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Dynamics of systems with fluctuating sample space

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Abstract. The purpose of this review paper is to summarize progress on a recently introduced topic: Sample space reducing (SSR) stochastic processes. The concept of SSR has been coined to offer an alternative route that can explain the origin of power law features associated with complex systems. Understanding the emergence of scale-free behavior remains a topic of continuing interest, specially in the statistical physics community, due to its appearance in diverse systems, besides the existence of different explanations. One crucial aspect of the SSR is that the problem can be solved analytically. This develops the topic even more attracting and offers a possibility to provide a deeper understanding. Eventually numerous generalization of the SSR has been advanced, and a shift in the focus was also seen from the scale-free statistics to different physically relevant forms. We here primarily discuss the following points: (i) Various ways to visualize the SSR and its applications, (ii) mention, for a comparison, other mechanisms that reveal the origin of the power law distribution, (iii) different generalizations with their importance and (iv) a brief discussion with future perspectives.

Keywords. Complex system; scale invariance; stochastic processes; cascade; records.

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1. Introduction

1.1 *Stochastic processes*

The dynamics of a wide variety of processes involve interaction with the evolving environment. The direct influence of such a complex interaction is mostly difficult to predict. The Brownian motion is naturally a fundamental example observed as typical trajectory of a tracer particle in a fluid. In such a situation, the evolution of the process is no longer deterministic. The dynamical update rules can be typically modeled by incorporating noise (additive or multiplicative). Amazingly, it has been noted that the quantification of large scale features can be correctly described, and analysis becomes more straightforward because dealing with unessential details can be avoided. This peculiar class of systems can be treated as stochastic processes. Applications of such processes span in almost every discipline.

A single trajectory of the stochastic process may not be useful to concretely characterize the process, and the observable of interest will be a random variable (discrete or continuous). Subsequently a large number of independent samples of the possible trajectories would be ordinarily required to accurately compute average and fluctuations, and eventually the entire probability distribution (marginal, joint and conditional). The Monte Carlo method can be conventionally used to simulate the process, and the characteristics of observables can be computed numerically. However, analytical treatment is substantially limited. Typically the long-time behavior is of particular interest, and if this behavior is time independent (or dependent), the process is termed stationary (or non-stationary). One crucial aspect that significantly simplifies analytical analysis is Markovian feature: the future state is only influenced by the present state and not the entire/partial past. Otherwise the process would be non-Markovian and such processes are typically challenging to track.

1.2 *Physical examples of the SSR stochastic processes*

In this review paper, our prime focus is on a specific class of stochastic processes: Sample space reducing (SSR). Before we present the standard definition of the process, it is worthwhile to first go through a few physical examples to recognize the unique nature of the process. As a first example, consider the event of ‘sentence formation’. The guiding principle for the sentence formation is undoubtedly the context and grammatical rules. The first word of a sentence can be picked from the sample space – all possible allowed words. As the first word is chosen, the sample space for the second word would be constrained. Similarly, when the next word is added satisfying the constraints, the region of sample space further changes and effectively reduces. As the process progresses, the size of sample space continues to decrease. Finally, the process stops because there is no further possibility to add any word. The same algorithm can be naturally extended from 'word formation' to 'predictive text algorithm' with an application to web search.

In the second notable example, consider 'fragmentation' event for a given object. For simplicity, the object is typically taken as a rod of unit length. Break the rod in two parts in a random manner. Further collect one piece and break this again. Repeat the same procedure until the size becomes too small to break. The description can be easily generalized for a two/three dimensional object.

In both these typical examples, one can note the following striking feature: the size or region of allowed sample space decreases as the complex process unfolds naturally. Therefore, the SSR represents naturally a path-dependent stochastic process. The dynamics of career represent, in addition, a notable example, where the possible job opportunities shrink with aging. In general, mostly complex systems that have been studied so far traditionally consider the fact that the set of all possible allowed state or state space is fixed. In the study of complex systems, not much considerable attention has been paid with regard to change in the size of state space as a function of time. Recently it has been shown that such processes are of immense importance and may offer an explanation of the emergent power law distribution [1].

1.3 *Applications and scope*

So far diverse applications of the SSR have been identified, and it is expected that more would be discovered in future. Striking examples typically include: (i) Computational linguistic [1, 2], (ii) fragmentation [3], (iii) diffusion on the weighted network [4], (iv) record statistics [5], (v) polymerization [6, 7], (vi) cascade processes (rumour or disease spreading) [8] and (vii) emergence of novelty [9, 10].

