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Probability distributions of nodal domains and amplitudes of wavefunctions

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Abstract. Zeros of a function encode significant information about the patterns they represent. Owing to non-separable or/and non-integrable nature of the classical system, the nodal curves display a complex morphology. As they must avoid to cross, the domains of positive or negative sign they enclose look very complex and intriguing. Probability distribution functions of the normalized-domain number will be presented for simple billiards. For some of these systems, the amplitude distribution functions are found. Several open problems will be discussed.

Keywords. Quantum chaos; billiards; nodal domains; probability distribution functions.

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1. Introduction

Classification and characterization of sound figures created upon excitation of glass plates by a violin bow gave birth to methods used today by engineers to design structures. Structures like bridges and cars to designing cavities for accelerators, and the like. Named after their discoverer, Chladni figures (1787) have attracted great attention by scientists, mathematicians, and engineers as their complete understanding is relevant to everyone. They are also beautiful. They originate from music. Sand or dust spread over these vibrating glass plates makes the patterns, expressed by the curves along which the plate is at rest. These curves are the 'zeros' of the wavefunctions. The first person to have made a headway with the problem of understanding Chladni figures was Sophie Germain (unpublished work, 1811–1815). Just for the case of a square (or a rectangular plate), the treatment of Germain was improved by Lagrange and Poisson, and finalized by Kirchhoff in 1850. Kirchhoff also solved the problem for circular plates. The remarkable point realized by Kirchhoff was that these figures correspond to eigenvalues and eigenfunctions of the biharmonic operator under free boundary conditions. In fact, the amplitude of flexural vibrations of stiff acoustic plates is [1] described by

$$\Delta^2\psi_j \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \psi_j = k_j^4\psi_j \tag{1}$$

for the jth resonance, or, with driving, by the Kirchhoff–Love equation:

$$\left(D\nabla^4 + \rho h\frac{\partial^2}{\partial t^2}\right)\psi(x, y, t) = F(x, y, t), \tag{2}$$

where D is the flexural rigidity, ρ is the mass density, h is the thickness and $F(x, y, t)$ is the effective force function. As opposed to the vibrations of membranes without internal stiffness, it is the square of the Laplace operator here. Attempts to solve these and related equations gave birth to computational methods by Timoshenko [2], culminating into the modern days finite element method, Ritz–Galerkin method [3, 4] and so on. Important advances in building submarines were made possible on this basis by Bubnov [5].

Mathematically, the solutions one looks for in the case of plates, and, the ones we obtain while solving the eigenvalue problems for the particle in a box system, are very similar. More details can be found by referring to the book by Timoshenko and a brilliant review [6]. This article presents open questions based on some of the known results [7], largely on billiards.

2. Nodal domain statistics

2.1 *Limiting distributions of nodal counts*

Eigenfunctions of a quantum billiard display an intricate pattern of nodal curves. These curves divide the domain into signed-regions. The number of domains, ν_j, and the order of energy levels in accordance with energy, labelled by j present a rather non-trivial relation. Following Courant, one studies the normalized number of nodal domains as

$$\xi_j = \frac{\nu_j}{j}, \qquad 0 < \xi_j \leq 1. \tag{3}$$

A limiting distribution can be constructed as

$$P(\xi) = \lim_{E\to\infty} P(\xi, I_g(E)) \tag{4}$$

by considering the energy levels in an interval $I_g(E) = [E, E + gE]$, $g > 0$. With the number of eigenvalues in $I_g(E)$ given by the Weyl formula, the distribution of ξ associated with $I_g(E)$ is

$$P(\xi, I_g(E)) = \frac{1}{N_I} \sum_{E_j \in I_g(E)} \delta\left(\xi - \frac{\nu_j}{j}\right). \tag{5}$$

For a rectangular billiard, the limiting distribution is [8]

$$P(\xi) = \begin{cases} \dfrac{1}{\sqrt{1-(\pi\xi/2)^2}}, & \text{for } \xi < 2/\pi, \\ 0, & \text{for } \xi > \xi_{\max} = 2/\pi. \end{cases} \tag{6}$$

The peak around $\xi \approx 0.64$ exactly coincides with the value of $2/\pi$ in eq. (6). Also for other separable, integrable billiards, a limiting distribution can be obtained.

