Chimera-like states generated by large perturbation of synchronous state of coupled metronomes

SERGEY BREZETSKIY¹, DAWID DUDKOWSKI¹, PATRYCJA JAROS¹, JERZY WOJEWODA¹, KRZYSZTOF CZOLCZYNSKI¹, YURI MAISTRENKO^{1,2} and TOMASZ KAPITANIAK^{1,*}

¹Division of Dynamics, Technical University of Lodz, Stefanowskiego 1/15, 90-924 Lodz, Poland

²Institute of Mathematics and Centre for Medical and Biotechnical Research, National Academy of Sciences of Ukraine, Tereshchenkivska st. 3, 01030 Kyiv, Ukraine

*Corresponding author. E-mail: tomaszka@p.lodz.pl

Abstract. Chimera states in systems of coupled identical oscillators are spatiotemporal patterns in which different groups of oscillators can exhibit coexisting synchronous and incoherent behaviors despite homogeneous coupling. Here, considering the network of coupled pendula, we find that the patterns of chimera-like states can be generated after the large perturbation (in which one or a few oscillators have been stopped for the moment) of the synchronous state of the whole network. We show that these chimera-like states can be observed in simple experiments with mechanical oscillators, which are controlled by elementary dynamical equations given by classical mechanics.

Keywords. Coupled oscillators; chimera states; networks.

PACS Nos 05.45.Xt

1. Introduction

The synchronization of dynamical systems has become a field of intense interest due to huge real-life applications, from social behavioral analysis to biological models like neural networks (e.g. [1–8] and references therein). Beside different types of synchronization, the coexistence of the phase locked oscillators with desynchronized and incoherent oscillators in the network of identical oscillators has been observed. This coexistence creates the spatiotemporal patterns known as chimera states [9–42]. These patterns are typical for the large networks of different topologies and have been reported both in simulations [9–24] and experiments [25–30, 40–42]. Recently, it has been shown that chimera states can also be observed in small networks [36–42]. One of the chimera-like state which has been recently observed is the solitary state. The term solitary comes from the Latin *solitarius* and can be understood as alone, lonely or isolated. Indeed, in a typical solitary state one can identify a group of units oscillating independently from the others (often with different frequencies), which are additionally separated (in space) from the main cluster of the remaining synchronized oscillators. Examples of solitary states

have been found in different types of networks, e.g., Kuramoto model [23], delayed-feedback systems [33], in Stuart–Landau equations [34] or in globally coupled identical oscillators with attractive and repulsive interactions [35]. Typically, chimera states can be observed for random initial conditions but they can also be generated by large perturbation (temporal switch-off of the excitation) of the synchronous state of the whole network [32].

The coexistence of the synchronous state with various chimera-like states raises the question of synchronization survivability under different perturbations which are always present in real-world systems [43, 44]. Here we show that the patterns of the chimera-like and solitary states can be observed both in simulations and experiments in the networks of mechanical oscillators after the large perturbation (in which one or a few oscillators have been stopped for the moment) of the synchronous state of the whole network. As proof of the concept, we use the network of coupled metronomes (Huygens clocks [45]), i.e., the system of coupled pendula which are excited by the clock's escapement mechanism [46–48].

This paper is organized as follows. In section 2 we introduce the considered model and its experimental

implementation. Section 3 presents the discussion of the obtained results. Final conclusions are given in section 4.

2. Model

We consider the system of n coupled pendula shown in figure 1a. Pendula of length l and mass m are coupled through the linear springs with stiffness coefficients k_x, k_{xx} ($k_{xx} = k_x/2$) and linear dampers with damping coefficients c_x, c_{xx} ($c_{xx} = c_x/2$). Pendula's displacements are given by the angles φ_i . Springs and dampers are connected to each pendulum at distance l_s from the pivot. Each pendulum is connected with the nearest neighbor (green springs) and the second nearest neighbor (red springs). Additionally, the motion of each pendulum is damped by the linear damper characterized by damping coefficient c_φ . The energy is transmitted to each pendulum by the escapement mechanism which generates excitation torque M_D (in the first stage when $0 < \varphi_i < \gamma_N$, then $M_D = M_N$, and when $\varphi_i < 0$, then $M_D = 0$; in the second stage for $-\gamma_N < \varphi_i < 0$, $M_D = -M_N$, and for $\varphi_i > 0$, $M_D = 0$). M_N and γ_N are respectively the moment of force generated by the escapement mechanism and the working angle of this mechanism. Both these parameters are determined by the metronomes' design [46,47].

