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$\mathcal{PT}$-symmetric nonlinear systems and their implication in optics

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Abstract. In this paper, we briefly review a few interesting dynamical aspects and applications of nonlinear $\mathcal{PT}$-symmetric systems. Being at the boundary between dissipative and conservative systems, these $\mathcal{PT}$-symmetric systems show many interesting characteristics and applications that cannot be seen in the usual dissipative and conservative ones. Among the variety of applications, we here focus on the applicability of $\mathcal{PT}$-symmetric systems in unidirectional light transport. This particular application clearly evidences the usefulness of the non-reciprocal nature and symmetry broken phase of a $\mathcal{PT}$-symmetric system and importantly the role of nonlinearities in optics.

Keywords. $\mathcal{PT}$-symmetry; spontaneous symmetry breaking; non-reciprocity.

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1. Introduction

Symmetries are the most fundamental properties of nature and are responsible for many physical phenomena that we observe. In 1998, Bender and Boettcher brought to notice an interesting type of symmetry, namely parity–time ($\mathcal{PT}$)-symmetry and showed that it can be responsible for purely real spectra of non-Hermitian operators. This discovery significantly impacted the literature, as it suggested the extension of quantum theory to a complex domain. Now, the concept of $\mathcal{PT}$-symmetry has gone far beyond quantum mechanics and has become an active area of research in many branches of physics including optics [1–7], plasmonics [8, 9], quantum optics [10, 11], Bose–Einstein condensation [12, 13], acoustics [14] and electronics [15, 16]. One of the main reasons for such a development is the realization of $\mathcal{PT}$-symmetric systems as a special class of dissipative systems with balanced loss and gain. Due to the balanced loss and gain nature, these systems are also said to lie at the boundary between the closed and open systems. They combine many features of conservative and dissipative systems and support novel dynamics or applications that cannot be achieved with the usual conservative or dissipative systems. Thus they open up new possibilities for flexible control and steering of physical processes. In optics, $\mathcal{PT}$-symmetric systems provide a hope for the development of novel all-optical devices enabling increased speed and energy efficient information processing on an optical chip. In this connection, nonlinearities can add more features to the $\mathcal{PT}$-symmetric systems and they can give rise to a wide variety of new phenomena. It has been shown that the nonlinear $\mathcal{PT}$-symmetric systems can give rise to many interesting features including the formation of localized modes, nonlinearity-induced $\mathcal{PT}$-symmetry breaking and time reversal suppression, all-optical switching and so on [17–20]. The literature evidences that the nonlinear $\mathcal{PT}$-symmetric systems can serve as powerful building blocks for the development of novel devices enabling directed energy transport.

Motivated by these contemporary developments, in this review, we briefly highlight the significance of nonlinearities and $\mathcal{PT}$-symmetric systems. In this connection, we here recall the basic aspects of $\mathcal{PT}$-symmetric systems. We explain the symmetry unbroken and broken phases of $\mathcal{PT}$-symmetric systems and also the bifurcations underlying the symmetry breaking phenomenon of these systems. From a practical point of view, not only the symmetry preserving phase but also the symmetry broken phase is pointed out to be useful. We here review a particular application of the $\mathcal{PT}$-symmetry broken phase, namely the unidirectional light transport and illustrate the role of nonlinearities over its application. Also, the literature shows that such a unidirectional light transport can be achieved with a simple nonlinear $\mathcal{PT}$-symmetric dimer where the non-reciprocal nature of the $\mathcal{PT}$-symmetric systems and the self-trapping nature of the nonlinearities combine to support such type of light transport. This form of directed light transport is a key mechanism for the construction of on-chip optical diodes or optical isolators. In this connection, many of the recent studies bring out possible difficulties in a normal $\mathcal{PT}$-symmetric dimer and ways to overcome these difficulties. In this review, we briefly bring out these developments under a single roof, including our own studies.

2. $\mathcal{PT}$-symmetric systems

Parity–time ($\mathcal{PT}$)-symmetric systems are the ones that show invariance neither under parity operation $\mathcal{P}$ nor under time reversal operation $\mathcal{T}$ but they show invariance under the combined operation of parity and time reversal operators [21]. For instance, in quantum theory, the Hamiltonian of the $\mathcal{PT}$-symmetric system commutes with the combined $\mathcal{PT}$-operator ($[\mathcal{H}, \mathcal{PT}] = 0$). Here, the parity ($\mathcal{P}$) and time reversal ($\mathcal{T}$) operators are respectively a linear and anti-linear operators and they do spatial inversion and time reversal operation, respectively. These operators are defined as,

$$\mathcal{P}: \hat{x} \to -\hat{x} \quad \text{and} \quad \hat{p} \to -\hat{p},$$

$$\mathcal{T}: \hat{x} \to \hat{x}, \quad \hat{p} \to -\hat{p} \quad \text{and} \quad i \to -i.$$

Note the change in the sign of i in the $\mathcal{T}$-operator, which is because of the fact that (as like $\mathcal{P}$) $\mathcal{T}$ is required to preserve the fundamental commutation relation $[\hat{x}, \hat{p}] = i$ in quantum mechanics. Considering the Hamiltonian operator of a one-dimensional quantum system,

$$\mathcal{H} = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x), \tag{1}$$

we can note that the system is $\mathcal{PT}$-symmetric (or the Hamiltonian commutes with the combined $\mathcal{PT}$-operator) only when the potential $V(x)$ is complex and is of the form $V(x) = V^*(-x)$. This implies that the real part of the potential is an even function of x, while the imaginary part is an odd function of x.