For non-separable, integrable billiards like an equilateral triangle, the right isosceles triangle, and the hemi-equilateral triangle billiard, it is not proved that a limiting distribution exists. A calculation of such a distribution requires one to express the energy levels as well as the number of domains in an analytical form. For these triangles, the expression for the number of domains is now known [9, 10]. For a class of eigenfunctions of a right isosceles triangle, a limiting distribution seems to exist, with two peaks, at $2/\pi$ and $2\sqrt{2}/\pi$ [11].

For chaotic billiards, assuming that the eigenfunctions can be expressed in terms of a random superposition of plane waves, there does exist a limiting distribution. However, for pseudointegrable billiards and KAM systems, results are not known.

2.2 *Geometric characterization of nodal domains*

2.2.1 *Area-to-perimeter ratio*: A nodal domain of an eigenfunction has geometric characteristics like a well-defined perimeter and area. The ratio of these two quantities turns out to be another statistically significant tool to detect signatures of the underlying dynamics. To address the morphology of the nodal lines, [12] considered the set of nodal domains of the jth eigenfunction of a billiard in a domain $\mathcal{D}$; this can be represented as the sequence $\{\omega_j^{(m)}\}$, $m = 1, 2, \ldots, \nu_j$. One can then define the ratio

$$\rho_j^{(m)} = \frac{\mathcal{A}_j^{(m)} \sqrt{E_j}}{L_j^{(m)}}, \tag{7}$$

where $\mathcal{A}_j^{(m)}$ and $L_j^{(m)}$ denote the area and perimeter of the nodal domain; the factor of the energy eigenvalue ensures the correct scaling. We look at the probability measure,

$$P_{\mathcal{D}}(\rho, E, g) = \frac{1}{N_I} \sum_{E_j \in I} \frac{1}{\nu_j} \sum_{m=1}^{\nu_j} \delta(\rho - \rho_j^{(m)}), \tag{8}$$

which hopefully tends to a limiting distribution. For a rectangular billiard, the distribution is of the form

$$P_{\text{rectangle}}(\rho) = \begin{cases} \dfrac{4}{\rho\sqrt{8\rho^2 - \pi^2}}, & \dfrac{\pi}{\sqrt{8}} \leq \rho \leq \dfrac{\pi}{2}, \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

This function, in addition to being independent of the aspect ratio of the billiard, is analytic and monotonically decreasing in the compact interval $[\pi/\sqrt{8}, \pi/2]$ but is discontinuous at the endpoints. All of these properties, including the support, are believed to be universal features for all two-dimensional separable surfaces. However, for integrable but non-separable and pseudointegrable billiards, the form of $P(\rho)$ is unknown. For chaotic Sinai and stadium billiards, numerics suggest a universal limiting distribution $P(\rho)$, which converges to that for the random-wave ensemble.

2.3 *Nodal volume statistics*

The nodal volume is the hypersurface volume of the nodal set of the jth eigenfunction, denoted by $\mathcal{H}_j$. We scale the volumes with the typical wavelength, $\sqrt{E_j}$, $E_j > 0$, and then define the rescaled dimensionless variable, $\sigma_j = \mathcal{H}_j/\mathcal{V}\sqrt{E_j}$, where $\mathcal{V}$ is the volume of the manifold $\mathcal{M}$. It is this rescaled quantity on which Yau's conjecture places the bounds $c_1 \leq \sigma_j \leq c_2$ for $j \geq 2$, where c_1 and c_2 depend only on the manifold and the metric. We now present some recent results [13] on the nodal volume statistics for an s-dimensional cuboid — a paradigm of regular classical dynamics — and for boundary-adapted planar random waves — an established model for chaotic wave functions — in irregular shapes [14].