The SSR was initially introduced with a possible motivation to offer an alternative and straightforward explanation for the Zipf's law: How frequently an object appears as a function of its rank for a given set of objects. This frequency vs rank plot generally follows an inverse power law form as

$$P(i) \sim i^{-\lambda}, \tag{1}$$

with a typical value for the exponent $\lambda \approx 1$. For the SSR, this prominent feature is instantly recognized as a visitation probability. Notable examples of such a behavior range from how frequently a word naturally occurs in a given text to city size distribution [11]. Although the exponent λ is not always equal to 1, in many instances it can support a value between 0 and 1, and more generally the upper bound can indeed take a large value ($\lambda \gg 1$). In order to explain the variability of the exponent, other features for the SSR need to be included. This motivated several generalizations: One typical case is the noisy SSR process [1], where the size of sample space occasionally expands, and this can satisfactorily explain λ between 0 and 1. However, the entire spectrum of the critical exponent can be adequately understood when a new attribute, namely, cascade is appropriately combined with the SSR [8].

The scale-free statistics remain one aspect. However, many non-equilibrium systems externally driven and displaying dissipative features show diverse variety of limiting distributions in the steady state. Even these non-power law type distributions can also be understood if the driving rate (probability) of the system is properly designed to have a state dependency [12]. Other than the visitation probability, there is undoubtedly another interesting observable: The survival time or life span of the process. The study of the survival time statistic has revealed precisely a direct correspondence with record statistics for uncorrelated and correlated sequences of random events [5]. Thus, the possible scope of the SSR processes has been positively enhanced. Below these key aspects are clarified in considerable detail.

1.4 *Organization*

This review paper is carefully organized in the following way: Section 2 begins with a basic formalism and then various toy models are adequately discussed to visualize the SSR processes. As the scale-invariant feature is one of the key aspects associated with these processes, a brief description of underlying conventional mechanisms that can plausibly explain the emergence of power law features is given in section 3. Generalizations along several specific directions and their fundamental importance are properly discussed in section 4, while a direct correspondence with records statistics and a key feature survival time statistics is presented in section 5. Finally, section 6 concludes the paper with a discussion and future perspectives.

2. Ways to visualize SSR

In this section, we first recall the fundamental formalism and then present different models that depict the essence of the SSR processes. Specific description of a physical system begins by precisely defining its state or configuration, a set of information essentially required to specify the system at any instant of time. How transition from one state to another takes place as a function of time describes the governing dynamical rules, while the set of all possible allowed states forms a state space. Broadly speaking, the state can be typically classified as recurrent (will reappear in the steady state), forbidden (does not appear as a consequence of the dynamical update, hence this is not an allowed state), absorbing (once the system reaches here, its dynamics ceases or it is trapped) and transient (in long time or steady state, this would disappear). Consider, for example, a randomly wandering animal (searching for food) that can be modeled at the simplest level as a discrete two-dimensional random walk. The position of the walker as a function of time represents the state of the system, and as a result of dynamical evolution the walker goes from one state to another state. Putting constraints like specifying boundary conditions (absorbing or reflecting) on the lattice, the number of allowed states for the walker can be made finite and fixed.

Right now we turn our prime focus to appreciate how the SSR offers an explanation of the Zipf's law. Many toy models have been proposed to realize the SSR process such as random filtering, a ball bouncing downward on a staircase landscape, throwing fair dices with multiple numbered faces, a directed random hoping process on the set of positive integers, or as a relaxation process where particle jumps from higher energy levels to low energy levels.

2.1 *Random filter*

One of the simplest examples illustrating the concept of sample space reduction is as follows: Consider throwing a fair cubic dice that has six faces, typically labelled 1 to 6. The entire sample space of possible outcomes that are random events is {1, 2, 3, 4, 5, 6}, where each event is equally likely occurring with probability 1/6. What if a constraint is imposed or an additional information is provided: If the possible outcome should be even number, then the sample space would be progressively reduced in size as {2, 4, 6}. Moreover if one more additional constraint is set such as if the outcome should be less than 6, then the sample space would further reduce to {2, 4}. Even if one more constraint is set here as the outcome should be greater than 2, then the sample space would reduce to {4} that is there is only one element. Here we note that as more and more additional information or constraints are set up on the sample space, the size of the space keeps on reducing. In the illustrative example considered here, the size of sample space is reducing from 6 to 1 as $6 \to 3 \to 2 \to 1$. Note that each subsequent constraint is set in addition to existing constraints. Consequently, the size of sample space is strictly reducing. This situation can be termed state or SSR.

