Indian Academy of Sciences Conference Series (2017) 1:1
DOI: 10.29195/iascs.01.01.0005

Grad-type fourteen-moment theory for dilute granular gases

VINAY KUMAR GUPTA[1] and PRIYANKA SHUKLA[2,*]

[1]SRM Research Institute and Department of Mathematics, SRM University, Chennai 603 203, India
[2]Department of Mathematics, Indian Institute of Technology Madras, Chennai 600 036, India
[*]Corresponding author. E-mail: priyanka@iitm.ac.in

Abstract. Grad's moment method is applied to derive the system of fully nonlinear 14-moment equations for dilute granular gases. The derived system of 14-moment equations is solved analytically in a quasilinear setting to obtain the homogeneous cooling state solution, and it is shown that the nonlinear terms of scalar fourth moment have practically no effect on Haff's law. The linear stability of the homogeneous cooling state is analyzed through the 14-moment system by decomposing it into two independent longitudinal and transverse problems. The eigenmodes for the longitudinal system are compared with those obtained by an existing theory by Kremer and Marques [12].

Keywords. Kinetic theory; moment method; granular gases; homogeneous cooling state; linear stability.

PACS Nos 47.57.Gc; 51.10.+y; 47.45.Ab

1. Introduction

The most important characteristic of a granular matter is the *inelastic* collisions among its grains. This very feature of the granular matter is accountable for many interesting phenomena—for instance, pattern formation, clustering, fingering, mixing and segregation, jamming, density waves, etc. (see, e.g., [1–3] and references therein). Owing to the inelastic collisions among grains, a granular matter loses energy during each collision; consequently its *granular temperature*, which is a measure of the fluctuating kinetic energy, decays continuously until it becomes zero. Therefore, in contrast to elastic gases in which collisions render the gas to reach an equilibrium state, the inelastic collisions in *granular gases* [1] lead only to a dead state.

The analogy between granular and molecular gases has inspired several researchers to develop mathematical models for granular gases through the kinetic theory within the framework of the Boltzmann equation (see, e.g., [4–12] and references therein). In general, most of the works based on the kinetic theory of granular gases extend (or employ) the two well-known approximation techniques, namely the Chapman–Enskog (CE) expansion [13] and Grad's moment method [14], of the kinetic theory for monatomic (elastic) gases.

The present work revisits a recent work based on Grad's moment method: "Fourteen-moment theory for granular gases" by Kremer and Marques [12] and develops in an analogous way. Nonetheless, the main differences in the present work and Ref. [12] are the following.

1. In contrast to Ref. [12], the present work derives *fully* nonlinear Grad 14-moment (G14) equations for dilute granular gases.
2. Through the stability analysis of equilibrium points in a dynamical system [15], the present work deduces that it is justifiable to ignore the nonlinear terms of the scalar fourth order moment in the moment equations.
3. Unlike Ref. [12], which obtains the Navier–Stokes, and Fourier (NSF) laws for dilute granular gases by Maxwell iteration procedure, the present work determines them by performing a CE-like expansion on the moment equations.
4. Ref. [12] analyzes the linear stability of the homogeneous cooling state (HCS), essentially, via the 13-moment theory, since it considers the scalar fourth order moment as a constant. The present work studies the same problem with 14-moment theory and presents an interesting result on the unstable eigenmode of the longitudinal system.

5. In Ref. [12], it is not clear how the authors obtain the dimensionless governing equations having constant coefficients for perturbed quantities (see eqs (59–63) of Ref. [12]). The present work answers this question as well.

The rest of the article is organized as follows. A brief review of the kinetic theory for dilute granular gases is presented in section 2. The G14 equations for them are derived in section 3. The HCS of a freely cooling granular gas is investigated in section 4. The NSF laws for dilute granular gases are determined in section 5. The linear stability of HCS is analyzed in section 6. The conclusions are given in section 7.

2. Short review of kinetic theory

The state of a dilute granular gas can be described by a single-particle velocity distribution function $f(t,\boldsymbol{x},\boldsymbol{c}) \equiv f$, which is the fundamental quantity in the kinetic theory and obeys the (inelastic) Boltzmann equation [6, 8, 16]:

$$\frac{\partial f}{\partial t} + c_i \frac{\partial f}{\partial x_i} + F_i \frac{\partial f}{\partial c_i} = d^2 \int_{\mathbb{R}^3} \int_{S^2} \left(\frac{1}{e^2} f'' f_1'' - f f_1 \right) (\hat{\boldsymbol{k}} \cdot \boldsymbol{g})\, \Theta(\hat{\boldsymbol{k}} \cdot \boldsymbol{g})\, \mathrm{d}\hat{\boldsymbol{k}}\, \mathrm{d}\boldsymbol{c}_1. \quad (1)$$

Here $t, \boldsymbol{x}, \boldsymbol{c}$ denote the time, position vector and the instantaneous velocity of a particle; d is the diameter of a particle; e is the coefficient of (normal) restitution; $\hat{\boldsymbol{k}}$ is the unit vector directed from the center of one particle to that of other at the time of collision, consequently the integration limits of $\hat{\boldsymbol{k}}$ extend over the unit sphere S^2; the integration limits for the velocity (of the other collision partner) $\boldsymbol{c}_1$ extend over the whole $\mathbb{R}^3$ space; $\boldsymbol{g} = \boldsymbol{c} - \boldsymbol{c}_1$ is the relative velocity; Θ is the Heaviside step function; $\boldsymbol{F}$ is the external force per unit mass and usually do not depend on $\boldsymbol{c}$; $f_1 \equiv f(t,\boldsymbol{x},\boldsymbol{c}_1)$, $f'' \equiv f(t,\boldsymbol{x},\boldsymbol{c}'')$, $f_1'' \equiv f(t,\boldsymbol{x},\boldsymbol{c}_1'')$; subscript i denotes the index notation for vectors, and Einstein summation convention is assumed over repeated indices. The velocities $\boldsymbol{c}''$ and $\boldsymbol{c}_1''$ are the pre-collisional velocities in an inverse collision [16, 17]. In what follows, we shall mostly use the index notation for vectors and tensors and, henceforth, write a single integration symbol without limits (even for multiple integrals) to make the notations compact. Nevertheless, the integration over any velocity space will always stand for the volume integral over $\mathbb{R}^3$ and that over $\hat{\boldsymbol{k}}$ will always stand for the volume integral over the unit sphere S^2. The right-hand side (RHS) of the Boltzmann equation (1) is referred to as the (inelastic) *Boltzmann collision operator*.

Some of the physical quantities such as (mass) density $\rho(t,\boldsymbol{x}) \equiv \rho$, macroscopic velocity $v_i(t,\boldsymbol{x}) \equiv v_i$, *granular* temperature $T(t,\boldsymbol{x}) \equiv T$, stress $\sigma_{ij}(t,\boldsymbol{x}) \equiv \sigma_{ij}$ and heat flux $q_i(t,\boldsymbol{x}) \equiv q_i$, are related to the velocity distribution function in the form of its moments as follows.

$$\rho = m \int f \,\mathrm{d}\boldsymbol{c}, \quad (2)$$

$$\rho\, v_i = m \int c_i f \,\mathrm{d}\boldsymbol{c}, \quad (3)$$

$$\frac{3}{2}\rho\,\theta = \frac{3}{2} n\, T = \frac{1}{2} m \int C^2 f \,\mathrm{d}\boldsymbol{c}, \quad (4)$$

$$\sigma_{ij} = m \int C_{\langle i} C_{j\rangle} f \,\mathrm{d}\boldsymbol{c}, \quad (5)$$

$$q_i = \frac{1}{2} m \int C^2 C_i f \,\mathrm{d}\boldsymbol{c}, \quad (6)$$

where m is the mass of a particle; $n \equiv n(t,\boldsymbol{x}) = \rho/m$ is the number density; $C_i = c_i - v_i$ is the peculiar velocity; $\theta \equiv \theta(t,\boldsymbol{x}) = T/m$ is the granular temperature in energy units; and the angle brackets around the indices denote the symmetric and traceless part of the tensor [18]. Here, the definition of the granular temperature is adopted following Ref. [16, 19], although some authors also refer $\theta = T/m$ as the granular temperature (defined as fluctuating kinetic energy per unit mass) (see, e.g., [20, 21]). It is noteworthy that in case of monatomic ideal gases, T in eq. (4) is replaced with $k_B T_{\mathrm{th}}$, where k_B denotes the Boltzmann constant and T_{th} is the thermodynamic temperature.