In connection with the above, the complex nature of these $\mathcal{PT}$-symmetric potentials may raise a question of the physical realizability and it was resolved by the isomorphism that exists between the paraxial equation of diffraction in optics and the Schrödinger equation. The paraxial equation of diffraction in optics can be written as [4],

$$i\frac{\partial E}{\partial z} + \frac{1}{2k}\frac{\partial^2 E}{\partial x^2} + k_0 n(x) E = 0, \tag{2}$$

where E is the electric field envelope, $k_0 = 2\pi/\lambda_0$, λ_0 is the wavelength in vacuum, $k = k_0 n_0$, n_0 is the background refractive index and $n(x)$ is the refractive index profile of the optical medium. Comparing the eq. (2) with the Schrödinger equation, one can note the isomorphism between them. Thus the parity–time symmetric systems can be realized in optics when the complex refractive index profile satisfies $n(x) = n^*(-x)$. It is well known in the literature that the real part of the refractive index of the optical medium is responsible for dispersion while the imaginary part of the refractive index is responsible for the gain or loss in optical medium. So it is evident that by suitably balancing the gain and loss in the medium, one can have optical $\mathcal{PT}$-symmetric systems. Due to this reason, $\mathcal{PT}$-symmetric systems can be interpreted as non-isolated physical systems with balanced loss and gain and they can be considered as a special class of dissipative systems that lie at the boundary between the open (dissipative) and closed (conservative) systems. Thus the understanding of $\mathcal{PT}$-symmetric systems as systems with balanced loss and gain has led to the exploration of $\mathcal{PT}$-symmetry in a variety of fields including Bose–Einstein condensates, mechanics, electronics and so on. To illustrate the above, Bender *et al.* have cited a simple example of the source–sink model [22]. The schematic diagram of the model is shown in figure 1. In this example, a box that is placed at the negative x region acts as a source and a box that is placed at the positive x region acts as a sink. To illustrate the $\mathcal{PT}$-symmetric nature of the system, we first do parity operation over the system (that is, we do a reflection about $x = 0$). The second row in the figure shows the system after parity operation. From this, we observe that the source and sink boxes are now interchanged. After the parity operation, we perform a time reversal operation as shown in figure 1. During time reversal operation, a system that emits energy seems to absorb energy and a system that absorbs energy seems to emit or radiate energy. Thus source will turn into sink and sink will turn into source which is shown in the third row of the figure. One can see that the system resulting from the parity and time reversal operations look the same as that of the original and it indicates the $\mathcal{PT}$-symmetric nature. Note that the system is not invariant with respect to $\mathcal{P}$-operation (or $\mathcal{T}$-operation) alone. On the other hand, the system mentioned in figure 1 is not isolated, the boxes are coupled with each other. When the coupling between the boxes are not strong enough, the system cannot be in equilibrium and it can be in equilibrium only when the coupling is strong [22]. Thus the system exists in two phases: (i) unbroken $\mathcal{PT}$-phase (the phase in

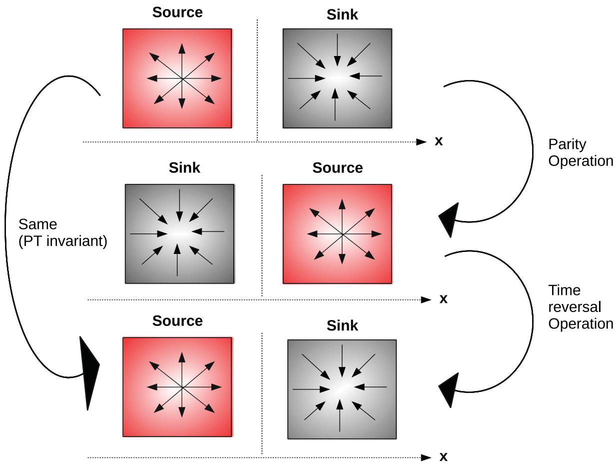

Figure 1. Source–sink model as a simple demonstrative example of $\mathcal{PT}$-symmetric system.

which the system is in equilibrium and the energy eigenvalues corresponding to the system are real) and (ii) broken $\mathcal{PT}$-phase (the phase in which the system is not in equilibrium and the energy eigenvalues corresponding to the system become complex). Thus the system exhibits a spontaneous symmetry breaking phase transition when the coupling strength is varied.

3. Spontaneous symmetry breaking in $\mathcal{PT}$-symmetric systems

In this connection, the phenomenon of spontaneous symmetry breaking has been observed in many of the $\mathcal{PT}$-symmetric systems. It has also been reported that the symmetry broken phase is equally important from an application point of view. Thus the studies on the spontaneous symmetry breaking and the associated bifurcations leading to such phenomena are interesting. In this connection, the tangent or saddle-node type bifurcation in $\mathcal{PT}$-symmetric systems is more interesting and this type of symmetry breaking may underly the non-reciprocal dynamics observed in $\mathcal{PT}$-symmetric systems [23]. To cite a simple example showing spontaneous symmetry breaking through tangent bifurcation, one can consider the $\mathcal{PT}$-symmetric mechanical system, namely modified Emden equation whose dynamical form can be written as [24]