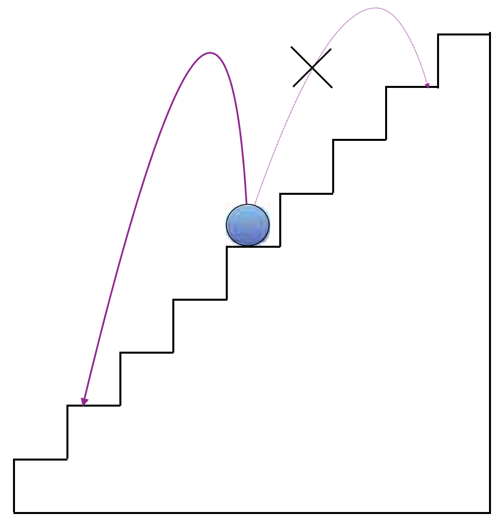

Figure 1. Bouncing ball realization of the SSR process with $N = 9$. The ball is at $x = 5$ at some instance of time. At a next time step it can only move downward to any site with transition probability $1/4$. The upward movement is forbidden. This is completely logical for such landscape as the ball is bouncing downward under the influence of gravitational force. Thus, the path is directed and acyclic.

On the other hand, if each constraint is independently imposed on the sample space, then it is not necessary that sample space should progressively reduce every time. However, it may occasionally expand and clearly this provides a better way to think of fluctuating or noisy SSR processes. For the above-mentioned example, the size of sample space for the noisy SSR varies as: $6 \to 3 \to 5 \to 4$.

As a concrete example, consider the following scenario: Given all courses to be offered through Swayam portal, a relevant topic can be searched by typically applying successive filters in a random order. The number of courses in the display list decreases as more and more filters are applied.

2.2 *Bouncing ball representation*

Consider a staircase landscape as shown in figure 1 and a ball bouncing downward with random steps in the following manner. The ball is initially at height N and it would instantly jump to any site downward with equal probability. As a result of the jump, if the ball reaches a site K, then at the next time step the ball can only jump downward to any site between $[1, K-1]$ with transition probability $1/(K-1)$. If now the ball is at height L here $L < K$, then at the next time step the ball can jump to any site between $[1, L-1]$ with transition probability $1/(L-1)$. The same process will be repeated until the ball reaches a ground level or a site with height 1.

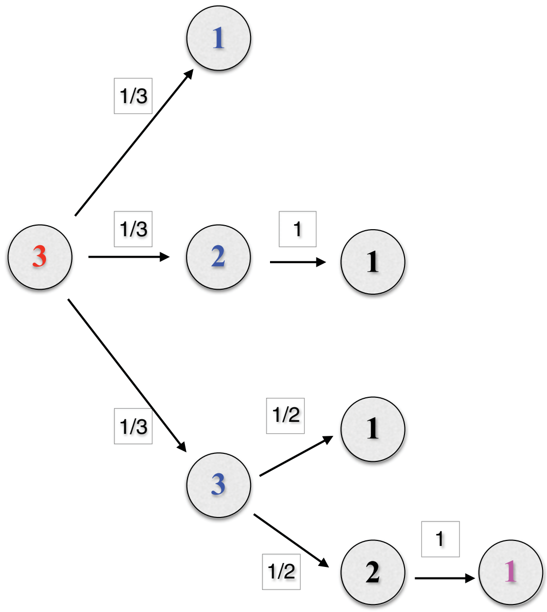

Figure 2. Pictorial representation of all possible paths that can originate with a system size $N = 3$ for the SSR process. A dice with $N = 3$ faces is thrown at $t = 0$ and the possible outcomes would be 1, 2 and 3 occurring with equal probability 1/3. A typical path would be, for example, $3 \rightarrow 2 \rightarrow 1$ with survival time $\tau = 3$. The visiting probabilities are as follows: $P_3(1) = 1/3 + 1/3.1 + 1/3.1/2 + 1/3.1/2.1 = 1$, $P_3(2) = 1/3 + 1/3.1/2 = 1/2$ and $P_3(3) = 1/3$.

2.3 *Rolling dice representation*

In the throwing dice case, consider N fair dices. Think that ith dice has i number of faces numbered between 1 and i. We start by throwing a dice that has N faces. The possible outcome would be a number between 1 to N occurring with equal probability. If the possible outcome is K, again throw a dice with $K - 1$ faces in the same manner. The subsequent outcome would be $1 \leq L \leq K - 1$, with probability $1/(K - 1)$. The same procedure is repeated until the outcome becomes 1. Here the process is stopped [1]. A typical realization is shown in figure 2.