Since the governing equation for f is the Boltzmann equation (1), the governing equation for any physical quantity defined above can be obtained by multiplying the Boltzmann equation (1) with a suitable monomial of velocity components and integrating the resulting equation over the velocity space subsequently. To derive the governing equations for physical quantities defined above, let us multiply the Boltzmann equation (1) with a generic variable $\psi \equiv \psi(t,\boldsymbol{x},\boldsymbol{c})$, which we shall specify later. Subsequent integration of the resulting equation over the velocity space leads to the so-called *transfer equation* for ψ:

$$\frac{\mathrm{D}}{\mathrm{D}t}\int \psi f \,\mathrm{d}\boldsymbol{c} + \underline{\frac{\partial}{\partial x_i}\int \psi\, C_i f \,\mathrm{d}\boldsymbol{c}} + \frac{\partial v_i}{\partial x_i}\int \psi f \,\mathrm{d}\boldsymbol{c} - \int \left(\frac{\partial \psi}{\partial t} + c_i \frac{\partial \psi}{\partial x_i} + F_i \frac{\partial \psi}{\partial c_i} \right) f \,\mathrm{d}\boldsymbol{c} = \mathcal{P}(\psi), \quad (7)$$

where $\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i}$ is the material derivative, the underline denotes the flux term, and

$$\mathcal{P}(\psi) = \frac{d^2}{2}\int\left(\psi' + \psi_1' - \psi - \psi_1\right) f f_1 \times (\hat{\boldsymbol{k}}\cdot\boldsymbol{g})\,\Theta(\hat{\boldsymbol{k}}\cdot\boldsymbol{g})\,\mathrm{d}\hat{\boldsymbol{k}}\,\mathrm{d}\boldsymbol{c}\,\mathrm{d}\boldsymbol{c}_1 = d^2\int\left(\psi' - \psi\right) f f_1 (\hat{\boldsymbol{k}}\cdot\boldsymbol{g})\,\Theta(\hat{\boldsymbol{k}}\cdot\boldsymbol{g})\,\mathrm{d}\hat{\boldsymbol{k}}\,\mathrm{d}\boldsymbol{c}\,\mathrm{d}\boldsymbol{c}_1, \quad (8)$$

with $\psi' \equiv \psi(t, \boldsymbol{x}, \boldsymbol{c}')$ etc., is the *production term* or the (inelastic) *Boltzmann collision integral* corresponding to the moment $\int \psi f\,\mathrm{d}\boldsymbol{c}$. While writing eqs (8), the symmetry properties of the Boltzmann collision operator have been used (see, e.g., [16, 17]). It is important to note from the transfer equation (7) that irrespective of the value of ψ chosen, the transfer equation (7) will always contain an additional unknown of one more order due to the (underlined) flux term in the transfer equation (7). Inserting $\psi = m, m\,c_i, \frac{1}{2}m\,C^2$ in the transfer equation (7) one by one, it yields the mass, momentum and energy balance equations, which—respectively—read

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\frac{\partial v_i}{\partial x_i} = 0, \quad (9)$$

$$\rho\frac{\mathrm{D}v_i}{\mathrm{D}t} + \frac{\partial \sigma_{ij}}{\partial x_j} + \frac{\partial(n\,T)}{\partial x_i} - \rho\,F_i = 0, \quad (10)$$

$$\frac{3}{2}n\frac{\mathrm{D}T}{\mathrm{D}t} + \frac{\partial q_i}{\partial x_i} + \sigma_{ij}\frac{\partial v_i}{\partial x_j} + n\,T\frac{\partial v_i}{\partial x_i} = -\mathscr{D}, \quad (11)$$

where

$$\mathscr{D} = -\frac{m\,d^2}{4}\int\left[(C')^2 + (C_1')^2 - C^2 - C_1^2\right] f f_1 \times (\hat{\boldsymbol{k}}\cdot\boldsymbol{g})\,\Theta(\hat{\boldsymbol{k}}\cdot\boldsymbol{g})\,\mathrm{d}\hat{\boldsymbol{k}}\,\mathrm{d}\boldsymbol{c}\,\mathrm{d}\boldsymbol{c}_1 \quad (12)$$

is the dissipation term resulting from the energy loss due to inelastic collisions and vanishes identically for elastic collisions. While deriving the momentum and energy balance equations (10) and (11), the mass balance equation (9) has also been utilized. Note that the RHSs of eqs (9) and (10) vanish due to the conservation of mass and momentum; however, the energy is not conserved due to dissipative collisions resulting into nonzero RHS in eq. (11).

Note that the stress and heat flux in the momentum and energy balance equations (10) and (11) typically have nonzero collisional contributions in addition to the usual kinetic contribution given by eqs (5) and (6) [17, 19, 22, 23]. Nevertheless, the kinetic contributions to the stress and heat flux in the case of dilute granular gases dominate over their respective collisional contributions, and therefore the collisional contributions to the stress and heat flux can be neglected for dilute granular gases [16]. In any case, since the momentum and energy balance equations (10) and (11) are obtained here by taking the velocity moments of the distribution function, only kinetic contributions to the stress and heat flux will emerge in the equations. The collisional contributions to stress and heat flux, if required, must be computed separately [19].

Clearly, the system of eqs (9–11) in hydrodynamic variables ρ, v_i and T is not closed, since it contains the additional unknowns σ_{ij}, q_i and $\mathscr{D}$, and in order to deal with this system further, it needs to be closed. Here, we employ Grad's moment method [14] in order to obtain a closed system of equations.

2.1 *Grad's moment method*

The function ψ in the transfer equation (7) can be chosen in infinitely many ways. Consequently, the transfer equation (7) leads to an infinite system of coupled first order partial differential equations (PDEs), which is of no use unless truncated after a finite number of equations. Obviously, the truncated system of equation will contain additional unknowns due to the (underlined) flux term in the transfer equation (7), hence will not be closed. To obtain a finite and closed system of moment equations, Grad's [14] idea was to approximate the velocity distribution function f with a finite linear combination of the N-dimensional Hermite polynomials [24] in peculiar velocity and to compute the unknown coefficients in the approximation in terms of the considered moments by satisfying their definitions with the approximated distribution function. Finally, the additional unknowns in the truncated system are computed using this distribution function. This method of obtaining a closed set of moment equations is referred to as *Grad's moment method* and its details can be found in [14] and in many standard textbooks (see, e.g., [18]). At the truncation level of first 13 moments (in 3-dimensions), the method yields the well-known Grad 13-moment (G13) equations, which form a closed system of equations for the physical quantities ρ, v_i, T, σ_{ij} and q_i.

Here, we shall derive the system of Grad 14-moment (G14) equations for a dilute granular gas, which in addition to the physical quantities consists of a fully contracted fourth rank tensor

$$w = m\int C^4 f\,\mathrm{d}\boldsymbol{c} \quad (13)$$

as a variable. It is imperative to include this variable into the system in case of granular gases, as it has been concluded in Ref. [12] and we shall also show in section 5. that the non-Fourier contribution to the heat flux (the term not proportional to the temperature gradient in heat flux) is a consequence of the inclusion of this variable.