2.3.1 *s-Dimensional cuboid*: The normalized eigenfunctions of an s-dimensional cuboidal Dirichlet billiard with sides of lengths $\{a_\ell, \ell = 1, 2, \ldots, s\}$ and volume $\mathcal{V} = \prod_{\ell=1}^{s} a_\ell$ are

$$\psi_{\mathbf{n}}(\mathbf{q}) = \frac{(2\pi)^{s/2}}{\mathcal{V}^{1/2}} \prod_{\ell=1}^{s} \sin\left(\frac{\pi n_\ell q_\ell}{a_\ell}\right), \tag{10}$$

$\{n_\ell\}$ being positive integers. The corresponding energies and rescaled nodal volumes are

$$E_{\mathbf{n}} = \pi^2 \sum_{\ell=1}^{s} \frac{n_\ell^2}{a_\ell^2}, \quad \sigma_{\mathbf{n}} = \frac{1}{\sqrt{E_{\mathbf{n}}}} \sum_{\ell=1}^{s} \frac{n_\ell - 1}{a_\ell}. \tag{11}$$

Let us define an asymptotic mean value of $\sigma_{\mathbf{n}}$ in a spectral interval $E_{\mathbf{n}} \in [E, E + \Delta E]$ of width ΔE near E,

$$\langle \sigma_{\mathbf{n}} \rangle_{[E,E+\Delta E]} = \frac{1}{N_{[E,E+\Delta E]}} \sum_{\mathbf{n} \in \mathbb{N}^s} \sigma_{\mathbf{n}} \chi_{[E,E+\Delta E]}(E_{\mathbf{n}}),$$

where χ is the characteristic function on the interval and $N_{[E,E+\Delta E]}$ is the number of eigenfunctions with energies in $[E, E + \Delta E]$. For the asymptotic behaviour, ΔE can be chosen to be $gE^{1/4}$, $g > 0$, without the loss of generality. Weyl's law for the cumulative level density is

$$N_{\mathrm{Weyl}}(E) = \frac{\zeta_s \mathcal{V}}{2^s \pi^s} E^{s/2} - \frac{\zeta_{s-1} \mathcal{S}}{2^{s+1} \pi^{s-1}} E^{(s-1)/2} + O(E^{(s-2)/2}), \tag{12}$$

where $\zeta_s = \pi^{s/2}/\Gamma(1 + s/2)$ is the volume of an s-dimensional unit sphere and $\mathcal{S} = 2\mathcal{V} \sum_{\ell=1}^{s} a_\ell^{-1}$ is the $(s-1)$-dimensional volume of the surface of the s-cuboid. Using eq. (12) to obtain $N_{[E,E+\Delta E]}$ and employing the Poisson summation formula, the mean value is found to be [13]

$$\langle \sigma_{\mathbf{n}} \rangle_{[E,E+\Delta E]} = \frac{2\zeta_{s-1}}{\pi \zeta_s} \left(1 - \beta_s \frac{\mathcal{S}}{\mathcal{V}} E^{-1/2} + O(E^{-3/4})\right),$$

$$\beta_s = \frac{\pi(s-1)\zeta_{s-2}}{2\, s\zeta_{s-1}} + \frac{\pi\zeta_s}{4\zeta_{s-1}} - \frac{\pi(s-1)\zeta_{s-1}}{2\, s\zeta_s}. \tag{13}$$

Utilizing the higher moments, the limiting distribution

$$P_s(\sigma) = \lim_{E\to\infty} \langle \delta(\sigma - \sigma_{\mathbf{n}}) \rangle_{[E,E+\Delta E]} \tag{14}$$

can be calculated for any s. The limiting distributions thus evaluated by [13] are non-zero only over a finite interval. For instance, $P_2(\sigma)$ is non-zero over $[1/\pi, \sqrt{2}/\pi]$ only wherein it varies as $4/\sqrt{2 - \pi^2\sigma^2}$. This observation is also in line with Yau's conjecture [15].

2.3.2 *Random wave model*: For the eigenfunctions of a chaotic billiard [13],

$$\langle \sigma \rangle_G = \rho_{\mathrm{bulk}} \left(1 - \frac{\mathcal{S}}{\mathcal{V}} \frac{\log k}{32\pi k} + O(k^{-1})\right), \qquad s = 2,$$

$$= \rho_{\mathrm{bulk}} \left(1 - \frac{\mathcal{S}}{\mathcal{V}} \frac{I_s}{32\pi k} + O(k^{-1})\right), \qquad s \geq 3,$$

where $\rho_{\mathrm{bulk}} = \Gamma((s+1)/2)/[\sqrt{\pi s}\Gamma(s/2)]$ is the constant nodal density of the standard random wave model (RWM) without boundaries and I_s are constants ($I_3 \simeq 0.758, I_4 = 0.645$). The limiting distribution of nodal volumes is now sharply peaked for a finite energy interval and converges to $P(\sigma) = \delta(\sigma - \rho_{\mathrm{bulk}})$. This is to be contrasted with the finite support for the cuboid's distribution — the distinct characters of $P(\sigma)$ therefore differentiate between chaotic and regular manifolds.