$$\ddot{x} + \alpha x\dot{x} + \beta x^3 + \lambda x = 0. \tag{3}$$

In the above, α denotes the strength of nonlinear damping and β is the strength of cubic nonlinearity. In the presence of the nonlinear damping term ($x\dot{x}$ term), the above equation is $\mathcal{PT}$-symmetric ($x \to -x$, $t \to -t$). It is also interesting to note that the above position dependent damping term $x\dot{x}$ incorporates balanced loss and gain to the system [24]. If $\alpha > 0$ ($\alpha < 0$), the system will have loss (gain) along the positive x-axis and it will have gain (loss) along the negative x-axis. The amount of gain present along $x < 0$ is balanced by the loss present along $x > 0$ [24]. Due to this balance between gain and loss, we can observe non-isolated periodic orbits (as in the case of conservative systems) in a particular parametric region. (We here wish to add that the periodic oscillations observed in the loss and gain free systems or conservative systems are non-isolated ones where the system oscillates with different amplitudes and may be with different frequencies with respect to different initial conditions. Considering the dissipative systems, periodic oscillations cannot be observed in systems with fully gain or fully loss terms. Periodic oscillations are observed in certain dissipative systems having both loss and gain. In such a case, when the loss and gain are unbalanced, we will observe only isolated periodic orbits as observed in various limit cycle oscillators. But when the loss and gain are balanced as in the case of $\mathcal{PT}$-symmetric systems, we can observe non-isolated periodic orbits as it was observed in conservative systems [24].)

To show the occurrence of the spontaneous symmetry breaking phenomenon, we here present the details of their dynamical behavior [24]. Rewriting system (3) as a system of two-coupled first-order ordinary differential equations,

$$\begin{aligned} \dot{x} &= x_1, \\ \dot{x}_1 &= -\alpha x x_1 - \beta x^3 - \lambda x, \end{aligned} \tag{4}$$

it can be seen that the system has a trivial equilibrium point E_0: $(x^*, x_1^*) = (0, 0)$ and a pair of non-trivial equilibrium points $E_{1,2}$: $(\pm\sqrt{-\lambda/\beta}, 0)$ (which exist only if $\lambda < 0$ or $\beta < 0$). Considering $\beta = 1.0 > 0$, the results of the linear stability analysis are presented in figure 2, where the real part of the eigenvalues corresponding to the Jacobian matrix is plotted with respect to λ. The figure shows that for $\lambda > 0$, E_0 has pure imaginary eigenvalues which give rise to non-isolated periodic oscillations in the system. By decreasing λ below 0, we find the appearance of E_1 and E_2 with opposite stabilities where E_1 is stable and E_2 is unstable (E_0 is of saddle type). Considering the equilibrium point E_0, it retains its form under $\mathcal{PT}$-operation whereas E_1 transforms to E_2 and E_2 transforms to E_1 under $\mathcal{PT}$-operation. So in the region $\lambda > 0$, the observed periodic oscillations around the equilibrium point E_0 retain their structure under $\mathcal{PT}$-operation. But, for $\lambda < 0$, the trajectories around E_1 get transformed

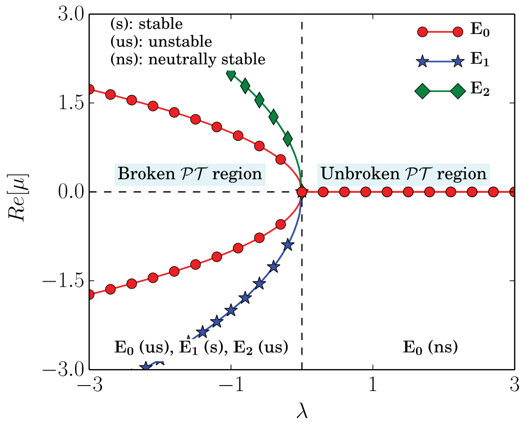

Figure 2. Plot of the real part of the eigenvalues associated with the equilibrium points E_0, E_1 and E_2 of system (3) for the value of $\alpha = 2.0$ and $\beta = 1.0$.

to trajectories around E_2 and vice-versa under the $\mathcal{PT}$-operation. Thus the $\mathcal{PT}$-symmetry is broken spontaneously for $\lambda < 0$, and here the symmetry is broken through a tangent or saddle-node like bifurcation. For a particular integrable case of the system, $\beta = \alpha^2/9$, the observed spontaneous symmetry breaking phenomenon of the system can also be interpreted using the exact solution [24].

The symmetry broken states of $\mathcal{PT}$-symmetric systems have an important implication in optics and the spontaneous symmetry breaking through the tangent bifurcation is also useful for the non-reciprocal nature of certain $\mathcal{PT}$-symmetric systems. Even though the $\mathcal{PT}$-symmetric systems offer many interesting applications, we present in the following an interesting simple application and show the role of nonlinearities over this application.

4. Unidirectional light transport with symmetry broken states and nonlinearities

To elucidate the application of symmetry broken states, we briefly discuss the work of Ramezani *et al.* [25] and illustrate the dynamics of a nonlinear $\mathcal{PT}$-symmetric system that supports unidirectional light transport. The nonlinear $\mathcal{PT}$-symmetric dimer that they have considered is of the form,

$$i\frac{d\psi_1}{dz} - i\gamma\psi_1 + \beta|\psi_1|^2\psi_1 + \epsilon\psi_2 = 0,$$

$$i\frac{d\psi_2}{dz} + i\gamma\psi_2 + \beta|\psi_2|^2\psi_2 + \epsilon\psi_1 = 0, \quad (5)$$

where ψ_1 and ψ_2 represent the complex electric field amplitudes in the amplifying and lossy waveguide channels, respectively. z represents the dimensionless propagation distance, γ is the gain or loss strength, β is the strength of the self-trapping nonlinearity and ϵ is the strength of coupling due to evanescent fields. Now, we discuss how the interplay of (non-reciprocal dynamics arising from) $\mathcal{PT}$-symmetry and (self-trapping phenomena associated with) Kerr type nonlinearities gives rise to interesting unidirectional light transport.

