits a static response for horizontal wind speed $U \leq U_c$, where U_c corresponds to the bifurcation point. Beyond U_c, self-sustained limit cycle oscillations occur. Under noisy inflow, the onset of limit cycle oscillations is preceded by a regime of intermittent oscillations. The noisy parameter, the mean horizontal wind speed U_m is assumed to be a uniformly distributed noise as in the case of the noisy logistic map. The variance of the noise is 1.333. The Poincare sections of the vertical displacement or 'plunge' of the airfoil yield a discrete time map, which can be analysed using the techniques described so far. Figure 2 shows that the onset conditions hold well for the aeroelastic system as well. The iterate values of the Poincare map of the oscillator is denoted by x_n and a laminar phase is characterised by the fact that $|x| \leq \alpha$, α being an arbitrary threshold. This indicates that the results derived for 1-d maps in this paper are applicable to more complex systems.

5. Conclusion

A simple and generic approach to obtain onset conditions for intermittency due to multiplicative noise is presented in this paper. The proposed conditions can be simplified to facilitate analytical prediction of intermittency with a change in the mean parameter value. Further, the conditions enable characterization of intermittency in engineering systems directly from the acquired data.

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