The described system can be experimentally implemented using n metronomes (we have used $n=20$ metronomes) whose pendula are connected by the spring elements as shown in figure 1b. The set of 20 coupled metronomes shown in figure 1b has been used in our experiments. We have used Wittner Maelzel metronomes (Model No. 802K) covering a frequency range of 40 (largo) to 208 (prestissimo) tics per minute, with a standard deviation of relative frequencies of $\sim 1\%$. Depending on the adjusted frequency, each metronome ticks for a duration of approximately 25 min (when fully wound up). We have measured the angular displacements of the metronomes' pendula φ_i to quantitatively analyze the behavior of the coupled metronomes. The metronomes have been placed on a circle of radius of 0.70 m as shown in figure 1b. As the spring elements we have used rubber bands (with cross-section radius 0.0001 m) of lengths 0.22 m and 0.65 m connecting respectively the nearest and the second nearest neighbors. The estimated stiffness coefficient k_x is equal to 0.484 N/m and the damping coefficient c_x is equal to 0.12 Ns/m. The motion of the set of coupled metronomes has been recorded with two Phantom v711 cameras capable of high speed image acquisition. Each camera has been set to a recording speed of 100 or 150 fps in order to cover more than 1500 periods of the pendula's

Figure 1. (a) n pendula coupled on the ring through springs and dampers, (b) experimental implementation of the system of figure 1a with 20 metronomes whose pendula are coupled by spring elements, (c) coexisting attractors of each uncoupled pendulum (metronome).

oscillations. The markers have been applied at the arms of the metronomes for further investigation with motion analysis program. TEMA software by Image

Systems has been applied to consecutive movies (image sequences) which delivered digital values of displacement (angles φ_i), velocity and acceleration obtained from tracing the markers on the arm of the selected metronome. Such time series allow for the estimation of the averaged frequencies of the selected metronomes (using eq. (2)). This experimental setup allows the successive registration of the displacements of two metronomes' pendula.

Each uncoupled pendulum is multistable and has three attractors: fixed points A_1^+ , A_1^- ($\pm \sin^{-1} \frac{M_N}{mgl}$, 0) and limit cycle A_2 shown in figure 1c. The boundaries between the basins of attraction of A_1^+ and A_1^- attractors and A_2 attractor are given by the ellipsoid $\frac{\varphi_0^2}{\gamma_N^2} + \frac{\varphi_0^2}{\omega^2 \gamma_N^2} = 1$, where ω is the frequency of oscillations and γ_N is the constant given by the design of the escapement mechanism. The basins of attraction of attractors A_1^+ , A_1^- and A_2 are shown respectively in yellow and green colors. The boundaries between the basins of A_1^+ and A_1^- attractors have not been determined as they play no role in the explanation of the observed behavior.

The dynamics of the system of coupled pendula shown in figure 1a is given by:

$$\begin{aligned} m l^2 \ddot{\varphi}_i + c_\varphi \dot{\varphi}_i + m g l \sin \varphi_i + k_x l_s^2 (\varphi_i - \varphi_{i-1}) \\ + k_x l_s^2 (\varphi_i - \varphi_{i+1}) + k_{xx} l_s^2 (\varphi_i - \varphi_{i-2}) + k_{xx} l_s^2 (\varphi_i - \varphi_{i+2}) \\ + c_x l_s^2 (\dot{\varphi}_i - \dot{\varphi}_{i-1}) + c_x l_s^2 (\dot{\varphi}_i - \dot{\varphi}_{i+1}) + c_{xx} l_s^2 (\dot{\varphi}_i - \dot{\varphi}_{i-2}) \\ + c_{xx} l_s^2 (\dot{\varphi}_i - \dot{\varphi}_{i+2}) = M_N, \end{aligned} \quad (1)$$

where $i = 1, 2, \dots, n$, $\varphi_0 = \varphi_n$, $\varphi_{n+1} = \varphi_1$. System (1) is symmetrical on the ring, i.e., pendulum i is coupled with pendula $i+1$ and $i-1$ (local coupling), and in addition, with $i-2$ and $i+2$ (nonlocal coupling).