In each independent realization, the outcome 1 is always observed or visited and hence the visiting probability to visit state 1 would be 1. If we start with $N = 2$, then the possible outcomes would be 1 or 2 with 1/2 probability. Subsequently if the outcome is 1 the process is stopped or if the outcome is 2 then again take a dice with 1 face and throw. Here certainly the possible outcome would be 1. We see that the outcome 2 is occurred once for two configurations, and the visiting probability for this case would be 1/2. Similarly, it can be easily checked that for outcome to be i the

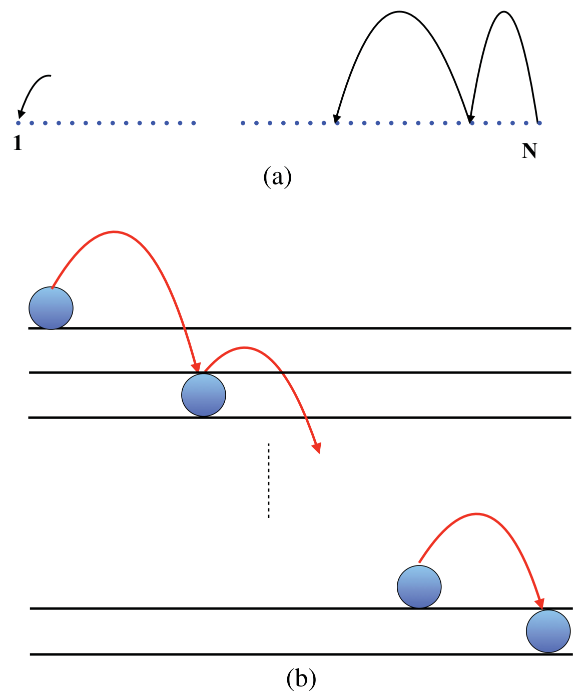

Figure 3. (**a**) The SSR as a hopping process on an integer line and (**b**) as a relaxation process showing transitions from higher to lower energy levels.

visiting probability varies as $P_N(i) = 1/i$. This shows how frequently an event occurs. This is basically Zipf's law.

Method of induction can be used to complete the proof for the above result. Assume that the visitation probability for $N' = N - 1$ is $P'_N(i) = 1/i$. If the process starts from a state N, it can go to state i in a single step with probability $1/N$. Or if it goes to an intermediate site j such that $i < j \leq N$ with probability $1/N$, then the state i will be visited with probability $P_{j-1}(i)$. This leads to a recursion relation:

$$P_N(i) = \frac{1}{N}\left(1 + \sum_{i<j\leq N} P_{j-1}(i)\right). \tag{2}$$

By assumption $P_{j-1}(i) = 1/i$ with $i < j \leq N$. Then the recursive equation can be simplified to show that

$$P_N(i) = \frac{1}{N}\left(1 + \sum_{i<j\leq N} \frac{1}{i}\right) \tag{3}$$

$$= \frac{1}{N}\left(1 + \frac{N-i}{i}\right). \tag{4}$$

or

$$P_N(i) = \frac{1}{i}. \tag{5}$$

2.4 *Random hopping on a discrete line*

Consider a directional hoping process $X(t)$ on a set of integers [1, N], starting from $X(0) = N$ and stopping when the process reaches 1, that is, $X(\tau) = 1$ [5]. At each elementary update, the process randomly hops from $X(t)$ to any position between 1 and $X(t) - 1$ with a uniform distribution. For $t_2 > t_1$, $X(t_2) < X(t_1)$. The same procedure is repeated till the process stops. Figure 3a shows a sample trajectory. The time τ is a discrete random variable and represents the survival time or life span of the process. For the SSR, $\tau \in [1, N]$.

2.5 *Relaxation process*

The SSR process can also be viewed as a relaxation process [12], where the particle loses energy as a consequence of relaxation by transitioning to lower energy levels from a higher energy level spontaneously and in random manner. To bring the particle to a higher energy level, the process is externally driven. Figure 3b shows a pictorial representation.

3. Mechanisms that explain power law

What invariably makes a decaying power law distribution an striking feature: This is essentially an emergent property typically observed in complex systems, where a large number of functional units collectively interact to produce such behavior. The most appealing characteristic of such distribution is that this is scale invariant or scale free, meaning there is a lack of a characteristic length scale. When the curve is plotted on a log–log scale, a straight line behavior with a negative slope is typically observed where the slope is the critical exponent characterizing the distribution. In other words, the smaller size events would be observed most commonly and the larger size events occur rarely, while the events with an intermediate size would be seen with moderate frequency. Mathematically, this is precisely a homogeneous function: Under a scaling of the argument, the qualitative behavior of the function remains invariant.