3. Grad's 14-moment equations

The system of 14-moment equations—in the variables ρ, v_i, T, σ_{ij}, q_i and w—for a dilute granular gas are obtained by replacing ψ in the transfer equation (7) with m, $m\,c_i$, $\frac{1}{2}m\,C^2$, $m\,C_{\langle i}C_{j\rangle}$, $\frac{1}{2}m\,C^2C_i$, $m\,C^4$ successively in a straightforward way. The first three choices for ψ yield the mass, momentum and energy balance equations (9–11). The remaining choices for ψ yields the governing equations for σ_{ij}, q_i and w. Following Kremer and Marques [12], we replace variable w in the 14-moment equations with a new variable

$$\Delta = \frac{1}{15\,\rho\,\theta^2} m \int C^4\big(f - f_M\big)\,\mathrm{d}\boldsymbol{c} = \frac{w}{15\,\rho\,\theta^2} - 1, \qquad (14)$$

which denotes the dimensionless non-equilibrium part of w. Here,

$$f_M \equiv f_M(t,\boldsymbol{x},\boldsymbol{c}) = n\left(\frac{1}{2\,\pi\,\theta}\right)^{3/2}\exp\left(-\frac{C^2}{2\,\theta}\right) \qquad (15)$$

is the equilibrium (Maxwellian) distribution function in the elastic limit. The remaining choices for ψ and the above replacement gives the governing equations for σ_{ij}, q_i and Δ, which read

$$\frac{\mathrm{D}\sigma_{ij}}{\mathrm{D}t} + \frac{\partial u^0_{ijk}}{\partial x_k} + \frac{4}{5}\frac{\partial q_{\langle i}}{\partial x_{j\rangle}} + \sigma_{ij}\frac{\partial v_k}{\partial x_k} + 2\sigma_{k\langle i}\frac{\partial v_{j\rangle}}{\partial x_k} + 2\rho\theta\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} = \mathcal{P}^0_{ij}, \qquad (16)$$

$$\begin{aligned}\frac{\mathrm{D}q_i}{\mathrm{D}t} &+ \frac{1}{2}\frac{\partial u^1_{ij}}{\partial x_j} + \frac{5}{2}\theta\left[\theta\left(\rho\frac{\partial\Delta}{\partial x_i} + \Delta\frac{\partial\rho}{\partial x_i}\right) + \rho\left(1+2\Delta\right)\frac{\partial\theta}{\partial x_i}\right] - \frac{5}{2}\theta\frac{\partial\sigma_{ij}}{\partial x_j} - \sigma_{ij}\frac{\partial\theta}{\partial x_j} \\ &- \sigma_{ij}\frac{1}{\rho}\left(\frac{\partial\sigma_{jk}}{\partial x_k} + \theta\frac{\partial\rho}{\partial x_j}\right) + u^0_{ijk}\frac{\partial v_j}{\partial x_k} + \frac{7}{5}q_i\frac{\partial v_j}{\partial x_j} + \frac{7}{5}q_j\frac{\partial v_i}{\partial x_j} + \frac{2}{5}q_j\frac{\partial v_j}{\partial x_i} = \frac{1}{2}\mathcal{P}^1_i,\end{aligned} \qquad (17)$$

$$\begin{aligned}15\rho\theta^2\frac{\mathrm{D}\Delta}{\mathrm{D}t} &+ \frac{\partial u^2_i}{\partial x_i} - 20\,\theta\left(1+\Delta\right)\left(\frac{\partial q_i}{\partial x_i} + \sigma_{ij}\frac{\partial v_i}{\partial x_j}\right) + 4u^1_{ij}\frac{\partial v_i}{\partial x_j} - 8\frac{q_i}{\rho}\left[\frac{\partial\sigma_{ij}}{\partial x_j} + \frac{\partial(\rho\theta)}{\partial x_i}\right] \\ &= \mathcal{P}^2 - 20\,\theta\left(1+\Delta\right)\left(\frac{1}{2}\mathcal{P}^1\right).\end{aligned} \qquad (18)$$

Eqs (9–11) and (16–18) form a system of 14-moment equations. Nevertheless, the system is not yet closed since it contains the additional unknowns

$$\left.\begin{aligned} u^0_{ijk} &= m\int C_{\langle i}C_jC_{k\rangle}f\,\mathrm{d}\boldsymbol{c}, \\ u^1_{ij} &= m\int C^2C_{\langle i}C_{j\rangle}f\,\mathrm{d}\boldsymbol{c}, \\ u^2_i &= m\int C^4C_i f\,\mathrm{d}\boldsymbol{c}, \end{aligned}\right\} \qquad (19)$$

as well as the unknown production terms $\frac{1}{2}\mathcal{P}^1$, $\mathcal{P}_{ij}$, $\frac{1}{2}\mathcal{P}^1_i$ and $\mathcal{P}^2$, which have the general form:

$$\mathcal{P}^a_{i_1\ldots i_n} = m\,d^2\int\left[(C')^{2a}C'_{\langle i_1}\ldots C'_{i_n\rangle} - C^{2a}C_{\langle i_1}\ldots C_{i_n\rangle}\right] \times ff_1(\hat{\boldsymbol{k}}\cdot\boldsymbol{g})\,\Theta(\hat{\boldsymbol{k}}\cdot\boldsymbol{g})\,\mathrm{d}\hat{\boldsymbol{k}}\,\mathrm{d}\boldsymbol{c}\,\mathrm{d}\boldsymbol{c}_1. \qquad (20)$$

Notice that $\mathcal{P}^0 = \mathcal{P}^0_i = 0$ due to mass and momentum conservation, and $\frac{1}{2}\mathcal{P}^1 = -\mathscr{D}$.

All the unknowns are expressed in terms of the considered variables by replacing f in eqs (19) and (20) with G14 distribution function which reads

$$f_{|\mathrm{G14}} = f_M\left[1 + \frac{1}{2}\frac{\sigma_{ij}}{\rho\,\theta^2}C_iC_j + \frac{1}{5}\frac{q_i}{\rho\,\theta^2}C_i\left(\frac{C^2}{\theta} - 5\right) + \Delta\left(\frac{15}{8} - \frac{5}{4}\frac{C^2}{\theta} + \frac{1}{8}\frac{C^4}{\theta^2}\right)\right]. \qquad (21)$$

We omit the details of finding the G14 distribution function for the sake of conciseness. Interested readers are referred to Ref. [12] and standard textbooks (e.g. [18, 25]), for the details of computing Grad distribution function. With the G14 distribution function (21), the additional unknowns become

$$u^0_{ijk} = 0, \quad u^1_{ij} = 7\,\theta\,\sigma_{ij}, \quad u^2_i = 28\,\theta\,q_i \qquad (22)$$

and the production terms also become known, although they are not easy to evaluate by hand. We have implemented the computational strategy for the evaluation of the production terms detailed in [26] into the computer algebra software MATHEMATICA® and obtained the fully nonlinear production terms. Inserting the variables (22) and computed production terms into eqs (9–11) and (16–18), one obtains the system of fully nonlinear G14 equations for dilute granular gases, which reads

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\frac{\partial v_i}{\partial x_i} = 0, \qquad (23)$$

$$\frac{\mathrm{D}v_i}{\mathrm{D}t} + \frac{1}{\rho}\left(\frac{\partial \sigma_{ij}}{\partial x_j} + T\frac{\partial n}{\partial x_i}\right) + \frac{1}{m}\frac{\partial T}{\partial x_i} = F_i, \tag{24}$$

$$\frac{\mathrm{D}T}{\mathrm{D}t} + \frac{2}{3n}\left(\frac{\partial q_i}{\partial x_i} + \sigma_{ij}\frac{\partial v_i}{\partial x_j}\right) + \frac{2}{3}T\frac{\partial v_i}{\partial x_i} = -\frac{(1-e^2)}{3}\frac{1}{\tau}T \times\left(1 + \frac{3\Delta}{16} + \frac{9\Delta^2}{1024} + \frac{1}{40}\frac{\sigma_{ij}\sigma_{ij}}{\rho^2\theta^2} + \frac{1}{200}\frac{q_i q_i}{\rho^2\theta^3}\right), \tag{25}$$

$$\frac{\mathrm{D}\sigma_{ij}}{\mathrm{D}t} + \frac{4}{5}\frac{\partial q_{\langle i}}{\partial x_{j\rangle}} + \sigma_{ij}\frac{\partial v_k}{\partial x_k} + 2\sigma_{k\langle i}\frac{\partial v_{j\rangle}}{\partial x_k} + 2\rho\theta\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} = -\frac{(1+e)(3-e)}{5}\frac{1}{\tau}\left[\left(1-\frac{\Delta}{32}\right)\sigma_{ij} + \frac{1}{14}\frac{\sigma_{k\langle i}\sigma_{j\rangle k}}{\rho\,\theta} + \frac{1}{100}\frac{q_{\langle i}q_{j\rangle}}{\rho\,\theta^2}\right], \tag{26}$$