The following parameter values have been used: $m = 0.044$ kg, $l = 0.011$ m, $l_s = 0.005$ m, $B = 0.0000974$ kgm², $c_\varphi = 0.0000107$ Nms, $M_D = 0.00011$ Nm, $\gamma_N = 10^\circ$, $c_x = 0.12$ Ns/m, $k_x = 0.484$ N/m. The frequency of the uncoupled metronome's pendulum is equal to $\omega = 6.97$ s⁻¹. With these parameters values the escapement mechanism generates oscillations of the uncoupled pendulum with amplitude $A \approx 0.407$ rad $\approx 23^\circ$. The 4th order Runge-Kutta method has been used for the integration of eq. (1). The phases $\theta_{1-20}(t)$ of the metronomes are obtained from numerical and experimental time series $\varphi_{1-20}(t)$ using Hilbert transformation: $\theta_{1-20}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\varphi_{1-20}(\tau)}{t - \tau} d\tau$. Further, the phases are used to obtain the averaged frequencies over time T :

$$\bar{\omega}_{1-20} \approx [\theta_{1-20}(T) - \theta_{1-20}(0)]/T. \quad (2)$$

The averaging has been performed over the time interval $T = 1500$ s.

In the considered system the chimera states can be observed due to: (i) the self-excited nature of its oscillations, (ii) the multistability (coexistence of A_1^+ , A_1^- and A_2 attractors) of each metronome, and (iii) sufficiently small coupling (too small to force synchronization via the energy transfers between metronomes [27–29]). Generally, in the described system the state of complete synchronization of all pendula coexists with various chimera-like states, including solitary states. Chimera-like states can be obtained also from random initial conditions when initial conditions of metronomes belong to the basins of different attractors (A_2 , A_1^+ and A_1^-).

3. Results and discussion

For nonzero coupling stiffness $k_x > 0$ we observe a broad range of parameters and initial conditions (whole basin of A_2 attractor, i.e., green region in figure 1c) in which the synchronization of all metronomes emerges. We have assumed that all metronomes are synchronized when the phase differences between the metronomes' pendula are zero (in numerical simulations) or close to zero (in experiments), i.e., $\varphi_i(t) - \varphi_j(t) = 0$ (in numerical simulations) and $\varphi_i(t) - \varphi_j(t) \approx 0$ (in experiments), $i, j = 1, 2, \dots, n$, $i \neq j$. For sufficiently small coupling stiffness k_x (smaller than the threshold value k_{th}) the synchronous state can be damaged by large perturbation. Here we adopt the perturbation in which one pendulum (or a few pendula) is (are) stopped for a moment, i.e., when $\varphi_i = 0$, φ_i is set to 0.

In the case when only one pendulum is perturbed, the threshold value of stiffness k_{th} is equal to 0.77 N/m ($k_{xx} = 3.95$ N/m, $c_x = 3.32$ Ns/m, $c_{xx} = 1.66$ Ns/m). For $k_x < k_{th}$ the perturbed pendulum oscillates with a small amplitude due to the energy transfer from the unperturbed pendula and its escapement mechanism is switched off. When $k_x > k_{th}$ the energy transfer allows the oscillations with a sufficiently large amplitude to switch on the escapement mechanism and after a transient time the perturbed pendulum is synchronized with the rest of the pendula. Generally, the coupling threshold, k_{th} depends both on the number of pendula being perturbed in the network and on the distribution of the perturbed units on the ring.

The post-perturbation behavior of system (1) is illustrated in figures 2–4. These figures present snapshots taken at the moment when φ_1 is maximum. Insets in figures 2 and 3 present the Poincare maps showing the positions of the selected pendula at the time when pendulum 1 reaches the maximum positive value of

Figure 2. Phase evolutions of system (1). The vertical axis gives the pendula positions at the time when pendulum 1 reaches the maximum positive value of displacement. On the horizontal axis we have the pendula's numbers. The blue (light grey) intervals show the transient evolution and the navy blue (dark grey) points or intervals give the pendula positions after the decay of transients. The insets present the Poincaré maps showing the positions of the selected pendula at the time when pendulum 1 reaches the maximum positive value of displacement, i.e., $\varphi_1 = \max_t \varphi_1(t)$, $\dot{\varphi}_1 = 0$. **(a)** Pendulum 7 has been stopped, the inset presents the Poincaré map of pendulum 7, **(b)** pendula 7, 8, 17, 18 have been stopped, the inset presents the Poincaré map of pendulum 18, **(c)** pendula 16–20 have been stopped, the inset presents the Poincaré map of pendula 16 (black dots), 17 (red dots) and 18 (green dot).

displacement, i.e., $\varphi_1 = \max_t \varphi_1(t)$, $\dot{\varphi} = 0$. The frequency plots (right panels of figures 3 and 4) show the mean frequencies of each metronome $\bar{\omega}_i$ normalized by the frequency of the uncoupled metronome.