In the past several novel mechanisms have been undoubtedly discovered that can plausibly explain the possible origin of power law distribution [11, 13]. To gain a contrasting picture, we recall some of the conventional mechanisms used commonly in many applications.

3.1 *Generic critical phenomena*

Many natural systems undergoing a phase transition can be tuned by varying an external control parameter to reach in the vicinity of the transition point that is called critical point. The remarkable transition could be abrupt (first order) or smooth (second order). When the transition is continuous, the phase of the system is indistinguishable from the phases observed slightly away from the critical point. Since the correlation length of the system is infinite at the critical point, diverging behavior in the observables remains a typical feature. Typical systems comprise the following: Specific Ising models describe the continuous phase transition in magnetic systems [14], and geometrical phase transition can be described by percolation phenomena. The water retention capacity of random surfaces is sufficiently understood by invasion percolation [15, 16], while disease spreading or forest fire spread can be modeled as directed percolation (DP) [17, 18]. Also the complex dynamics of natural time-varying images show the critical properties and the critical exponents reveal that the system belongs to DP universality class [19]. The criticality and avalanches are also observed in the Kuramoto model [20].

3.2 *Self-organized criticality*

There are a range of naturally driven dissipative systems, where without tuning of an external control parameter, the system itself organizes to reside at the critical point, reflecting a kind of self-organization [21–24]. The system exhibits a non-equilibrium steady state such that in the critical state, that is an attractor, observables characterizing the system behavior show a power law distribution. The system is slowly driven and responds instantly with fluctuations recognized as avalanches, indicating a cascading effect. Thus, the existence of distinct separation of the time scale or multiple time scale can be typically observed. Common instances represent landslides or avalanches modeled as sandpile [22], earthquake [25], forest fire [26], evolution of species [27], critical branching processes [28], number theoretic division model [29], the interface depinning transition [30], Pólya urn [31], stellar processes [32] and neuronal systems [33–39], to name just a few. In the presence of quenched disorder, magnetic skyrmions exhibit avalanche dynamics where the gradient of the system induces driving [40].

3.3 *Combination of exponentials*

Consider a random event characterized by an exponential distribution. Take another observable related to the first variable in an exponential manner. In that case the second variable would show a power law distribution.

More precisely, take a random event x with an exponentially decaying distribution as:

$$P(x) \sim \exp(-ax), \tag{6}$$

where a is the decay rate. The event in which we are interested is related to x as

$$y = \exp(bx), \tag{7}$$

where b is a constant, then it can be seen using probability chain rule

$$P(x)dx = P(y)dy, \tag{8}$$

where

$$P(y) \sim y^{-(1+\mu)} \quad \text{with } \mu = a/b. \tag{9}$$

For this distribution to be well normalized, the variable y should be greater than y_0, where y_0 is the minimum value and $\mu > 0$.

3.4 *Preferential attachment*

The preferential attachment clearly describes the 'rich get richer' phenomena. In the context of complex networks, a specific class of networks with a power law type degree distribution is known as the scale-free network [41]. One of the generative mechanisms for networks with a power law type degree distribution is based on the preferential attachment (a new node forms a link with the existing nodes that are popular). However, the popularity is one aspect. The fitness (talent) of the nodes represents another feature and fitness-based links of a network can also lead to a scale-free network [42, 43]. This mechanism is termed as 'good get richer'.

3.5 *Multiplicative processes with constraint*

If the event of interest is described as a multiplication of several independent random variables with Gaussian distribution, then from central limit theorem it can be justly observed that the unconstrained multiplicative process results in log-normal distribution. However, it has been observed that for a constrained multiplicative process where the size of the event should be larger than a threshold size, the distribution of size shows a clear power law distribution [44]. Clearly, it is noticed that an additional attribute should be combined with the multiplicative process to generate a power-law behavior [45, 46]. This attribute could be a boundary constraint [44], random stop [45], reset [47], or additive noise [48].

3.6 *Nonlinearity: Inverse of a quantity*

Take $y = 1/x$ where x is the random variable with a uniform distribution. Then one finds $P(y) \sim y^{-2}$. This in general can be described as: Consider

$$y = 1/x^a, \tag{10}$$

then

$$P(y) \sim y^{-(1+\mu)} \tag{11}$$

with $\mu = 1/a$. Again, $\mu > 0$.