$$\frac{\mathrm{D}q_i}{\mathrm{D}t} + \frac{5}{2}\theta\left[\theta\left(\rho\frac{\partial \Delta}{\partial x_i} + \Delta\frac{\partial \rho}{\partial x_i}\right) + \rho\left(1+2\Delta\right)\frac{\partial \theta}{\partial x_i}\right] + \theta\frac{\partial \sigma_{ij}}{\partial x_j} + \frac{5}{2}\sigma_{ij}\frac{\partial \theta}{\partial x_j} - \sigma_{ij}\frac{1}{\rho}\left(\frac{\partial \sigma_{jk}}{\partial x_k} + \theta\frac{\partial \rho}{\partial x_j}\right) + \frac{7}{5}q_i\frac{\partial v_j}{\partial x_j} + \frac{7}{5}q_j\frac{\partial v_i}{\partial x_j} + \frac{2}{5}q_j\frac{\partial v_j}{\partial x_i} = -\frac{(1+e)}{60}\frac{1}{\tau} \times\left[\left\{(49-33e) + (19-3e)\frac{\Delta}{32}\right\}q_i + \frac{3(7+e)}{10}\frac{\sigma_{ij}q_j}{\rho\,\theta}\right], \tag{27}$$

$$\frac{\mathrm{D}\Delta}{\mathrm{D}t} + \frac{8}{15}\frac{1}{\rho\theta}\left(1-\frac{5}{2}\Delta\right)\left(\frac{\partial q_i}{\partial x_i} + \sigma_{ij}\frac{\partial v_i}{\partial x_j}\right) + \frac{4}{3}\frac{q_i}{\rho\theta^2}\frac{\partial \theta}{\partial x_j} - \frac{8}{15}\frac{q_i}{\rho^2\theta^2}\left(\frac{\partial \sigma_{ij}}{\partial x_j} + \theta\frac{\partial \rho}{\partial x_i}\right) = \frac{(1+e)}{15}\frac{1}{\tau} \times\left[(1-e)(1-2e^2) - (81-17e+30e^2-30e^3)\frac{\Delta}{16} + (1873-2001e+30e^2-30e^3)\frac{\Delta^2}{1024} + (1-e)\frac{45\,\Delta^3}{512} + \frac{1}{4}\left\{\frac{(23+9e+30e^2-30e^3)}{30} + (1-e)\Delta\right\}\frac{\sigma_{ij}\sigma_{ij}}{\rho^2\theta^2} - \frac{1}{20}\left\{\frac{(31+33e-30e^2+30e^3)}{30} - (1-e)\Delta\right\}\frac{q_i q_i}{\rho^2\theta^3}\right], \tag{28}$$

where

$$\tau = \frac{1}{4\sqrt{\pi}\,n\,d^2\sqrt{\theta}} \tag{29}$$

is the mean free time.

4. Homogeneous cooling state

The state of a granular gas when without applying any external force its granular temperature gradually decays (due to inelastic collisions) but its spatial homogeneity is preserved is termed as *homogeneous cooling state* [16]. The HCS of a freely cooling granular gas have been investigated theoretically via Navier–Stokes-level hydrodynamics [9, 16], via a 13-moment theory [12], and via molecular dynamic simulations [27]. It is also known from the Navier–Stokes-level hydrodynamics and molecular dynamic simulations that the HCS of a granular gas is not stable and leads to *inhomogeneous cooling state* (ICS) rendering clustering instabilities [28, 29]. The ICS of a granular gas is beyond the scope of the present article; nevertheless, the theory presented here can be utilized to investigate the ICS of a granular gas in the future.

Here, we study the HCS of a granular gas with the G14 equations. We assume that the granular gas is spatially homogeneous (i.e., $\frac{\partial(\cdot)}{\partial x_i} = 0$) and that there is no external force (i.e., $F_i = 0$). Furthermore, for simplicity, we drop the nonlinear terms which are products of vectors and/or tensors (i.e., the terms having $\sigma_{ij}\sigma_{ij}$, $\sigma_{ij}q_i$, $q_i q_j$ etc.) in the RHSs of the G14 equations (23–28). This means that we are focusing our attention on the early evolution stage of homogeneously cooling granular gas. The G14 equations in the zero external force case on dropping the nonlinear terms which are products of vectors and/or tensors read

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\frac{\partial v_i}{\partial x_i} = 0, \tag{30}$$

$$\frac{\mathrm{D}v_i}{\mathrm{D}t} + \frac{1}{\rho}\left(\frac{\partial \sigma_{ij}}{\partial x_j} + T\frac{\partial n}{\partial x_i}\right) + \frac{1}{m}\frac{\partial T}{\partial x_i} = 0, \tag{31}$$

$$\frac{\mathrm{D}T}{\mathrm{D}t} + \frac{2}{3n}\left(\frac{\partial q_i}{\partial x_i} + \sigma_{ij}\frac{\partial v_i}{\partial x_j}\right) + \frac{2}{3}T\frac{\partial v_i}{\partial x_i} = -\frac{(1-e^2)}{3}\frac{1}{\tau}T\left(1 + \frac{3\Delta}{16} + \frac{9\Delta^2}{1024}\right), \tag{32}$$

$$\frac{\mathrm{D}\sigma_{ij}}{\mathrm{D}t} + \frac{4}{5}\frac{\partial q_{\langle i}}{\partial x_{j\rangle}} + \sigma_{ij}\frac{\partial v_k}{\partial x_k} + 2\sigma_{k\langle i}\frac{\partial v_{j\rangle}}{\partial x_k} + 2\rho\theta\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} = -\frac{(1+e)(3-e)}{5}\frac{1}{\tau}\left(1-\frac{\Delta}{32}\right)\sigma_{ij}, \tag{33}$$

$$\frac{\mathrm{D}q_i}{\mathrm{D}t} + \frac{5}{2}\theta\left[\theta\left(\rho\frac{\partial\Delta}{\partial x_i} + \Delta\frac{\partial\rho}{\partial x_i}\right) + \rho\left(1+2\Delta\right)\frac{\partial\theta}{\partial x_i}\right] + \theta\frac{\partial\sigma_{ij}}{\partial x_j} + \frac{5}{2}\sigma_{ij}\frac{\partial\theta}{\partial x_j} - \sigma_{ij}\frac{1}{\rho}\left(\frac{\partial\sigma_{jk}}{\partial x_k} + \theta\frac{\partial\rho}{\partial x_j}\right) + \frac{7}{5}q_i\frac{\partial v_j}{\partial x_j} + \frac{7}{5}q_j\frac{\partial v_i}{\partial x_j} + \frac{2}{5}q_j\frac{\partial v_j}{\partial x_i} = -\frac{(1+e)}{60}\frac{1}{\tau}\left[(49-33e) + (19-3e)\frac{\Delta}{32}\right]q_i, \quad (34)$$

$$\frac{\mathrm{D}\Delta}{\mathrm{D}t} + \frac{8}{15}\frac{1}{\rho\theta}\left(1-\frac{5}{2}\Delta\right)\left(\frac{\partial q_i}{\partial x_i} + \sigma_{ij}\frac{\partial v_i}{\partial x_j}\right) + \frac{4}{3}\frac{q_i}{\rho\theta^2}\frac{\partial\theta}{\partial x_j} - \frac{8}{15}\frac{q_i}{\rho^2\theta^2}\left(\frac{\partial\sigma_{ij}}{\partial x_j} + \theta\frac{\partial\rho}{\partial x_i}\right) = \frac{(1+e)}{15}\frac{1}{\tau} \times\left[(1-e)(1-2e^2) - (81-17e+30e^2-30e^3)\frac{\Delta}{16} + (1873-2001e+30e^2-30e^3)\frac{\Delta^2}{1024} + (1-e)\frac{45\Delta^3}{512}\right]. \quad (35)$$