In all cases of figures 2a–c the perturbed pendula oscillate with small amplitudes (their escapement mechanisms are switched off). The oscillations of the perturbed pendula are correlated with the oscillations of the unperturbed pendula. In the case of periodic oscillations (figure 2a and b) the period of the perturbed pendula is equal to nT , where $n = 1, 2, \dots$, and T is the period of unperturbed pendula. Respectively, the perturbed pendula oscillate with periods 9 (figure 2a) and 2 (figure 2b) times larger than the unperturbed pendula as can be seen in the Poincaré maps shown in the insets. In the third case (figure 2c) the perturbed pendula 16–20 oscillate with small amplitude but the averaged frequencies of the perturbed and the unperturbed pendula calculated from eq. (2) are equal. Pendula 16 and 20 oscillate chaotically with a period close to $4T$, pendula 17 and 19 in the same way but with period close to $9T$, and pendulum 18 periodically with period T (see Poincaré map in the inset).

When more pendula are perturbed and two groups of excited pendula are separated by a number of perturbed (unexcited) pendula one can observe chimera-like states in which some pendula are uncorrelated with the rest, i.e., oscillate with different incommensurate frequencies. Examples of such chimera-like states are shown in figure 3a and b. In the case of figure 3a pendula 3 and 15–18 have been stopped. As the result of this perturbation their escapement mechanisms have been switched off and they have started to oscillate chaotically with small amplitudes. Two groups of excited pendula: (1, 2, 19, 20) and (4–14) oscillate with slightly different frequencies and their phases diverge from each other as can be seen in the Poincaré maps (middle part of figure 3a) and frequency plot (lower part of figure 3a). Perturbed (unexcited) pendula oscillate with different frequencies, but frequencies of pendula 15, 16 and 17, 18 are respectively very close to frequencies of the two group (4–14) and (1, 2, 19, 20). Larger differences in frequencies are observed in the case illustrated in figure 3b. After the perturbation of pendula 3, 13, 14, 17 and 18, one observes the creation of two clusters of excited pendula which consist respectively of pendula (1, 2, 15, 16, 19, 20) and (4–14). The pendula in both clusters are phase synchronized but oscillate with different incommensurate frequencies as can be seen in the frequency plot (lower part of figure 3b). The unexcited pendula oscillate with small amplitudes and frequencies different from the frequencies of both clusters.

Figure 3. Phase evolutions of system (1). Left panel: The vertical axis gives the pendula positions in time when pendulum 1 reaches the maximum positive value of displacement. On the horizontal axis we have the pendula's numbers. The blue (light grey) intervals show the transient evolution and the navy blue (dark grey) points or intervals give the pendula positions after the decay of transients. The insets present the Poincaré maps showing the positions of the selected pendula at the time when pendulum 1 reaches the maximum positive value of displacement, i.e., $\phi_1 = \max_t \phi_1(t)$, $\dot{\phi} = 0$ and the representative snapshot. Right panels: Mean frequencies of metronomes $\bar{\omega}_i$ normalized by the frequency of uncoupled metronome ω (yellow triangles and black dots indicate respectively numerical and experimental results). Experimentally observed metronomes have been chosen in such a way to capture the main features of the described phenomena. **(a)** Pendula 3 and 15–18 have been stopped, and the Poincaré map shows the trajectories of pendula 10 (red dots), 17 (black dots) and 1 (green dot) **(b)** pendula 3, 13, 14, 17 and 18 have been stopped, and the Poincaré map shows the trajectories of pendula 10 (green dots), 16 (black dots) 17 (pink dots) and 1 (red dot).