3.7 *First passage time distribution of random walk*

Consider a discrete time unbiased random walk on an integer line. At each time step, the walk moves by unit step either left or right randomly with equal probability. The walk starts from origin. The time duration between two consecutive origins is the quantity of interest. The time duration follows a power law distribution with an exponent 3/2 [49]. The exponent value in a range between 5/4 and 5/2 can be satisfactorily explained by a distinct class of random walks discussed in the context of a Pólya urn [31, 50]. A class of random walks with movable partial reflectors can generate the exponent between 3/2 and 2 [51].

4. Generalizations and their importance

4.1 *SSR with non-uniform prior probability*

In the original SSR, the transition from a state to another state out of the available states takes place with uniform probability. Interestingly, it has been observed that the Zipf's law obtained for the SSR is robust even for non-uniform prior probability of the states (these can be imagined as the width of a stair i) [4]. Consider $q(i)$ representing the prior probability of the state of a given system. Then the transition from j to i will be precisely

$$P(i|j) = \begin{cases} \dfrac{q(i)}{g(j-1)} & \text{if } i < j, \\ 0 & \text{otherwise,} \end{cases} \tag{12}$$

with $g(j-1) = \sum_{l<j} q(l)$. The prior probability is normalized, i.e. $\sum_i q(i) = 1$. The visiting probability satisfies the relation

$$p(i) = \sum_{i<j\leq N} P(i|j)p(j). \tag{13}$$

It was shown that eventually

$$p(i) = \frac{q(i)}{g(i)} p(1). \tag{14}$$

Take a polynomial prior such as $q(i) \sim i^{\alpha}$ with $\alpha > -1$. In this case, one can easily show that

$$p(i) = \frac{p(1)}{i}. \tag{15}$$

4.2 *Noisy SSR*

Since the SSR process is strictly reducing, in general, it is not a necessary condition. The generalized case is the noisy SSR process where from time to time the size of the state space may expand [1]. In order to realize this, consider an unconstrained random hopping on integers between 1 and N. The walker can instantly jump to any possible site uniformly. Mixing the SSR process and unconstrained random hoping would undoubtedly result in the noisy SSR process, where the SSR process is executed with probability λ. Strikingly, it has been observed that the occupation probability denoting on an average how many times a site is visited behaves as $p(i) \sim i^{-\lambda}$ with $\lambda \in [0, 1]$. Thus the noisy SSR processes provide an explanation of more general form of the Zipf's law.

Note that a site may be visited only once for a particular realization of the SSR process, while for the noisy SSR process there is no such restriction. Recall that the visiting probability distribution $P(i)$ represents the probability to visit ith site and the occupation probability $p(i)$ denotes on an average how many times the site is visited. For the SSR, these two are related by a normalization factor as $P(i) = p(i)/p(1)$, but for the noisy SSR these may not be proportional. The occupation probability $p(i)$ can be computed analytically. Note that a transition from j to i occurs as

$$P(i|j) = \begin{cases} \dfrac{\lambda}{j-1} + \dfrac{1-\lambda}{N} & \text{for } i < j, \\ \dfrac{1-\lambda}{N} & \text{for } i \geq j > 1, \\ \dfrac{1}{N} & \text{for } i \geq j = 1, \end{cases} \tag{16}$$

where the first line is the contribution from the strictly SSR process, the second line represents the contribution due to unconstrained random walk and the last line denotes restart. As the process is Markovian and renewal ensures stationarity, with the help of this transition probability, the occupation probability can be written as

$$p(i) = \sum_{j=1}^{N} P(i|j)p(j). \tag{17}$$

Equations (16) and (17) lead to a recursion relation that on iteration results in

$$p(i+1) - p(i) = -\frac{\lambda}{i} p(i+1). \tag{18}$$

After a rearrangement and iteration, this equation can be written as

$$\frac{p(i)}{p(1)} = \prod_{j=1}^{i-1} \left(1 + \frac{\lambda}{j}\right)^{-1}, \tag{19}$$

or

$$\frac{p(i)}{p(1)} = \exp\left[-\sum_{j} \log\left(1 + \frac{\lambda}{j}\right)\right], \tag{20}$$