On introducing the scaling

$$\left.\begin{aligned} t_* &= \frac{t}{\tau_0}, \quad n_* = \frac{n}{n_0}, \quad v_i^* = \frac{v_i}{\sqrt{\theta_0}}, \quad T_* = \frac{T}{T_0}, \\ \sigma_{ij}^* &= \frac{\sigma_{ij}}{n_0 T_0}, \quad q_i^* = \frac{q_i}{n_0 T_0\sqrt{\theta_0}}, \quad \Delta_* = \frac{\Delta}{n_0 T_0 \theta_0}, \end{aligned}\right\} \quad (36)$$

with $\tau_0 = 1/(4\sqrt{\pi}\, n_0\, d^2\sqrt{\theta_0})$, $\theta_0 = T_0/m$; and $n_0 = n(0)$ and $T_0 = T(0)$ being the initial values of the number density and granular temperature, respectively. With the above scaling, the G14 equations in the HCS reduce to

$$\frac{\mathrm{d}n_*}{\mathrm{d}t_*} = 0, \quad (37)$$

$$\frac{\mathrm{d}v_i^*}{\mathrm{d}t_*} = 0, \quad (38)$$

$$\frac{\mathrm{d}T_*}{\mathrm{d}t_*} = -\frac{(1-e^2)}{3}\, n_* T_*^{3/2}\left(1 + \frac{3\Delta}{16} + \frac{9\Delta^2}{1024}\right), \quad (39)$$

$$\frac{\mathrm{d}\sigma_{ij}^*}{\mathrm{d}t_*} = -\frac{(1+e)(3-e)}{5} n_*\sqrt{T_*}\left(1 - \frac{\Delta}{32}\right)\sigma_{ij}^*, \quad (40)$$

$$\frac{\mathrm{d}q_i^*}{\mathrm{d}t_*} = -\frac{(1+e)}{60} n_*\sqrt{T_*}\left[(49-33e) + (19-3e)\frac{\Delta}{32}\right]q_i^*, \quad (41)$$

$$\frac{\mathrm{d}\Delta}{\mathrm{d}t_*} = \frac{(1+e)}{15} n_*\sqrt{T_*}\left[(1-e)(1-2e^2) - (81-17e+30e^2-30e^3)\frac{\Delta}{16} + (1873-2001e+30e^2-30e^3)\frac{\Delta^2}{1024} + (1-e)\frac{45\,\Delta^3}{512}\right]. \quad (42)$$

Eqs (37) and (38) lead to trivial solution for n^* and v_i^*:

$$n^*(t_*) = n_*(0) = 1 \quad \text{and} \quad v_i^*(t_*) = v_i^*(0) = 0, \quad (43)$$

since initial velocity is zero in the HCS.

4.1 *Haff's law*

By a heuristic approach, Haff [30] discovered that in the HCS, the decay rate of the granular temperature of a freely cooling granular gas is given by

$$\frac{\mathrm{d}T}{\mathrm{d}t} \propto -\mathring{n}\, d^2(1-e^2)T^{3/2}, \quad (44)$$

where $\mathring{n}$ is the average number density. The solution of eq. (44) gives the solution for the granular temperature of a freely cooling granular gas:

$$T(t) = \frac{T(0)}{(1+t/\tau_\circ)^2}, \quad (45)$$

where $\tau_\circ^{-1} \propto \mathring{n}\, d^2(1-e^2)\sqrt{T_\circ}$ is an inverse time scale, see, e.g., [16, 30]. Eq. (45) is termed as *Haff's law*.

A quick comparison of eq. (39) with eq. (44) reveals that Haff's law can be obtained from eq. (39), provided Δ is constant or

$$\frac{\mathrm{d}\Delta}{\mathrm{d}t_*} = 0. \quad (46)$$

With a constant value of Δ ($= \alpha$, let us say)—which is to be obtained from condition (46) and eq. (42)—eq. (39) with solution $(43)_1$ for n_* and initial condition $T_*(0) = 1$ yields Haff's law for evolution of the dimensionless granular temperature T_*:

$$T_*(t_*) = \frac{1}{\left[1 + \frac{(1-e^2)}{6}\left(1 + \frac{3}{16}\alpha + \frac{9\,\alpha^2}{1024}\right)t_*\right]^2}. \quad (47)$$

Notice that, by virtue of condition (46), the constant α in (47) is an *equilibrium point* of eq. (42) (see chapter 2 of the textbook [15])—or, in other words, α is the root of cubic equation

$$(1-e)(1-2e^2) - (81-17e+30e^2-30e^3)\frac{\Delta}{16} \\ \underline{+(1873-2001e+30e^2-30e^3)\frac{\Delta^2}{1024}} \\ \underline{+(1-e)\frac{45\,\Delta^3}{512}} = 0. \tag{48}$$

One interesting case arises if we drop the nonlinear (underlined) terms in eq. (48). In this case, there is only one root of eq. (48), and therefore in this case

$$\alpha = \frac{16\,(1-e)\,(1-2e^2)}{81-17e+30e^2-30e^3} = a_2, \tag{49}$$

which is same as the coefficient of second Sonine polynomial $S_2(v^2)$ obtained while performing CE expansion (see e.g., [16, 31]). Furthermore, if we also drop the nonlinear (underlined) term in eq. (47), one obtains Haff's law (or granular temperature) as obtained in Ref. [12]. Indeed, we show below that the underlined terms in eqs (47) and (48) have practically no effect on Haff's law.

A simple stability analysis of the equilibrium points (see chapter 2 of the textbook [15]) of eq. (48) shows that only one equilibrium point of eq. (48) is stable whereas the other two are unstable. The inset of figure 1 illustrates the stable equilibrium point (by black faced circle) and unstable equilibrium points (by white faced circles) for $e = 0.95$ on a phase portrait. We shall, therefore, ignore the unstable equilibrium points and consider only the stable equilibrium point of eq. (48). Note that the stable equilibrium point tends to zero in the limit $e \to 1$. To see the effect of the nonlinear (underlined) terms in eq. (48), we plot—in the main panel of figure 1—the stable equilibrium point of eq. (48) (by continuous line) and the single equilibrium point given by eq. (49) (with '+' symbols) over e. Clearly, the line and symbols coincide with each other for $0.35 \lesssim e \leq 1$. For $e \lesssim 0.35$, the line and symbols slightly deviate from each other. Nevertheless, the maximum relative difference between the values represented by line and symbols is approximately 8.5% which is for $e = 0$. Despite this difference in the values of α, it turns out—but not shown here—that for any fix e, the granular temperature profiles from eq. (47) (with or without the underlined term) for the two values of α (corresponding to line and symbol in the main panel of figure 1) differ from each other only negligibly, see [32]. Therefore, it is justifiable to drop Δ^2 and Δ^3 terms in eqs (39) and (42).

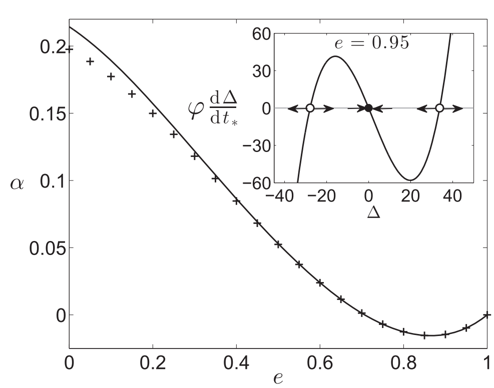

Figure 1. Equilibrium points of eq. (42). The line in the main panel depicts the stable root of the cubic equation (48) while the '+' symbol denotes the single root (49) when the nonlinear terms are dropped in eq. (48). The inset exhibits the stable (black faced circle) and unstable (white faced circles) roots of the cubic equation (48) for $e = 0.95$. Here, $\varphi = \frac{15}{(1+e)\,n_*\sqrt{T_*}} > 0$ (cf. eq. (42)), thus does not play any role in analyzing the stability of equilibrium points.