The mechanism of the creation of the chimera-like states is described in figure 4a–c. In the case shown in figure 4a pendula 10 and 12 have been stopped. The amplitude of the unperturbed pendulum 11 decreases. This pendulum transfers the energy to the neighboring unexcited pendula to keep them oscillating with a small amplitude. Larger amounts of energy are dissipated in the dampers connecting pendulum 11 with pendula 10 and 12 (this energy is proportional to the pendula's velocity differences). The amount of the dissipative energy is not sufficient to change its

frequency and all excited pendula oscillate with the same frequency different from the frequency of unexcited ones. The signal transferred from the unperturbed pendula 9 and 13 (through the coupling to the second nearest neighbors) allows the preservation of synchronization [44]. In the case of figure 4b and c the number of neighboring unexcited pendula is larger (respectively 4 and 6). Pendulum 11 loses more energy and has not received any signal from the unperturbed pendula. As a result it oscillates with a different frequency and synchronization is not preserved.

Our numerical results have been partially confirmed by the experiments in which we have been able to estimate the frequencies of the selected metronomes (metronomes 9 and 18 in figure 3a, metronomes 7 and 15 in figure 3b and metronomes 11 and 17 in figure 4a–c). Experimentally observed metronomes have been chosen in such a way so as to capture the main features of the described phenomena. In all cases the experimental results confirm frequency differences of the selected metronomes and the creation of chimera-like states.

4. Conclusions

In summary, we have constructed the simple experimental setup to show the emergence of chimera-like and solitary states after a large perturbation of synchronous state in the network of the coupled pendula. The nodes in the network are nonlocally coupled pendula (Huygens' clocks realized by metronomes). We observe the formation of the coexisting coherent group (or groups) of metronomes (different groups can oscillate with different incommensurate frequencies) and incoherent single oscillators. This behavior is observed experimentally and confirmed in numerical simulations. It seems that chimera-like and solitary states are common in networks of coupled oscillators sensitive to large perturbations.

Acknowledgements

We thank B. Jagiello for technical assistance in the experiments. This work has been supported by the Polish National Science Centre, MAESTRO Programme - Project No 2013/08/A/ST8/00/780.

