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Interconnections among analytical methods for two-coupled first-order integrable systems

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Abstract. We recall six different analytical methods, namely extended Prelle–Singer procedure, Lie point symmetries, λ-symmetries, adjoint symmetries, Jacobi last multiplier and Darboux polynomial methods which are used to identify integrability quantifiers of nonlinear ordinary differential equations (ODEs). We point out how these methods are interconnected in the case of two-coupled first-order nonlinear ODEs. We demonstrate the interconnections with an example.

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1. Introduction

In recent years, attempts have been made to connect various analytical methods that exist in the literature to identify integrable quantifiers of nonlinear ordinary differential equations (ODEs) [1]. Even though the methods are different they essentially look for the following quantities, namely symmetries, integrating factors and integrals of motion. As far as coupled first-order nonlinear ODEs are concerned, the widely used analytical methods are the Jacobi last multiplier method, Darboux polynomial method, Lie point symmetries, adjoint symmetries, λ-symmetries and extended Prelle–Singer (PS) method. In this paper, we interconnect these methods in the case of coupled first order ODEs. We organize our presentation as follows. In the following section, we recall each one of these methods and mention the determining equations for the integrability quantifier in the considered method. The interconnection between these methods is discussed elaborately in section 3. We demonstrate the interconnections with an example in section 4. Finally, we present our conclusion in section 5.

2. Analytical methods for two-coupled first-order ODEs

In this section, we present six analytical methods which are widely used to derive the solution of the given two-coupled first-order ODEs. We plan to start our analysis from the extended Prelle–Singer method.

2.1 *Extended Prelle–Singer method* [2]

Let us consider a system of two-coupled first-order ODEs of the form

$$\dot{x} = \phi_1(t, x, y), \quad \dot{y} = \phi_2(t, x, y), \tag{1}$$

where ϕ_i, $i = 1$ and 2, are analytic functions where over dot denotes total differentiation. The aim of the PS-procedure is to obtain two functions, namely integrating factors R and K from which an integral can be constructed. The method of deriving determining equations for the functions R and K is given in Ref. [2]. The determining equations for the above two-coupled first-order ODEs (1) are given by

$$R_t + \phi_1 R_x + \phi_2 R_y = -(R\phi_{1x} + K\phi_{2x}), \tag{2a}$$
$$K_t + \phi_1 K_x + \phi_2 K_y = -(R\phi_{1y} + K\phi_{2y}), \tag{2b}$$
$$R_y = K_x. \tag{2c}$$

The integral of motion can be obtained from,

$$I = r_1 + r_2 - \int \left[K + \frac{d}{dy}(r_1 + r_2) \right] dy, \tag{3}$$

where $r_1 = \int (R\phi_1 + K\phi_2)dt$ and $r_2 = -\int (R + d/dx\,(r_1))dx$.

The determining eqs (2a)–(2c) are difficult to solve. So, in Ref. [2], the authors introduced a transformation

$$R = SK, \tag{4}$$

where S is a null form so that the determining eqs (2a)–(2c) become

$$S_t + \phi_1 S_x + \phi_2 S_y = -\phi_{2x} + (\phi_{2y} - \phi_{1x})S + \phi_{1y}S^2,$$
$$K_t + \phi_1 K_x + \phi_2 K_y = -K(S\phi_{1y} + \phi_{2y}),$$
$$K_x = SK_y + KS_y, \tag{5}$$

The variables S and K are decoupled in the new set of equations which are easier to solve than the earlier set of eqs (2a)–(2c).

2.2 *Lie point symmetries* [3]