4.2 *Relaxation of moments in the HCS*

We again consider the G14 equations (37)–(42) in the HCS but without Δ^2 and Δ^3 terms in eqs (39) and (42). We have seen that the number density n_* and velocity v_i^* in the HCS are given by eqs (43) trivially since their governing equations (37) and (38) are not coupled with the remaining eqs (39–42), which, on the other hand, are coupled and need to be solved numerically to see the relaxation of T_*, σ_{ij}^*, q_i^* and Δ with time t_*. We have solved these equations numerically with initial conditions $T_*(0) = \sigma_{ij}^*(0) = q_i^*(0) = \Delta_*(0) = 1$ and found that the granular temperature profile coincides with that obtained via Haff's law (47) (with or without the underlined term) and the other moments—σ_{ij}^*, q_i^* and Δ—decay faster than the granular temperature. We skip showing these results here because these results are exactly same as shown in figure 1 of Ref. [12].

5. Navier–Stokes, and Fourier laws

In this section, we compute the constitutive relations for the stress and heat flux in 5-moment theory which considers the governing equations only for mass, momentum and energy, i.e., eqs (9–11). These equations consist of the stress σ_{ij} and heat flux q_i as additional unknowns. The constitutive relations for the stress and heat flux in 5-moment theory for granular gases are typically procured by CE expansion, see, e.g. [8, 11]. Recently, Kremer and Marques Jr. [12] employed Maxwellian iteration procedure to obtain these constitutive relations via the G14-moment equations for dilute granular gases. Here, we apply a CE-like expansion [18] on the balance equations for σ_{ij}, q_i and Δ (i.e., on (33)–(35)) to compute the constitutive relations for the stress and heat flux.

To this end, the RHSs of eqs (33–35) are divided by ε, which denotes the Knudsen number and replaced by 1 in the end of the procedure, see section 6.3 of [18]. The division by ε on the RHSs of eqs (33–35) is equivalent to considering dimensionless equations whereas replacing ε by 1 in the final results is equivalent to writing them back in the dimensional form [18]. Next, the non-equilibrium variables $\Psi \in \{\sigma_{ij}, q_i, \Delta\}$ are expanded in powers of the Knudsen number ε as

$$\Psi = \Psi_{|0} + \varepsilon\,\Psi_{|1} + \varepsilon^2\,\Psi_{|2} + \dots \tag{50}$$

and these expansions are inserted in eqs (33–35) after dividing their RHSs by ε. Finally, the coefficients of each power of ε on both sides of the equations are compared. Comparison of the coefficients of ε^{-1} on both sides of these equations yields $\sigma_{ij|0} = q_{i|0} = 0$ and $\Delta_{|0} = a_2$. Clearly, $\Delta_{|0} = 0$ in the elastic limit ($e = 0$) but nonzero for granular flows ($e \neq 0$). Next, comparison of the coefficients of ε^0 on both sides of equations for stress and heat flux (eqs (33) and (34) with RHSs divided by ε and variables replaced with expansions (50)) yields $\sigma_{ij|1}$ and $q_{i|1}$, which are the first order approximations for stress and heat flux, and on replacing ε by 1, they read

$$\sigma_{ij} = -2\mu\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} \qquad \text{and} \qquad q_i = -\kappa\frac{\partial T}{\partial x_i} - \lambda\frac{\partial n}{\partial x_i} \tag{51}$$

with the transport coefficients

$$\left.\begin{aligned}
\mu &= \frac{5}{4\sqrt{\pi}\,d^2}\frac{1}{(1+e)(3-e)\left(1-\frac{a_2}{32}\right)}\sqrt{m\,T},\\
\kappa &= \frac{75}{2\sqrt{\pi}\,d^2}\frac{1+2\,a_2}{(1+e)\left[49-33e+\frac{(19-3e)\,a_2}{32}\right]}\sqrt{\frac{T}{m}},\\
\lambda &= \frac{75}{2\sqrt{\pi}\,d^2}\frac{a_2}{(1+e)\left[49-33e+\frac{(19-3e)\,a_2}{32}\right]}\frac{T}{n}\sqrt{\frac{T}{m}}.
\end{aligned}\right\} \tag{52}$$

Eqs $(51)_1$ and $(51)_2$ are the Navier–Stokes law and the Fourier law, respectively. The transport coefficients μ and κ are the shear viscosity and thermal conductivity, respectively. The term with the coefficient λ in eq. $(51)_2$ is solely due to the inclusion of Δ into the Grad moment system. Moreover, this term in eq. $(51)_2$ is exclusive for granular gases and it vanishes identically for monatomic ideal gases (i.e., for $e = 1$) as $a_2 = 0$ for them. The transport coefficients (52) obtained here are exactly same as those obtained in Ref. [12] via the Maxwell iteration procedure.

6. Linear stability analysis

In this section, we analyze the stability of the HCS due to small perturbations through the G14 equations derived in section 3. The amplitudes of these perturbations are assumed to be sufficiently small so that the linear analysis remains valid.

For the linear stability analysis, we decompose all the field variables into their reference values—i.e., their respective solutions in the HCS—and perturbations from their respective solutions in the HCS, i.e., we define

$$\left.\begin{aligned}
n(t,\boldsymbol{x}) &= n_0\big[1+\tilde{n}(t,\boldsymbol{x})\big],\\
T(t,\boldsymbol{x}) &= T_H(t)\big[1+\tilde{T}(t,\boldsymbol{x})\big],\\
v_i(t,\boldsymbol{x}) &= v_H(t)\,\tilde{v}_i(t,\boldsymbol{x}),\\
\sigma_{ij}(t,\boldsymbol{x}) &= n_0\,T_H(t)\,\tilde{\sigma}_{ij}(t,\boldsymbol{x}),\\
q_i(t,\boldsymbol{x}) &= n_0\,T_H(t)\,v_H(t)\,\tilde{q}_i(t,\boldsymbol{x}),\\
\Delta(t,\boldsymbol{x}) &= a_2 + \tilde{\Delta}(t,\boldsymbol{x}),
\end{aligned}\right\} \tag{53}$$

where n_0 is the constant number density and $T_H(t)$ is the granular temperature in the HCS; a_2 is the constant solution for Δ in the HCS; the adiabatic sound speed in the HCS $v_H(t) = \sqrt{5T_H(t)/(3m)}$ is taken as the reference speed; and any quantity with tilde denotes the dimensionless perturbation from its solution in the HCS. As σ_{ij} and q_i decay faster than the granular temperature (see figure 1 of Ref. [12]), we have assumed that σ_{ij} and q_i are approximately zero in the HCS.

Inserting the field variables from (53) into the G14 equations (30)–(35) without Δ^2 and Δ^3 terms (recall that these terms do not have any significant effect), and neglecting all the nonlinear terms of perturbed quantities, we obtain a system of linear first order PDEs for the perturbed quantities consisting of time-dependent coefficients . Nevertheless, it is possible to convert this system to a new one having constant coefficients as follows. We introduce a time scale and a length scale

$$\tau_H(t) = \frac{1}{4\sqrt{\pi}\,n_0 d^2}\sqrt{\frac{m}{T_H(t)}} \quad \text{and} \quad L(t) = \tau_H(t)\,v_H(t), \tag{54}$$

respectively, to make the space variables dimensionless (e.g., $\tilde{x}_i = x_i/L$), and a dimensionless time $\tilde{t}$ such that

$$\tau_H(t)\frac{\partial}{\partial t}(\cdot) = \frac{\partial}{\partial \tilde{t}}(\cdot). \tag{55}$$

This along with eq. $(54)_1$ and the HCS solution for the temperature yields

$$\tilde{t} = \frac{1}{\tau_H(0)} \ln\left(1 + \frac{t}{\tau_1}\right), \tag{56}$$

where

$$\tau_1 = \frac{\tau_H(0)}{\frac{(1-e^2)}{6}\left(1 + \frac{3a_2}{16}\right)}, \quad \tau_H(0) = \frac{1}{4\sqrt{\pi}\, n_0 d^2}\sqrt{\frac{m}{T_H(0)}}.$$