$$\sim \exp\left(-\sum_{j=1}^{i-1} \frac{\lambda}{j}\right) \tag{21}$$

$$\sim \exp(-\lambda \log(i)) \tag{22}$$

$$= i^{-\lambda}. \tag{23}$$

4.3 *Cascade SSR*

Any scaling exponent for the Zipf's law can be explained by a combination of the SSR with a cascade effect [8]. The detail of the underlying dynamics is as follows: consider a set of ordered N states with labels $1, 2, \ldots, N$. A transition from $j \rightarrow i$ occurs only if $j > i$. In terms of the bouncing ball representation, the process starts with one ball at state N and stops when it reaches 1. The transition occurs from N to any state between $[1, N-1]$ randomly with uniform probability or more generally with a prior probability not necessarily uniform. For example, the ball reaches the Kth state, then the ball will split in μ balls and each ball executes a SSR independently. After each jump, each ball again splits in μ balls. And the multiplicative process continues until all balls reach state 1. When $\mu = 1$, the cascade SSR reduces simply to the SSR and when $\mu < 1$, the process corresponds to the noisy SSR process.

The visiting probability can be computed analytically. The argument follows as: Given that q_i denotes prior probability of state i, the cumulative prior probability is

$$g(k) = \sum_{i \leq k} q_i. \tag{24}$$

For the SSR, the transition probability from state j to i reads

$$P(i|j) = \begin{cases} \dfrac{q(i)}{g(j-1)} & \text{if } i < j, \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

As the number of balls is not conserved in the SSR cascade, the expected number of jumps from $j \rightarrow i$ is approximated as

$$n(j \rightarrow i) = \mu P(i|j). \tag{26}$$

In a single realization SSR cascade, the expected number of visits to the state i is denoted as n_i. Then

$$n_i = \sum_{j>i} n(j \to i) n_j \tag{27}$$

$$= \mu q_i \sum_{j>i} \frac{n_j}{g(j-1)}. \tag{28}$$

Subtracting $n_{i+1} - n_i$ and rearranging, one can write

$$\frac{n_{i+1}}{q_{i+1}} \left(1 + \mu \frac{q_{i+1}}{g(i)} \right) = \frac{n_i}{q_i}. \tag{29}$$

Iterating this equation and after a little bit of algebra it can be shown that

$$n_i = \frac{n_1}{q_1^{1-\mu}} \left(\frac{q_i}{g(i)^\mu} \right). \tag{30}$$

When $q_i = 1/(N-1)$, i.e., equal prior probabilities for all states, the expected visiting probability behaves as

$$n_i \sim i^{-\mu}. \tag{31}$$

4.4 *SSR driven with state dependent noise level*

A remarkable combination of driving and the SSR as a relaxation process can adequately explain a wide variety of non-power law type distributions [12]. In the absence of driving, the SSR relaxes by jumping from high states to low states and eventually gets trapped in an attractor. In order to instantly lift the process from the attractor, an external driving is naturally required. If the process is driven or restarted from the attractor with $\lambda = 1$ (slow driving), the process corresponds to the standard SSR. If the gradual relaxation of the SSR is interrupted at each time with a driving rate $r = 1 - \lambda$, the visiting probability typically becomes $p(k) \sim k^{-\lambda}$. The rare case $\lambda = 0$ (driving at every step) prominently represents a pure Bernoulli process.

If the driving rate is explicitly state dependent, then with probability $r = 1 - \lambda(k)$ the process is driven or restarted whenever it reaches state k. The range of the driving rate is $0 < r < 1$. Given $q_i > 0$ a normalized prior probability of state i, the transition probability for a transition from $k \to i$ is

$$P(i|k) = \begin{cases} \lambda(k) \dfrac{q_i}{g(k-1)} + [1 - \lambda(k)] q_i & \text{if } i < k, \\ [1 - \lambda(k)] q_i & \text{otherwise,} \end{cases} \tag{32}$$

where $g(k)$ is the cumulative distribution of the prior probabilities. Then following the similar calculation steps as performed in the previous subsections, it can be noted that

$$\frac{p_{\lambda,q}(i+1)}{q_{i+1}} \left(1 + \lambda(i+1) \frac{q_{i+1}}{g(i)} \right) = \frac{p_{\lambda,q}(i)}{q_i}. \tag{33}$$

This can be iteratively written as

$$p_{\lambda,q}(i) = \frac{q_i}{Z_{\lambda,q}} \prod_{1<j\leq i} \left(1 + \lambda(j) \frac{q_j}{g(j-1)} \right)^{-1}, \tag{34}$$

where $Z_{\lambda,q}$ is the normalization constant. With an approximation and in the continuum version, one obtains

$$p_{\lambda,q}(x) = \frac{q(x)}{Z_{\lambda,q}} \exp \left[- \int_1^x \lambda(y) \frac{q(y)}{g(y)} dy \right]. \tag{35}$$