4.1 *Linear dynamics*

To begin, let us consider the linear case of (5). The linearized version of (5) can be written as,

$$i\frac{d\Psi}{dt} = \mathcal{H}\Psi, \quad (6)$$

where $\Psi = [\psi_1 \;\; \psi_2]^T$ and the Hamiltonian $\mathcal{H}$ takes the form,

$$\mathcal{H} = \begin{pmatrix} i\gamma & -\epsilon \\ -\epsilon & -i\gamma \end{pmatrix}. \quad (7)$$

The eigenvalues corresponding to the Hamiltonian are found to be $\lambda = \pm\sqrt{\epsilon^2 - \gamma^2}$. It is clear that the eigenvalues are real when $\gamma < \epsilon$ and they become complex (or pure imaginary) when $\gamma > \epsilon$. One can observe a $\mathcal{PT}$-phase transition while increasing the value of γ, where the $\mathcal{PT}$-symmetry is unbroken for $\gamma < \epsilon$ and it is broken for $\gamma > \epsilon$. $\gamma = \epsilon$ represents an exceptional point. It should be noted that the eigenvalues are always real in the Hermitian case, that is for $\gamma = 0$. Using figure 3, one can compare the dynamics that can be observed in the Hermitian ($\gamma = 0$) and non-Hermitian ($\gamma \neq 0$) cases and check the reciprocal or non-reciprocal nature of their dynamics. For this purpose, the beam propagation pattern in the Hermitian and non-Hermitian cases are presented for two different initial conditions in figure 3, where the two initial conditions are given by (i) $|\psi_1(0)|^2 = 1$ and $|\psi_2(0)|^2 = 0$ (input beam along the first waveguide alone) and (ii) $|\psi_1(0)|^2 = 0$ and $|\psi_2(0)|^2 = 1$ (input beam along the second waveguide alone). For these two initial conditions, the field dynamics in the Hermitian case is shown in figures 3a and b. These figures show the existence of power oscillations in the Hermitian or conservative case. A close examination of the beam propagation pattern observed in figures 3a and b can reveal the existence of left–right symmetry. Because, the beam dynamics of the first waveguide in figure 3a (in which the input beam is injected along the first waveguide) looks similar to the dynamics observed in the second waveguide shown in figure 3b (in which the input beam is injected along the second waveguide). Similarly, the dynamics observed in the second waveguide of figure 3a looks similar to the one observed in the first waveguide corresponding to figure 3b. This shows the reciprocal nature of the beam dynamics.

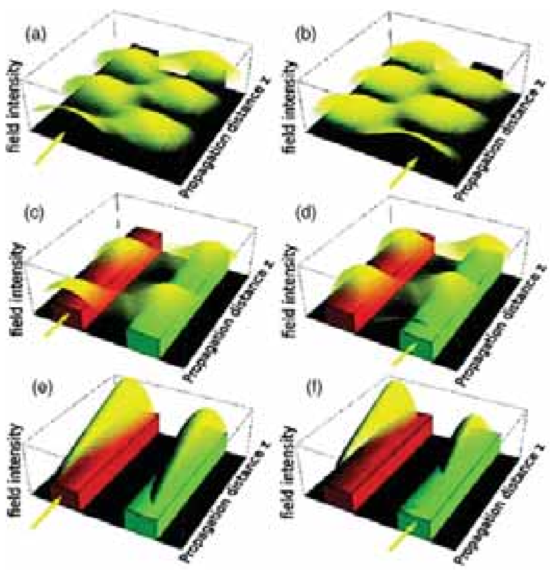

Figure 3. Dynamics in the linear case of system (5) for $\epsilon = 1$. (**a**) and (**b**) are plotted for the linear Hermitian case ($\gamma = 0$) and (**c**)–(**f**) for the linear non-Hermitian case. The red colored waveguide at the left represents the gain site and the green colored waveguide at the right represents the loss site. (**c**) and (**d**) are plotted for $\gamma = 0.4$ (that is, $\gamma < \gamma_{\rm PT} = \epsilon = 1$ which lies in the symmetry unbroken region) and (**e**) and (**f**) are plotted for $\gamma = 1.5, \gamma > \gamma_{\rm PT}$. In (**a**), (**c**) and (**e**), input light is given only along the left waveguide channel. In (**b**), (**d**) and (**f**), input light is given only along the second waveguide channel. Courtesy: H. Ramezani, T. Kottos, R. El-Ganainy and D. N. Christodoulides, *Phys. Rev. A* **82**, 043803 (2010).