With these definitions of dimensionless space and time, the governing equations for the perturbed quantities read

$$\frac{\partial \tilde{n}}{\partial \tilde{t}} + \frac{\partial \tilde{v}_i}{\partial \tilde{x}_i} = 0, \tag{57}$$

$$\frac{\partial \tilde{v}_i}{\partial \tilde{t}} + \frac{3}{5}\left(\frac{\partial \tilde{\sigma}_{ij}}{\partial \tilde{x}_j} + \frac{\partial \tilde{n}}{\partial \tilde{x}_i} + \frac{\partial \tilde{T}}{\partial \tilde{x}_i}\right) - \frac{1}{2}\xi_1 \tilde{v}_i = 0, \tag{58}$$

$$\frac{\partial \tilde{T}}{\partial \tilde{t}} + \frac{2}{3}\left(\frac{\partial \tilde{q}_i}{\partial \tilde{x}_i} + \frac{\partial \tilde{v}_i}{\partial \tilde{x}_i}\right) + \xi_0 \tilde{\Delta} + \xi_1\left(\tilde{n} + \frac{1}{2}\tilde{T}\right) = 0, \tag{59}$$

$$\frac{\partial \tilde{\sigma}_{ij}}{\partial \tilde{t}} + \frac{4}{5}\frac{\partial \tilde{q}_{\langle i}}{\partial \tilde{x}_{j\rangle}} + 2\frac{\partial \tilde{v}_{\langle i}}{\partial \tilde{x}_{j\rangle}} + \xi_2 \tilde{\sigma}_{ij} = 0, \tag{60}$$

$$\frac{\partial \tilde{q}_i}{\partial \tilde{t}} + \frac{3}{2}\left[\frac{\partial \tilde{\Delta}}{\partial \tilde{x}_i} + a_2 \frac{\partial \tilde{n}}{\partial \tilde{x}_i} + (1 + 2a_2)\frac{\partial \tilde{T}}{\partial \tilde{x}_i}\right] + \frac{3}{5}\frac{\partial \tilde{\sigma}_{ij}}{\partial \tilde{x}_j} + \xi_3 \tilde{q}_i = 0, \tag{61}$$

$$\frac{\partial \tilde{\Delta}}{\partial \tilde{t}} + \frac{8}{15}\left(1 - \frac{5}{2}a_2\right)\frac{\partial \tilde{q}_i}{\partial \tilde{x}_i} + \xi_4 \tilde{\Delta} = 0, \tag{62}$$

where the coefficients

$$\xi_0 = \frac{1}{16}(1 - e^2), \tag{63}$$

$$\xi_1 = \frac{1}{3}(1 - e^2)\left(1 + \frac{3}{16}a_2\right), \tag{64}$$

$$\xi_2 = \frac{2}{15}(1 + e)\left[(2 + e) - \frac{3(13 - 11e)}{64}a_2\right], \tag{65}$$

$$\xi_3 = \frac{1}{60}(1 + e)\left[(19 - 3e) - \frac{(161 - 177e)}{32}a_2\right], \tag{66}$$

$$\xi_4 = \frac{1}{240}(1 + e)(81 - 17e + 30e^2 - 30e^3) \tag{67}$$

depend only on the parameter e, the coefficient of restitution.

Now, we assume a normal mode solution of the form

$$\tilde{\Psi} = \bar{\Psi}\, \exp\left[\mathrm{i}(\boldsymbol{k} \cdot \tilde{\boldsymbol{x}} - \omega \tilde{t})\right], \quad \Psi \in \{n, v_i, T, \sigma_{ij}, q_i, \Delta\}, \tag{68}$$

for system (57)–(62). Here, i is imaginary unit; $\boldsymbol{k}$ and ω are the dimensionless wavevector and frequency of the disturbance, respectively; and bar denotes the complex amplitude of the corresponding perturbed field variable. For the temporal stability analysis, to be analyzed here, the wavevector $\boldsymbol{k}$ is assumed to be real and the frequency ω is assumed to be complex. The imaginary part of frequency, $\mathrm{Im}(\omega)$, determines whether the amplitude of the disturbance grows or decays in time. The form of the normal mode solution (68) intimates that the solution will not grow with time if $\mathrm{Im}(\omega) \le 0$ and vice versa. Consequently, stability requires $\mathrm{Im}(\omega) \le 0$.

If we assume that the wavevector of the disturbance is parallel to the x-axis (i.e., $\boldsymbol{k} = k\hat{\boldsymbol{x}}$, where $k = |\boldsymbol{k}|$ is referred to as the wavenumber and $\hat{\boldsymbol{x}}$ is the unit vector in x-direction), we get two independent eigenvalue problems, namely the longitudinal and transverse problems for the amplitude of the disturbance. The longitudinal problem reads

$$\begin{bmatrix} \omega & -k & 0 & 0 & 0 & 0 \\ -\frac{3k}{5} & \omega - \mathrm{i}\frac{\xi_1}{2} & -\frac{3k}{5} & -\frac{3k}{5} & 0 & 0 \\ \mathrm{i}\xi_1 & -\frac{2k}{3} & \omega + \mathrm{i}\frac{\xi_1}{2} & 0 & -\frac{2k}{3} & \mathrm{i}\xi_0 \\ 0 & -\frac{4k}{3} & 0 & \omega + \mathrm{i}\xi_2 & -\frac{8k}{15} & 0 \\ -\frac{3k\, a_2}{2} & 0 & -\frac{3k(1+2a_2)}{2} & -\frac{3k}{5} & \omega + \mathrm{i}\xi_3 & -\frac{3k}{2} \\ 0 & 0 & 0 & 0 & -\frac{8k}{15}\left(1 - \frac{5}{2}a_2\right) & \omega + \mathrm{i}\xi_4 \end{bmatrix} \begin{bmatrix} \bar{n} \\ \bar{v}_x \\ \bar{T} \\ \bar{\sigma}_{xx} \\ \bar{q}_x \\ \bar{\Delta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{69}$$

while the transverse problem reads

$$\begin{bmatrix} \omega - \mathbb{i}\frac{\xi_1}{2} & -\frac{3k}{5} & 0 \\ -k & \omega + \mathbb{i}\xi_2 & -\frac{2k}{5} \\ 0 & -\frac{3k}{5} & \omega + \mathbb{i}\xi_3 \end{bmatrix} \begin{bmatrix} \bar{v}_y \\ \bar{\sigma}_{xy} \\ \bar{q}_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{70}$$

For nontrivial solutions of the eigenvalue problems (69) and (70), the determinants of both the coefficient matrices must vanish. This condition leads to the dispersion relations—a relation between the wavenumber k and frequency ω, which is usually exploited to express ω as a function of k or vice versa—for the longitudinal and transverse problems. As we are interested in the temporal stability analysis, we assume that the wavenumber k is real and solve the vanishing determinant conditions to express the frequency as $\omega \equiv \omega(k)$.

Note that the coefficient matrix for the transverse problem (70) is exactly same as that in Ref. [12]. Consequently, the behavior of eigenmodes for the transverse problem is same as that found in Ref. [12]. Therefore, we shall not discuss the transverse problem here.

As the sign of imaginary part of frequency, $\mathrm{Im}(\omega)$, decides the stability of eigenmodes, (by solving the vanishing determinant condition) we plot the $\mathrm{Im}(\omega)$ for the longitudinal problem (69) over the wavenumber k for the coefficient of restitution $e = 0.75$ in figure 2. There are six eigenmodes in figure 2 corresponding to the six eigenvalues of the coefficient matrix in the longitudinal problem (69). All the six modes are distinctly identified for small wavenumbers, however as the wavenumber increases two pairs of modes coincide. It is clear from figure 2 that out of the six eigenmodes, only one eigenmode has $\mathrm{Im}(\omega) > 0$ for small wavenumbers and, hence, is unstable for small wavenumbers; the remaining five eigenmodes are stable for all wavenumbers. It is worth pointing out—but not shown here—that in the elastic case ($e = 1$), all six eigenmodes are always stable for all wavenumbers. Note that the unstable eigenmode in figure 2 also becomes stable for $k \gtrsim 0.19$, as the $\mathrm{Im}(\omega)$ for this eigenmode also becomes negative beyond this value of wavenumber. Such a wavenumber at which $\mathrm{Im}(\omega)$ changes its sign is referred to as the *critical wavenumber*. Thus the critical wavenumber for the longitudinal problem in case of $e = 0.75$ is approximately 0.19, see figure 2.