In this method one should construct a condition for the given differential equation to be invariant under a one-parameter group of infinitesimal transformations [3]. Solving this invariance condition one obtains the symmetries of the given equation from which one can derive first integrals, integrating factors and so on. The invariance condition of eq. (1) is given by

$$\xi\frac{\partial\phi_1}{\partial t} + \eta_1\frac{\partial\phi_1}{\partial x} + \eta_2\frac{\partial\phi_1}{\partial y} - \eta_1^{(1)} = 0,$$
$$\xi\frac{\partial\phi_2}{\partial t} + \eta_1\frac{\partial\phi_2}{\partial x} + \eta_2\frac{\partial\phi_2}{\partial y} - \eta_2^{(1)} = 0, \tag{6}$$

where $\eta_1^{(1)} = \dot{\eta}_1 - \dot{x}\dot{\xi}$, $\eta_2^{(1)} = \dot{\eta}_2 - \dot{y}\dot{\xi}$. The associated Lie point symmetry vector field can be written as $v = \xi\partial/\partial t + \eta_1\partial/\partial x + \eta_2\partial/\partial y$. By introducing characteristics Q_1 and Q_2,

$$Q_1 = \eta_1 - \dot{x}\xi = \eta_1 - \phi_1\xi, \quad Q_2 = \eta_2 - \dot{y}\xi = \eta_2 - \phi_2\xi, \tag{7}$$

the determining eq. (6) can be written as

$$D[Q_1] = \phi_{1x}Q_1 + \phi_{1y}Q_2, \quad D[Q_2] = \phi_{2x}Q_1 + \phi_{2y}Q_2, \tag{8}$$

where D is the total derivative operator and it is given by $D = \partial/\partial t + \phi_1\partial/\partial x + \phi_2\partial/\partial y$.

2.3 *Jacobi last multiplier* [4]

The Jacobi last multiplier method is one of the oldest methods [4] in the theory of ODEs. This method also helps to derive the integrals. The determining equation for the Jacobi last multiplier of (1) can be written as

$$D[\log M] + \phi_{1x} + \phi_{2y} = 0, \tag{9}$$

where D is the total differential operator. The solutions of eq. (9) give the Jacobi last multipliers M. The first integral can be constructed by evaluating their ratios, that is, $I = M_1/M_2$.

2.4 *Darboux polynomials* [5]

The Darboux method is yet another powerful method to study the integrability properties of nonlinear ODEs. Darboux polynomials are the solutions of the following determining equation:

$$D[f] = \alpha(t, x, y)f, \tag{10}$$

where D is the total differential operator and $\alpha(t, x, y)$ is the cofactor. Once a sufficient number of Darboux polynomials are determined then the associated first integral can be determined by evaluating the ratio of two Darboux polynomials which have the same cofactor. The product of Darboux polynomials also acts as Darboux polynomials.

2.5 *Adjoint symmetries* [6]

Adjoint symmetries are solutions of the adjoint equation of the linearized symmetry conditions (8), that is

$$D[\Lambda_1] = -(\Lambda_1\phi_{1x} + \Lambda_2\phi_{2x}),$$
$$D[\Lambda_2] = -(\Lambda_1\phi_{1y} + \Lambda_2\phi_{2y}). \tag{11}$$

The adjoint symmetries can be used to identify the integrating factor and to derive integrals.

2.6 *λ-Symmetries approach* [7]

λ-Symmetries help to integrate the ODEs in case the considered equation does not admit Lie point symmetries but shown to be integrable. λ-Symmetries determining equation for the two-coupled first-order ODEs (1) are given by

$$\xi\frac{\partial\phi_1}{\partial t} + \eta_1\frac{\partial\phi_1}{\partial x} + \eta_2\frac{\partial\phi_1}{\partial y} - \eta_1^{[\lambda,(1)]} = 0,$$
$$\xi\frac{\partial\phi_2}{\partial t} + \eta_1\frac{\partial\phi_2}{\partial x} + \eta_2\frac{\partial\phi_2}{\partial y} - \eta_2^{[\lambda,(1)]} = 0, \tag{12}$$

where $\eta_1^{[\lambda,(1)]}$ and $\eta_2^{[\lambda,(1)]}$ are the λ-first prolongations and they are defined by $\eta_i^{[\lambda,(1)]} = \eta_i^{(1)} + \lambda(\eta_i - \phi_i\xi), i = 1, 2$. In terms of characteristics Q_1 and Q_2, we can rewrite the invariance condition (12) as

$$D[Q_1] = \phi_{1x}Q_1 + \phi_{1y}Q_2 - \lambda Q_1,$$
$$D[Q_2] = \phi_{2x}Q_1 + \phi_{2y}Q_2 - \lambda Q_2. \tag{13}$$

In the case $\lambda = 0$, eq. (13) gives the Lie point symmetries by determining eq. (8).