Now considering the linear non-Hermitian case, the dynamics is pictured out in the unbroken and broken $\mathcal{PT}$-regions separately in figures 3c–f. Figures 3c and d show the dynamics in the unbroken $\mathcal{PT}$-region ($\gamma < \epsilon$) and figures 3e and f present the beam evolution in the broken $\mathcal{PT}$-region ($\gamma > \epsilon$). In the unbroken $\mathcal{PT}$-phase, similar to the conservative or Hermitian case, the system exhibits power oscillations. However, in the broken $\mathcal{PT}$-phase, the power grows up in both the waveguides as shown in figures 3e and f. It is to be noted that unlike the Hermitian case, the dynamics in the non-Hermitian case (figures 3c–f) shows non-reciprocal nature with respect to the symmetry axis of the system [25]. In more clear words, the beam propagation pattern of the non-Hermitian case differs depending on whether the initial excitation is on the left or right waveguide. The beam propagation pattern observed in the first (second) waveguide of figures 3c and e will not exactly match with the pattern observed in the second (first) waveguide of figures 3d and f, respectively. This is in contrast with the $\gamma = 0$ case (figures 3a and b), where the beam propagation is

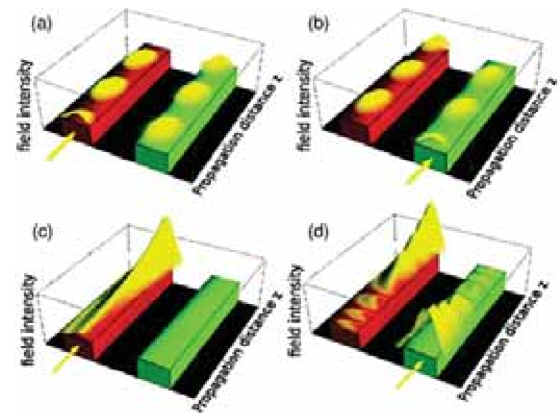

Figure 4. Dynamics in the nonlinear case of system (5). The red colored waveguide at the left represents the gain site and the green colored waveguide at the right represents the loss site. (**a**) and (**b**) show the beam dynamics in the unbroken $\mathcal{PT}$-region and (**c**) and (**d**) show the beam dynamics in the broken $\mathcal{PT}$-region. (**a**) and (**c**) correspond to an initial excitation at the left-waveguide channel and (**b**) and (**d**) correspond to an initial excitation at the right-waveguide channel. Courtesy: H. Ramezani, T. Kottos, R. El-Ganainy and D. N. Christodoulides, *Phys. Rev. A* **82**, 043803 (2010).

insensitive to the initial conditions. The following discussions will illustrate that this non-reciprocal nature, when coupled with the self-trapping nature of the nonlinearity, can result in interesting unidirectional light transport.

4.2 *Nonlinearities for unidirectional light transport*

In the presence of nonlinearity, the field dynamics in the symmetry unbroken and broken regions of system (5) are presented respectively in figures 4a, b and c, d. As it was discussed previously, here also, we consider a set of initial conditions, namely (i) $|\psi_1(0)|^2 = 1$ and $|\psi_2(0)|^2 = 0$ (for figures 4a and c) and (ii) $|\psi_1(0)|^2 = 0$ and $|\psi_2(0)|^2 = 1$ (for figures 4b and d). In the unbroken $\mathcal{PT}$-phase, figures 4a and b illustrate the existence of power oscillations (as observed in the linear case). In the broken $\mathcal{PT}$-phase, figures 4c and d show that (for both input configurations) the power in the gain site alone grows and it gets saturated to a higher value whereas the power in the loss site reaches zero as $z \to \infty$. Importantly, while looking at the dynamics in both the unbroken and broken $\mathcal{PT}$-regions, we note that the dynamics in both the regions are non-reciprocal and such nature can be seen more obviously from the dynamics in the broken $\mathcal{PT}$-region. In the latter case, one cannot find similarity in the field dynamics corresponding to the first waveguide/gain site of figure 4c and the one corresponding to the second waveguide/loss site of figure 4d. This is because of

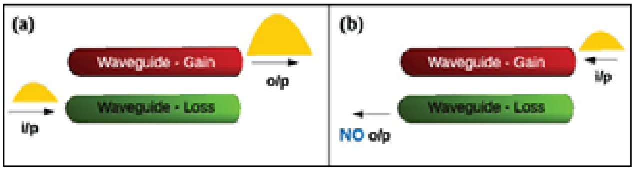

Figure 5. Illustration of unidirectional light propagation in the $\mathcal{PT}$-symmetric nonlinear dimer. (**a**) Shows the power transfer in the forward direction from the loss to gain waveguide and (**b**) shows that there is no power transfer in the reverse direction from the gain to the loss waveguide.

the reason that the output field always leaves the sample from the waveguide with gain (red-colored) independent of the input beam pattern and the output field at the lossy waveguide approaches zero for longer waveguides. It is important to note that in the case of the linear $\mathcal{PT}$-symmetric dimer (see figures 3e and f), the beam intensity at the lossy waveguide never goes to zero and it increases exponentially.