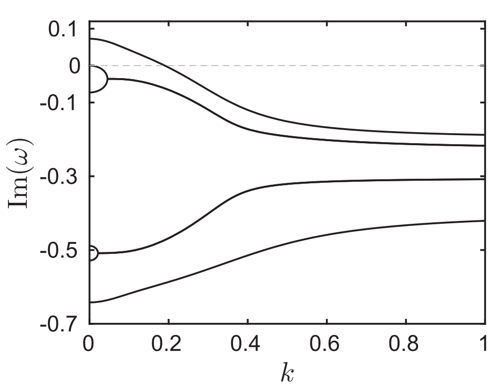

Figure 2. Imaginary parts of longitudinal eigenmodes plotted over the wavenumber k for the $e = 0.75$.

In order to see the behavior of the critical wavenumber on varying the coefficient of restitution, we plot the critical wavenumber for the longitudinal problem (69) over all possible values of the coefficient of restitution in figure 3. The region in the left and below of the curves in figure 3 is the stable region whereas that in the right and above the curves is the unstable region. Note that the critical wavenumber from the G13 theory (shown by red color) in figure 3 is obtained by the theory of Kremer and Marques [12], where the authors assumed a constant value (HCS solution) for Δ while studying the linear stability analysis of the HCS. It may be noticed from figure 3 that the critical wavenumber from the G13 theory closely follows that from the G14 theory for $0.35 \lesssim e \leq 1$. However, as the coefficient of restitution decreases further the critical wavenumbers from both the theories deviate from each other. In particular, the G13 theory (red curve in figure 3) does not give a critical wavenumber for $0 \leq e \lesssim 0.21$, this means that the unstable eigenmode from the G13 theory remains unstable below a certain value of the coefficient of restitution; however, the G14 theory of the present work (blue curve in figure 3) demonstrates that the unstable eigenmode turns stable above a critical wavenumber for all values of the coefficient of restitution.

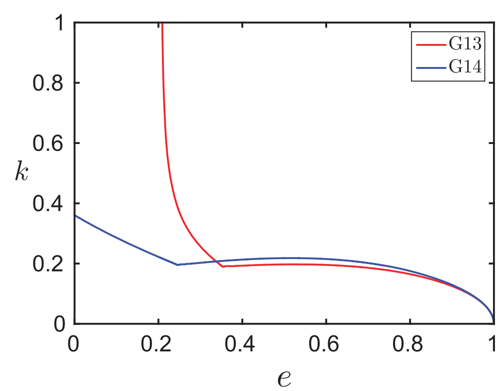

Figure 3. Critical wavenumber k for the longitudinal problem (69)—obtained via the G13 (red color) and G14 (blue color) systems—plotted over the coefficient of restitution e. The eigenmode is unstable below or left of the curves while stable above or right of the curves. The G13 curve is equivalent to that of Ref. [12].

7. Conclusion

The fully nonlinear G14 equations for dilute granular gases have been derived through Grad's moment method. The HCS of a freely cooling granular gas has been investigated with the quasilinear G14 equations, and it has been shown by the stability analysis of fixed points in a dynamical system that the nonlinear terms of the fully contracted fourth moment (Δ) has only negligible effect on Haff's law, and hence can be neglected. The constitutive relations for the stress and heat flux (the NSF laws) have been determined by performing a CE-like expansion on the G14 equations. The constitutive relations obtained in the present work agree with those obtained in Ref. [12] by Maxwell iteration procedure.

The linear stability of the HCS has been analyzed through the G14 system (where, unlike Ref. [12], Δ has not been taken as a constant) by decomposing it into the longitudinal and transverse problems. Nevertheless, the attention has been paid only to the longitudinal problem, since the transverse problem is exactly same as that in Ref. [12]. It has been found that the one out of six eigenmodes for the longitudinal problem is unstable for $e \neq 0$. It has also been unveiled that the unstable eigenmode from the G13 theory of Ref. [12] remains unstable below a certain coefficient of restitution whereas that from the G14 theory of the present work becomes stable above some critical wavenumber for all values of the coefficient of restitution. Thus, it can be recommended that the fully contracted fourth moment (Δ) ought to be considered as a field variable, at least while analyzing the stability of the HCS.

Acknowledgements

The authors thank Prof. Manuel Torrilhon from RWTH Aachen University, Germany, for many fruitful discussions. VKG gratefully acknowledges the financial support from SRM University through the project "GRAIN", and PS gratefully acknowledges financial support from IIT Madras in the form of New Faculty Initiation Grant (MAT/15-16/833/NFIG/PRIY) and New Faculty Seed Grant (MAT/16-17/671/NFSC/PRIY).

References

[1] I Goldhirsch, *Annu. Rev. Fluid Mech.* **35**, 267 (2003)
[2] P Shukla and M Alam, *Phys. Rev. Lett.* **103**, 068001 (2009)
[3] M Alam, V Chikkadi and V K Gupta, *Eur. Phys. J. ST* **179**, 69 (2009)
[4] J T Jenkins and M W Richman, *Arch. Ration. Mech. Anal.* **87**, 355 (1985)
[5] J T Jenkins and M W Richman, *Phys. Fluids* **28**, 3485 (1985)
[6] A Goldshtein and M Shapiro, *J. Fluid Mech.* **282**, 75 (1995)
[7] J J Brey, F Moreno and J W Dufty, *Phys. Rev. E* **54**, 445 (1996)
[8] N Sela and I Goldhirsch, *J. Fluid Mech.* **361**, 41 (1998)
[9] J J Brey, J W Dufty, C S Kim and A Santos, *Phys. Rev. E* **58**, 4638 (1998)
[10] M Bisi, G Spiga and G Toscani, *Phys. Fluids* **16**, 4235 (2004)
[11] V K Gupta, *Kinetic Theory and Burnett Order Constitutive Relations for a Smooth Granular Gas*, Master's thesis (JNCASR, Bangalore, India 2011)
[12] G M Kremer and W Marques, Jr., *Kinet. Relat. Models* **4**, 317 (2011)
[13] S Chapman and T G Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, Cambridge, UK, 1970)
[14] H Grad, *Comm. Pure Appl. Math.* **2**, 331 (1949)
[15] S H Strogatz, *Nonlinear Dynamics and Chaos* (Addison-Wesley Press, Reading, Massachusetts, 1994)
[16] N V Brilliantov and T Pöschel, *Kinetic Theory of Granular Gases* (Oxford University Press, New York, 2004)
[17] K K Rao and P R Nott, *An Introduction to Granular Flow* (Cambridge University Press, New York, 2008)
[18] H Struchtrup, *Macroscopic Transport Equations for Rarefied Gas Flows* (Springer, Berlin, 2005)
[19] V Garzó, *Phys. Fluids* **25**, 043301 (2013)
[20] C K K Lun, S B Savage, D J Jeffrey and N Chepurniy, *J. Fluid Mech.* **140**, 223 (1984)
[21] C S Campbell, *Annu. Rev. Fluid Mech.* **22**, 57 (1990)
[22] J J Brey, J W Dufty and A Santos, *J. Stat. Phys.* **87**, 1051 (1997)
[23] V Garzó and J W Dufty, *Phys. Rev. E* **59**, 5895 (1999)
[24] H Grad, *Comm. Pure Appl. Math.* **2**, 325 (1949)
[25] G M Kremer, *An Introduction to the Boltzmann Equation and Transport Processes in Gases* (Springer, Berlin, 2010)
[26] V K Gupta and M Torrilhon, *AIP Conf. Proc.* **1501**, 67 (2012)
[27] J J Brey, M J Ruiz-Montero and D Cubero, *Phys. Rev. E* **54**, 3664 (1996)
[28] I Goldhirsch and G Zanetti, *Phys. Rev. Lett.* **70**, 1619 (1993)
[29] R Brito and M H Ernst, *Europhys. Lett.* **43**, 497 (1998)
[30] P K Haff, *J. Fluid Mech.* **134**, 401 (1983)
[31] T P C van Noije and M H Ernst, *Granular Matter* **1**, 57 (1998)
[32] V K Gupta, P Shukla and M Torrilhon, *J. Fluid Mech.* (accepted), arXiv preprint, 1701.09052 (2017)