3. The interconnections

To explore the interlink among the aforementioned methods, we consider the extended PS procedure for two-coupled first-order ODEs. As the starting point by introducing suitable transformations in the extended PS method quantities, namely integrating factors R and null forms S, we interconnect the other methods with the extended PS procedure.

3.1 *Transformations*

To connect the other methods with the extended PS procedure, we introduce the following transformations:

$$S = -\frac{D[V]}{V}\frac{1}{\phi_{1y}} + \frac{\phi_{1x}}{\phi_{1y}}, \quad K = \frac{V}{F} \tag{14}$$

in the determining eq. (5), where V and F are functions of t, x and y. The resultant expression reads

$$D^2[V] - \left(\frac{D[\phi_{1y}]}{\phi_{1y}} + \phi_{2y} + \phi_{1x}\right)D[V] + \left(\phi_{1x}\frac{D[\phi_{1y}]}{\phi_{1y}} - D[\phi_{1x}] + \phi_{2y}\phi_{1x} - \phi_{1y}\phi_{2x}\right)V = 0, \tag{15}$$

$$D[F] = (\phi_{2y} + \phi_{1x})F. \tag{16}$$

Equation (16) is the Darboux polynomial determining equation with cofactor $(\phi_{2y} + \phi_{1x})$.

3.2 *Extended PS method with other methods*

3.2.1 *Connection* 1: *Lie point symmetries and extended PS method:* By taking the derivative of the ratio of the characteristics Q_2 and Q_1 and using eq. (8), we arrive at

$$D[g] = \phi_{2x} + (\phi_{2y} - \phi_{1x})g - \phi_{1y}g^2, \tag{17}$$

where $g = Q_2/Q_1$. By comparing eq. (17) with the first eq. in (5), we can identify the null form S as the ratio of the characteristics (Q_2/Q_1), that is

$$S = -g = -\frac{Q_2}{Q_1}. \tag{18}$$

The expression (18) relates the extended PS procedure with the Lie point symmetry method.

3.2.2 *Connection* 2: *Adjoint symmetries and extended PS method:* To relate adjoint symmetries with the extended PS method, we compare the adjoint symmetry equation (11) with the PS method determining eqs (2a)–(2b). This comparison provides the connection between adjoint symmetries and the integrating factors R and K, that is

$$\Lambda_1 = R, \quad \Lambda_2 = K, \tag{19}$$

provided Λ_1 and Λ_2 satisfy eq. (2c).

3.2.3 *Connection* 3: *λ-Symmetries and extended PS method:* To obtain the interconnection between the λ-symmetries and extended PS method, we consider the choice $\xi = 0$, $\eta_1 = 1$ and $\eta_2 = \beta$ in (13), which in turn provides the following expressions:

$$\lambda = \phi_{1x} + \phi_{1y}\beta, \quad D[\beta] = \phi_{2x} + (\phi_{2y} - \phi_{1x})\beta - \phi_{1y}\beta^2. \tag{20}$$

Now by comparing eqs (20) and (17), we find that $\beta = g$. Since $g = -S$, the PS method quantity S can now be interlinked to the λ-symmetry with

$$\beta = -S. \tag{21}$$

From the known expression of β we can obtain λ through the first relation given in (20). The λ-symmetry can be written as

$$\tilde{V} = \frac{\partial}{\partial x} - S\frac{\partial}{\partial y}. \tag{22}$$

This establishes the connection between the λ-symmetries and the extended PS method.

3.2.4 *Connection* 4: *Darboux polynomials and extended PS method:* By comparing eq. (16) with (10), we find $F = f$ and $\alpha = \phi_{2y} + \phi_{1x}$. From the above expression, we establish the interconnection between the Darboux polynomial method and the extended PS procedure as

$$F = \frac{V}{K}. \tag{23}$$

3.2.5 *Connection* 5: *Jacobi last multiplier and extended PS method:* To find an expression which relates the Jacobi last multiplier and extended PS method, we compare the expression (9) with (16) and we get $F = M^{-1}$. In other words, the expression which interrelates the Jacobi last multiplier with the PS procedure turns out to be

$$K = VM. \tag{24}$$

Thus one can obtain the Jacobi last multiplier for the given equation from the extended PS method itself.