3. Eigenfunction amplitude distributions

Shortly after Percival classified the eigenfunctions into regular and irregular for respectively integrable and chaotic systems, Berry suggested that by taking the eigenfunctions of chaotic systems as a random superposition of plane waves, the amplitudes are distributed normally. For integrable, pseudointegrable and KAM systems, there were no rigorous results until recently. Even now, we only know very little about a few low-lying eigenfunctions of a couple of integrable planar billiards. A systematic study undertaken in [16] is based on the characteristic functions and their Fourier transforms. This has given us new distribution functions, adding to the venerable repertoire known thus far. We summarize the results so that the reader may appreciate its forms.

For a square billiard of side length, π, the distribution function is found in terms of the complete elliptic integral of the first kind:

$$P(\Psi) = \frac{2}{\pi^2} K(1 - \Psi^2) \left[1 + \frac{\mathrm{sgn}\Psi}{mn}\right] \tag{15}$$

for m, n odd. It is $2/\pi^2 K(1 - \Psi^2)$ for other cases. The non-separable integrable case of a right isosceles triangle presents with great technical difficulty. The only cases known are for ground and first excited states. Let us present here the result for the ground state where the side length of the legs is taken as π, the characteristic function for which is

$$\varphi_\Psi(\xi) = \int_0^\pi \int_0^x \exp[i\xi(\sin x \sin 2y - \sin 2x \sin y)]dydx. \tag{16}$$

The Fourier transform of this gives the probability distribution function:

$$P(\Psi) = \frac{3(1 + \mathrm{sgn}\Psi)}{4\sqrt{2}\pi} \mathcal{G}^{0,4}_{4,4}\left(\frac{27\Psi^2}{64}\Big|^{-1/6,1/6;1/2;1/2}_{-1/4,0,0,1/4}\right),$$

$$|\Psi| < \sqrt{\frac{64}{27}}, \tag{17}$$

and zero elsewhere. For the ground state of the equilateral triangle, the distribution function is expressed in terms of hypergeometric functions.

4. A few future directions

Most of the systems exhibit mixed dynamics, following the dictates of the Kolmogorov–Arnold–Moser theorem. Almost nothing is known regarding the properties of nodal domains and nodal curves. It would be very useful to have at least numerical results which will allow us to understand the transitions in the limiting distributions.

Most of the plane polygonal billiards are pseudointegrable. Not only the analytic form for their eigenfunctions remains open, even the statistical properties of domains are not convincingly understood. The area distribution seems to follow the scaling as if the excited state eigenfunctions are similar to the random wave superposition [17]. However, more detailed investigation is necessary.

The complex and intriguingly beautiful morphology of nodal curves is modelled in terms of stochastic Loewner evolution (also called Schramm Loewner evolution, SLE). The driving term is Brownian for the SLE. For pseudointegrable and integrable planar billiards, it would be most interesting to choose a proper driving so that the nodal curves encode the required symmetry.

Difference equation formalism was discovered recently for counting the nodal domains [7, 9, 10]. This helped counting the domains for certain systems, throwing light on a problem which was open for a very long time. However, the empirical difference equations are asking for their origin. A connection of these results and questions with Hilbert's sixteenth problem will also be worth exploring.

As mentioned briefly in the text, there is a connection of statistics of nodal volumes with the well-known Yau's conjecture.

In the opinion of the author, one of the most important problems is to find a way to obtain exact solution of the Schrödinger equation for non-integrable billiards. The only known cases are [18–20]. These results are originated from the connections between random matrix theories and exactly solvable models for many-body systems. The other important direction to consider is to understand the relations ensuing from the billiard eigenfunctions and certain difference equations satisfied by the vertex models of statistical mechanics [21]. This connection was first pointed out by Gaudin [22] in the context of six- and eight-vertex models. To go further, we also need generalization of vertex models and the corresponding colouring problems — there has been new results in this direction very recently [23].

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