The above discussion shows the usefulness of nonlinearity which enables the localization of light in the gain site and also enables the intensity in the gain site to be saturated at a higher value as $z \to \infty$. The role of nonlinearity in enhancing the non-reciprocity (or in suppressing the time reversibility) was also illustrated clearly in [6]. This dynamic feature suggests the possibility to have unidirectional light transport because the power transport between one end of the loss waveguide and the other end of the gain waveguide is possible but the transport in the reverse direction is not possible (since power can be transferred in the forward direction from the loss to the gain waveguide but not in the backward direction from the gain to the loss waveguide). The above statement has also been illustrated in figure 5, where one can find that one end of the gain waveguide and the other end of the loss waveguide act as the input or output port. In figure 5a, the input beam is injected through the loss site and as we have pointed out earlier the output field now leaves through the gain site on the other end. When the input beam is injected to the gain site as shown in figure 5b, there will not be any output from the loss site at the other end. This novel unidirectional light transport of the $\mathcal{PT}$-symmetric nonlinear dimer is the key mechanism for establishing on-chip optical isolators (diodes). These characteristics enhanced the curiosity of many researchers and brought huge interest over the studies of nonlinear dynamics of $\mathcal{PT}$-symmetric dimers, trimers and oligomers [7, 26–32]. Apart from the theoretical interest, this type of coupler has also received more attention from the experimental point of view [4].

4.3 *Potential problems*

In the above consideration of unidirectional light transport, a few teething problems do exist. To name a few, we can state the following:

(i) balancing loss and gain in the system

(ii) controlling blow-up responses

(iii) obtaining finite power output

In the following, we mention a few ways that are reported to overcome these demerits. In particular, we highlight the role of nonlinearities in overcoming these issues.

4.4 *To overcome the loss–gain balancing problem*

To improve the practical realizability of the above couplers, studies were undertaken to achieve unidirectional light transport without the balanced loss–gain profile. In this connection, Alexeeva *et al.* have considered two lossy waveguides that are placed in an active medium [29]. Thus the medium enhances the evanescent field which couples the two-waveguide channels. The dynamical equation corresponding to such a situation can be simply written as,

$$i\frac{d\psi_1}{dz} = -i\gamma\psi_1 - \psi_2 - |\psi_1|^2\psi_1 + ia\psi_2$$

$$i\frac{d\psi_2}{dz} = -i\gamma\psi_2 - \psi_1 - |\psi_2|^2\psi_2 + ia\psi_1. \tag{8}$$

In the above equation, the first term indicates the lossy nature of the two-waveguide channels. The second and third terms respectively represent the evanescent field coupling and self-trapping nonlinearity. The last term in the above equation represents the presence of active medium and a represents the strength of active coupling. With this model, they have shown the possibility of unidirectional light transport and, in this case, there is no need to balance the gain and loss where the loss compensation can be performed with a finite band of gain coefficients. In addition, this type of optical couplers supports the finite power output and it does not allow abrupt growth of optical modes.

Recently, a still more general asymmetric active coupler is shown to be useful for the directed transport of light [28]. In this case, dissimilar waveguides with unbalanced loss and gain is considered to achieve stable finite power directed light transport. The dynamical equations corresponding to this set-up can be given by,

$$i\frac{d\psi_1}{dz} = -(\omega_1 + i\gamma_1)\psi_1 - \alpha(|\psi_1|^2 + \sigma|\psi_2|^2)\psi_1 - \frac{\epsilon}{2}\psi_2,$$

$$i\frac{d\psi_2}{dz} = -(\omega_2 + i\gamma_2)\psi_2 - \alpha(|\psi_2|^2 + \sigma|\psi_1|^2)\psi_2 - \frac{\epsilon}{2}\psi_1, \quad (9)$$

where ω_i represents the propagation constant of i^{th} waveguide ($i = 1, 2$). γ_1 and γ_2 respectively represent the strength of loss (or gain) in the first waveguide and strength of gain (or loss) in the second waveguide. We can also note the role of nonlinearities in system (9) where α and σ respectively represent the self-phase and cross-phase modulation terms. ϵ represents the evanescent field coupling. This system supports stable finite power during unidirectional light transport. The asymmetric nature of the system or the unbalanced loss–gain profile is responsible for such a stable finite power transport.

To overcome the need for balancing loss and gain, the abovementioned studies considered non-$\mathcal{PT}$-symmetric or asymmetric systems. But we can also achieve unidirectional light transport in loss–gain free $\mathcal{PT}$-symmetric systems. In our recent work [23], we have elucidated the above statement with $\mathcal{PT}$-symmetrically coupled nonlinear systems. A general form of the evolution equations corresponding to the latter type of $\mathcal{PT}$-symmetric dimers can be written as

$$i\frac{d\psi_1}{dz} = ia\psi_2 - \epsilon\psi_2 - \beta|\psi_1|^2\psi_1 + \alpha G(\psi_1, \psi_2, \psi_1^*, \psi_2^*),$$

$$i\frac{d\psi_2}{dz} = -ia\psi_1 - \epsilon\psi_1 - \beta|\psi_2|^2\psi_2 + \alpha G(\psi_2, \psi_1, \psi_2^*, \psi_1^*), \quad (10)$$

where the first term in the above equations represents the $\mathcal{PT}$-symmetric coupling or magneto-optic type coupling and ϵ represents the coupling due to the evanescent fields. β is the self-trapping nonlinearity and $G(\psi_1, \psi_2, \psi_1^*, \psi_2^*) = -G(-\psi_1, -\psi_2, -\psi_1^*, -\psi_2^*)$ can be any odd order nonlinear term defining the nonlinear interactions with the coupling strength α. Note that the above type of systems can have both conservative and dissipative situations. In the absence of the nonlinear interaction term, the system is obviously conservative and depending on the form of G, the system will be either conservative or dissipative.