4. Example

To demonstrate the interconnections which we derived in the previous section, as an example we consider the Ermakov–Pinney type eq. [4]

$$\dot{x} = y, \quad \dot{y} = \frac{3y^2}{2x} - 2x^3. \tag{25}$$

One can find the interlinks among analytical methods starting from any of the analytical methods. We start our analysis from Lie point symmetries. Equation (25) admits the following Lie symmetry vector fields:

$$v_1 = \frac{\partial}{\partial t}, \quad v_2 = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}. \tag{26}$$

The characteristics Q_1 and Q_2 read (vide eq. (7))

$$Q_{11} = -y, \quad Q_{21} = 2x^3 - \frac{3y^2}{2x}, \tag{27}$$

$$Q_{12} = -(ty + x), \quad Q_{22} = 2tx^3 - \frac{3ty^2}{2x} - 2y. \tag{28}$$

Using the relation $S = -Q_2/Q_1$, we can identify the functions S_1 and S_2 and by substituting them in eq. (5) for K, and solving the underlying equation, we obtain

$$S_1 = \frac{2x^3}{y} - \frac{3y}{2x}, \quad K_1 = -\frac{2y}{x^3}, \tag{29a}$$

$$S_2 = \frac{4tx^4 - 3ty^2 - 4xy}{2txy + 2x^2}, \quad K_2 = \frac{2txy + 2x^2}{x^4}. \tag{29b}$$

From the above expressions for S and K, we obtain the integrating factors in the form (vide eq. (4))

$$R_1 = \frac{3y^2}{x^4} - 4, \quad R_2 = \frac{4tx^4 - 3ty^2 - 4xy}{x^4}. \tag{30}$$

To obtain λ-symmetries, we use the first expression in eq. (20) along with $\beta = -S$. The λ-functions turn out to be

$$\lambda_1 = \frac{3y}{2x} - \frac{2x^3}{y}, \quad \lambda_2 = \frac{-4tx^4 + 3ty^2 + 4xy}{2x(ty + x)}. \tag{31}$$

The associated λ-symmetry vector field can be written as

$$\tilde{V}_1 = \frac{\partial}{\partial x} - \left(\frac{2x^3}{y} - \frac{3y}{2x}\right)\frac{\partial}{\partial y}, \tag{32}$$

$$\tilde{V}_2 = \frac{\partial}{\partial x} - \left(\frac{4tx^4 - 3ty^2 - 4xy}{2txy + 2x^2}\right)\frac{\partial}{\partial y}. \tag{33}$$

The function V can be evaluated using the first equation in eq. (14) which in turn leads to $V_1 = y, \quad V_2 = 2(ty + x)$. Using relation (23), Darboux polynomials can be fixed as

$$F_1 = -\frac{x^3}{2}, \; F_2 = x^3, \; \alpha_1 = \alpha_2 = \frac{3y}{x}. \tag{34}$$

The inverse of the Darboux polynomial provides the Jacobi last multipliers and they are given by

$$M_1 = -\frac{2}{x^3}, \quad M_2 = \frac{1}{x^3}. \tag{35}$$

The integrals can be constructed with the help of eq. (3). We find

$$I_1 = \frac{y^2}{x^3} + 4x, \quad I_2 = -\frac{4tx^4 + ty^2 + 2xy}{x^3}. \tag{36}$$

From (36) the general solution of eq. (25) can be identified in the form

$$x(t) = \frac{16I_1}{4I_1^2t^2 - 4I_1I_2t + I_2^2 + 64}, \tag{37}$$

$$y(t) = -\frac{64I_1^2(2I_1t - I_2)}{(4I_1^2t^2 - 4I_1I_2t + I_2^2 + 64)^2}, \tag{38}$$

where I_1 and I_2 are the integration constants.

5. Conclusion

In this paper, we have considered six different analytical methods which help to identify the integrability of a system of two-coupled first-order ODEs. By fixing suitable transformations in the extended PS procedure quantities, we have shown that one can interconnect the remaining methods with the extended PS procedure. We have demonstrated the theory with an example.

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