References

- [1] L M Pecora and T L Carroll, *Phys. Rev. Lett.* **64**(8), 821 (1990)
- [2] A S Pikovsky, M G Rosenblum and J Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences*, (Cambridge University Press, Cambridge, 2002)
- [3] S Boccaletti, J Kurths, G Osipov, D L Valladares and C S Zhou, *Phys. Rep.* **366**, 1 (2002)
- [4] L Kocarev and U Parlitz, *Phys. Rev. Lett.* **76**(11), 1816 (1996)
- [5] B F Kuntsevich and A N Pisarchik, *Phys. Rev. E* **64** 046221. 12 (2001)
- [6] M L Rabinovich, P Varona, A L Selverston and H D Abarbanel, *Rev. Mod. Phys.* **78**, 1213 (2006)
- [7] B Blasius, A Huppert and L Stone, *Nature* **399**, 354 (1999)
- [8] D J Watts and S H Strogatz, *Nature* **393**, 440 (1998)
- [9] M Panaggio and D Abrams, *Nonlinearity* **28**, 67 (2015)
- [10] Y Kuramoto and D Battogtokh, *Nonlinear Phen. Complex Syst.* **5**, 380 (2002)
- [11] D M Abrams and S H Strogatz, *Phys. Rev. Lett.* **93**, 174102 (2004)
- [12] D M Abrams, R Mirollo, S H Strogatz and D A Wiley, *Phys. Rev. Lett.* **101**, 084103 (2008)
- [13] A E Motter, *Nat. Phys.* **6**, 164 (2010)
- [14] L Omelchenko, Y Maistrenko, P Hövel and E Schöll, *Phys. Rev. Lett.* **106**, 234102 (2011)
- [15] L Omelchenko, B Riemenschneider, P Hövel, Y Maistrenko and E Schöll, *Phys. Rev. E* **85**, 026212 (2012)
- [16] C R Laing, *Physica D* **238**, 1569 (2009)
- [17] E A Martens, *Phys. Rev. E* **82**, 016216 (2010)
- [18] E A Martens, *Chaos* **20**, 043122 (2010)
- [19] M Wolfrum and O E Omel'chenko, *Phys. Rev. E* **84**, 015201 (2011)
- [20] G S Sethia, A Sen and F M Atay, *Phys. Rev. Lett.* **100**, 144102 (2008)
- [21] I Waller and R Kapral, *Phys. Rev. A* **30**, 2047 (1984)
- [22] A Zakharova, M Kapeller and E Scholl, *Phys. Rev. Lett.* **112**, 154101 (2014)
- [23] P Jaros, Y Maistrenko and T Kapitaniak, *Phys. Rev. E* **91**, 022907 (2015)
- [24] D Dudkowski, Y Maistrenko and T Kapitaniak, *Phys. Rev. E* **90**, 032920 (2014)
- [25] A M Hagerstrom, T E Murphy, R Roy, P Hövel, I Omelchenko and E Scholl, *Nat. Phys.* **8**, 658 (2012)
- [26] M R Tinsley, S Nkomo and K Showalter, *Nat. Phys.* **8**, 662 (2012)
- [27] E A Martens, S Thutupalli, A Fourriere and O Hallatschek, *Proc. Nat. Acad. Sci.* **110**, 10563 (2013)
- [28] L Larger, B Penkovsky and Y Maistrenko, *Phys. Rev. Lett.* **111**, 054103 (2013)
- [29] L Larger, B Penkovsky and Y Maistrenko, *Nat. Commun.* **6**, 7752 (2015)
- [30] T Kapitaniak, P Kuzma, J Wojewoda, K Czolczynski and Y Maistrenko, *Sci. Rep.* **4**, 6379 (2014)
- [31] S Olmi, E M Martens, S Thutupalli and A Torcini, *Phys. Rev. E* **92**, 03090(R) (2015)
- [32] D Dudkowski, Y Maistrenko and T Kapitaniak, *Chaos* **26**, 116306 (2016)
- [33] V Semenov, A Zakharova, Y Maistrenko and E Scholl, *Europhys. Lett.* **115**, 10005 (2016)
- [34] K Premalatha, V K Chandrasekar, M Senthilvelan and M Lakshmanan, *Phys. Rev. E* **94**, 012311 (2016)
- [35] Y Maistrenko, B Penkovsky and M Rosenblum, *Phys. Rev. E* **89**, 060901(R) (2014)
- [36] P Ashwin and O Burylko, *Chaos* **25**, 013106 (2015)
- [37] M J Panaggio, D M Abrams, P Ashwin and C Laing, *Phys. Rev. E* **93**, 012218 (2016)

- [38] Ch Bick and P Ashwin, *Nonlinearity* **29**, 1468 (2016)
- [39] Y Maistrenko, S Brezetsky, P Jaros, R Levchenko and T Kapitaniak, *Phys. Rev. E* **95**, 010203(R) (2017)
- [40] J Wojewoda, K Czołczynski, Y Maistrenko and T Kapitaniak, *Sci. Rep.* **6**, 34329 (2016)
- [41] D Dudkowski, J Grabski, J Wojewoda, P Perlikowski, Y Maistrenko and T Kapitaniak, *Sci. Rep.* **6**, 29833 (2016)
- [42] J D Hart, K Bansal, T E Murphy and R Roy, *Chaos* **26**, 094801 (2016)
- [43] F Hellmann, P Schultz, C Grabow, J Heitzig and J Kurths, *Sci. Rep.* **6**, 29654 (2016)
- [44] M Kapitaniak, M Lazarek, M Nielaczny, K Czołczynski, P Perlikowski and T Kapitaniak, *Sci. Rep.* **4**, 4391 (2014)
- [45] C Huygens [Letter to de Sluse]. Oeuvres Complètes de Christian Huygens (letters; no. 133 of 24 February 1665, no. 1335 of 26 February 1665, no. 1345 of 6 March 1665) (Société Hollandaise Des Sciences, Martinus Nijhoff, La Haye, 1665)
- [46] M Kapitaniak, K Czołczynski, P Perlikowski, A Stefanski and T Kapitaniak, *Phys. Rep.* **517**, 1 (2012)
- [47] K Czołczynski, P Perlikowski, A Stefanski and T Kapitaniak, *Physica A* **388**, 5013 (2009)
- [48] M Kapitaniak, P Brzeski, K Czołczynski, P Perlikowski, A Stefanski and T Kapitaniak, *Prog. Theo. Phys.* **128**, 1141 (2012)