In contrast to the $\mathcal{PT}$-symmetric systems with intrinsic loss and gain, the linear case of the system will have real eigenvalues (the Hamiltonian of the linear case is also Hermitian). Thus one might question whether

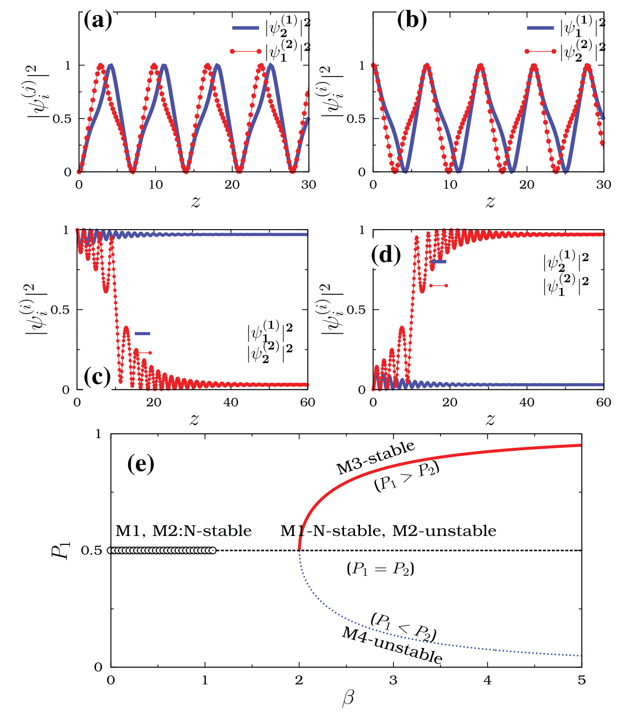

Figure 6. Non-reciprocal dynamics in the conservative case given in (10) for $G(\psi_1, \psi_2, \psi_1^*, \psi_2^*) = |\psi_1|^2|\psi_2|^2\psi_2$. (**a**) and (**b**) are plotted in the unbroken $\mathcal{PT}$-region for $k = 0.5$, $a = 0.5$, $\beta = 2.0$ and $\alpha = 1.0$. (**c**) and (**d**) are plotted in the broken $\mathcal{PT}$-region for $\epsilon = 0.5$, $a = 0.5$, $\beta = 4.0$ and $\alpha = 1.0$. (**e**) Shows tangent like bifurcation in the system for $\epsilon = 0.5$, $a = 0.5$, $P = 1.0$ and $\alpha = 1.0$.

it is possible to achieve non-reciprocal dynamics in these systems and particularly in conservative cases. With these questions, we have classified the cases which support reciprocal and non-reciprocal dynamics in [23]. The results show that all non-conservative cases of (10) admit non-reciprocal dynamics whereas only certain conservative cases show such dynamics. For instance, considering nonlinear interactions up to quintic order, we noticed that the G's of the form $G = |\psi_2|^2\psi_1^2\psi_2^*$ and $|\psi_1|^2|\psi_2|^2\psi_2$ show non-reciprocal dynamics and with other forms of G, the system shows reciprocal dynamics. The advantage of non-reciprocal conservative systems is the possibility to have a finite power output. While observing the bifurcations underlying spontaneous symmetry breaking in these conservative $\mathcal{PT}$-symmetric systems, we find that many of the reciprocal systems show pitchfork type symmetry breaking bifurcation and tangent type bifurcation is observed in non-reciprocal systems.

To elucidate the above, we have illustrated the non-reciprocal dynamics observed in the conservative case with $G = |\psi_1|^2|\psi_2|^2\psi_2$ in figure 6. In this figure, we have plotted the beam propagation patterns for the same two types of initial conditions: (i) $|\psi_1(0)|^2 = 1$ and $|\psi_2(0)|^2 = 0$ and (ii) $|\psi_1(0)|^2 = 0$ and $|\psi_2(0)|^2 = 1$. The resultant beam propagation pattern corresponding to the first (second) initial condition is represented by $|\psi_1^{(1)}|^2$ and $|\psi_2^{(1)}|^2$ ($|\psi_1^{(2)}|^2$ and $|\psi_2^{(2)}|^2$). As mentioned earlier in section 4.2, in figures 6a–d, the mismatch in the patterns of $|\psi_1^{(1)}|^2$ and $|\psi_2^{(2)}|^2$ (or in the patterns of $|\psi_2^{(1)}|^2$ and $|\psi_1^{(2)}|^2$) elucidate the non-reciprocal nature. Further, the symmetry breaking bifurcation in the system is also illustrated in figure 6e. In this figure, the modes $M1$ and $M2$ are symmetric whereas the modes $M3$ and $M4$ are asymmetric (with $|\psi_1|^2 \neq |\psi_2|^2$ or the powers $P_1 \neq P_2$). We can observe from the figure that the symmetric mode $M1$ is throughout neutrally stable and $M2$ loses its stability with an increase in the value of β. The figure also shows that near $\beta = 2.0$, two asymmetric modes appear where the mode $M3$ is stable and $M4$ is unstable (similar to the situation discussed in section 3). Thus the above non-reciprocal system shows spontaneous symmetry breaking through tangent like bifurcation and due to this bifurcation, for the asymmetric initial conditions, we observe only stabilization of the mode $M3$ with $P_1 > P_2$ and not the mode $M4$ with $P_1 < P_2$. Thus the system exhibits non-reciprocal type dynamics. Besides the above, our results also show that the self-trapping nonlinearity is essential to achieve unidirectional light transport [23] and in the absence of self-trapping nonlinearity, many of the systems do not show non-reciprocal dynamics. This again highlights the need for nonlinearities to enhance the features of $\mathcal{PT}$-symmetric systems.