Note that for the uniform prior probability distribution $q(x) = 1/N$. In this case, a relation between the driving rate and the visiting probability can be explicitly computed

$$\lambda(x) = -x \frac{d}{dx} \log p_{\lambda,q}(x). \tag{36}$$

With the help of eq. (36), for a given visiting probability, the driving rate's state dependency can be easily computed. Some examples are worked out below:

- Power law: $p(x) \sim x^{-\alpha}$,
$$\lambda(x) = -x \frac{d}{dx} [-\alpha \log x] = \alpha. \tag{37}$$
- Exponential distribution: $p(x) \sim \exp(-\beta x)$ with $\beta > 0$
$$\lambda(x) = -x \frac{d}{dx} [-\beta x] = \beta x. \tag{38}$$
- Stretched exponential: $p(x) \sim \exp\left(-(\beta/\alpha) x^\alpha\right)$ with $\alpha > 0, \beta > 0$
$$\lambda(x) = \beta x^\alpha. \tag{39}$$
- $\alpha = 2$ corresponds to the normal distribution.

5. Correspondence with records: Survival time statistics

In this section, the specific focus is on the survival time statistics and a correspondence with records in a sequence of random events. A recent study on the same topic explores the life span or survival time of the SSR process [5] which is a discrete random variable. The case of SSR is such that the exact results can be provided for a number of quantities such as visiting or occupation probability. The survival time probability distribution can also be computed analytically. It has been rigorously observed that both mean and variance of the survival time vary in a logarithmic manner with the system size and the asymptotic probability distribution is Gaussian [5].

In order to properly understand the correspondence between the survival time and the records for uncorrelated events, recall the definition of records. kth event forms a record if it is better than all the previously existing entries. Consider the case of record statistics in the independent identically distributed (iid) random variables [52]. The event is independent and uncorrelated, and equally likely to surpass all previous entries, consequently resulting a record at the kth entry with probability $1/k$. This is exactly similar to the visiting probability in the SSR process.

The survival time statistics immediately provides a clue that a mapping can be established between two seemingly different problems, namely the record statistics in the iid random variables and the survival time statistics in the SSR process. This explains the practical utility of the survival time in identifying a hidden connection, and further it becomes more apparent that the SSR processes have many applications. Clearly this correspondence provides a deeper understanding of the SSR process.

A conjecture supported with simulation studies suggests that there exists a correspondence between the survival time statistics for the noisy SSR processes and the record statistics for correlated time series [53]. It has been observed that the mean and standard deviations of the survival time behave as $\sim N/N^{\lambda}$, where N is the system size and the parameter λ can be continuously varied between 0 and 1. Interestingly, the survival probability distribution satisfies a simple sailing behavior of the form

$$\mathcal{P}_N(\tau) \sim N^{-\theta} J(\tau/N^{\theta}), \tag{40}$$

where J is a universal scaling function and

$$\theta = 1 - \lambda. \tag{41}$$

Two explicit examples of correlated sequences of random events have been identified where such a correspondence can be verified: (i) Drifted random walk with Cauchy distributed jumps [52] and (ii) fractional Brownian motion characterized by the Hurst exponent H that plays the role of θ [53, 54].

6. Discussion and future perspective

The SSR stochastic processes have been successfully proposed as an alternative route to explain the origin of a power law distribution associated with the complex system. Many applications have been identified from sentence formation in linguistic to fragmentation of objects and record statistics in a sequence of random events to cycle statistics in random permutation with uniform measure. Several generalizations, such as non-uniform prior probability of state, the noisy SSR, the SSR cascade and the SSR driven with state dependent noise, have certainly widen the scope of the concept of the SSR.

The extensive study of the survival time statistics has been shown to be extremely fruitful in discovering various new applications. Yet the survival time statistics in the generalized scenarios of the SSR, such as the SSR cascade and the SSR driven with the state dependent noise level, is still unexplored. Possibly, studies along this direction may further be useful to identify novel applications. Thus, the extent of how far the SSR is physically relevant certainly needs further exploration.

So far the discussion was restricted to understand a class of systems that can be described by stochastic processes. On the other hand, a wide variety of systems exhibiting unpredictable behavior can be deterministically described by a set of nonlinear equations of motion. This is the class of chaotic systems. For the chaotic systems, the dynamics are deterministic, although the future state is unpredictable and sensitive to initial conditions. To uncover the dynamics of phase space and their consequences, efforts should be made to identify class of systems, if that exists, where the phase space is evolving as a function of time.

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