4.5 *To control the blow-up responses*

One of the other problems met with the $\mathcal{PT}$-symmetric systems is that the blow-up type of response for certain initial conditions. The existence of such responses is more viable from the non-conservative nature of many of the $\mathcal{PT}$-symmetric dimers. For instance, Refs [31, 32] show explicitly the existence of blow-up solution in a $\mathcal{PT}$-symmetric dimer. Efforts have also been taken to control or overcome such blow-up type solution. In [29, 32], it has been shown that the above mentioned nonlinear actively coupled dimer, as given in eq. (8), has completely bounded dynamical regimes. In this case (8), the self-focusing type nonlinearity couples the linearly excitable mode to the linearly damped normal mode and thus suppresses blow-up type solutions. It is also more obvious that with the conservative $\mathcal{PT}$-symmetric systems [23], one can have bounded dynamics under all initial conditions.

In our recent article [33], we have shown that an interesting nonlinear coupling that is related to stimulated Raman scattering can control the blow-up response in a $\mathcal{PT}$-symmetric dimer (with competing linear and nonlinear loss–gain profile) of the form

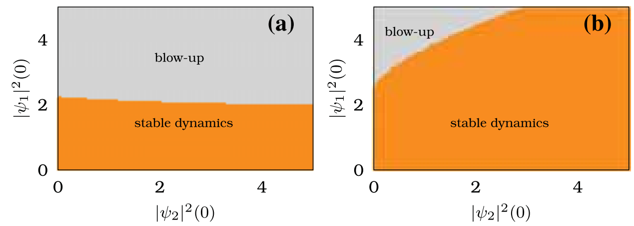

Figure 7. Role of nonlinear coupling in controlling blow-up responses. (**a**) is plotted for $\chi = 0.0$ and (**b**) is for $\chi = 0.3$ and in both the figures, we considered $\gamma = 1.0$, $\alpha = 0.5$, $\omega_1 = 1.0$, $\omega_2 = 0.8$ and $\epsilon = 0.5$. We here varied the initial conditions and pictured out the input power configurations that lead to blow-up. To draw (**a**) and (**b**), we considered $\psi_1(0) = u(0)+0i$ and $\psi_2(0) = v(0) + 0i$.

$$i\frac{d\psi_1}{dz} = \omega_1\psi_1 - i\gamma\psi_1 + i\alpha|\psi_1|^2\psi_1 - \epsilon\psi_2 - i\chi_1|\psi_2|^2\psi_1,$$
$$i\frac{d\psi_2}{dz} = \omega_2\psi_2 + i\gamma\psi_2 - i\alpha|\psi_2|^2\psi_2 - \epsilon\psi_1 + i\chi_2|\psi_1|^2\psi_2, \quad (11)$$

where ω_1 and ω_2 correspond to the propagation constants and ϵ is the evanescent field coupling. γ and α respectively represent the linear and nonlinear loss–gain strength. The nonlinear coupling that is useful for suppressing blow-up responses is given as the last term in the above equation. By choosing $\chi_1 = (\omega_1/\omega_2)\chi$ and $\chi_2 = \chi$, one can relate the above coupling to the coupling that is observed between the complex amplitudes of pump and the Stokes modes in the stimulated Raman scattering process [34, 35]. In such a situation, the pump signal has higher frequency than the Stokes signal so that $\omega_1 > \omega_2$. In [33], the roles of this nonlinear coupling in controlling the blow-up response and in directing light transport are demonstrated. For instance, the reduction in the blow-up region with respect to different input powers is illustrated in figure 7. The blow-up region in the absence and presence of the nonlinear coupling is shown respectively in figures 7a and b. From these figures, we find the reduction in the blow-up region by the introduction of nonlinear coupling.

4.6 *Obtaining finite power output*

The normal $\mathcal{PT}$-symmetric dimer given in (5) shows unstable dynamics in the symmetry broken region, that is, in the unidirectional light transport region. Thus achieving a finite power output in this region is also crucial from a practical point of view. In this connection, one may use conservative non-reciprocal $\mathcal{PT}$-symmetric structures to have a finite power output. But with limit cycle type systems, one can have output that is independent of the input powers. For the above purpose, one may use active couplers [29], asymmetric active couplers [28] or systems with competing linear and nonlinear loss–gain profiles [26, 33].

5. Conclusion

It is very clear that a combination of nonlinear science and $\mathcal{PT}$-symmetry can provide fertile ground for the control over energy transport in optical media and useful in the construction of novel devices. Besides the simplicity of the structure, the $\mathcal{PT}$-symmetric dimers discussed in this review show interesting dynamics. However, from a practical point of view, one needs to extend the studies to $\mathcal{PT}$-symmetric networks and their variants to obtain better control over light or energy transport and to construct on-chip optical devices. In this situation also, one has to study the ways to control the finiteness of the output field and suppress the blow-up type responses. In addition to the above-mentioned applications and scope, the $\mathcal{PT}$-symmetric systems are also fruitful in various other situations as well. For instance, $\mathcal{PT}$-symmetric defects in the periodic lattices control the transport properties of the lattice and help to achieve unidirectional invisibility, non-reciprocal or one way light transport and so on. Thus future researches along these lines may bring exciting results with robust control over light or energy transport.

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