Data assimilation with Chua's circuit

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Abstract. Data assimilation refers to a set of techniques used to combine observational information with numerical models for chaotic dynamical systems and provides a rich interface between dynamical and statistical methodologies in nonlinear dynamics. The main aim of this paper is to compare and contrast two extensively studied paradigms in each of these approaches: on one hand, the ensemble Kalman filter which is a statistical estimation technique, and on the other hand, chaotic synchronization that has been studied in many other contexts, by viewing synchronization as a data assimilation method. In particular, we study the sensitivity of these two methods to changes in observational noise and observational frequency, using both simulated observations and data obtained from an experimental realization of a commonly used low-dimensional dynamical system, namely, Chua circuit, in both the periodic as well as the chaotic regime.

Keywords. Synchronization; ensemble Kalman filter; data assimilation.

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1. Introduction

Data assimilation (DA) is a tool that improves the forecast of a future state of a dynamical system by taking into account the noisy observational data and a dynamical model of the system. Noisy observations are incorporated into the output of the numerical model and the most probable forecast is given. This tool is indispensable in many fields of earth sciences, e.g., weather forecasting, oceanography [1–4]. Various methods for data assimilation have been developed over the past few decades [5–8]. In particular, ensemble Kalman filter (EnKF) is a very commonly used method of sequential data assimilation. It is a Monte-Carlo approximation of the Kalman filter. [9, 10]

When two or more chaotic systems are coupled, under certain conditions the trajectories eventually get synchronized. Many different types of synchronization of chaotic systems have been studied extensively in recent years [11, 12]. For our purposes, we will focus on identical chaotic dynamical systems coupled at discrete time instances. This can be referred to as sporadic or impulsive synchronization. Synchronization of a numerical model with observations obtained from a physical system by means of unidirectional coupling of the observations to the model

can be thought of as a method of sequential data assimilation [13–19].

The main aim of this paper is to compare and contrast the ensemble Kalman filter with synchronization, for a low-dimensional chaotic dynamical system, using numerical as well as experimental data. In particular, we focus on the dependence of the two aforementioned methods on three parameters: (i) observational noise η , (ii) time period between two subsequent observations Δt_{obs} , and (iii) time duration for which the system was observed t_{fin} . The model we used for these studies is the low-dimensional LCR circuit called Chua's circuit [20, and references therein]. One of the main motivations in choosing this model is the ease of the experimental setup and also the presence of both chaotic and periodic dynamics within the experimentally realizable configurations.

Using the experimental data as well as the simulated data, we show that the EnKF is less sensitive to changes in the observational noise than synchronization. Thus more accurate observations lead to more accurate state estimates when using EnKF but the estimation error remains more or less constant even with increasing accuracy of the observations. We demonstrate the above statement by using simulated observations with varying noise levels in the observations.

We also note that the time between observations does not have much effect on either synchronization or on EnKF.

The paper is organized as follows. We first introduce the basic model and present its dynamics in section 2. We then briefly describe in section 3 the two main techniques that we use: synchronization and EnKF. The results are presented in section 4 followed by a summary of the main conclusions and some avenues for future investigations in the last section 5. The two appendices discuss the details of the experimental setup and the non-dimensionalization we use.

2. Basic model and dynamics

The Chua circuit is a basic LCR circuit along with a nonlinear circuit element that exhibits chaotic behaviour for a certain set of parameters. It has been extensively studied in many different contexts, including experimental and numerical studies of synchronization of coupled Chua circuits [20–25]. The schematic circuit is shown in figure 1. It consists of an inductor L with an internal resistance r , two capacitors C_1 and C_2 , a nonlinear circuit element N_R , and a variable resistor R which we use as the main control in the circuit determining whether it is chaotic or not.

The differential equations governing this circuit can be derived by using Kirchhoff's laws. They can be suitably non-dimensionalized to the following form, derived in detail in appendix B.

$$\begin{aligned} \frac{dX}{d\tau} &= \alpha(Y - X - \tilde{g}(X)), \quad \alpha = \frac{C_2}{C_1} \\ \frac{dY}{d\tau} &= X - Y + Z, \\ \frac{dZ}{d\tau} &= -\beta Y - \gamma Z, \quad \beta = \frac{R^2 C_2}{L}, \quad \gamma = \frac{r R C_2}{L}, \end{aligned} \quad (1)$$

where the nonlinear element has the following piecewise linear response:

Figure 1. Chua's circuit.

$$\tilde{g}(X) = \begin{cases} \tilde{m}_2 X + (\tilde{m}_0 - \tilde{m}_2)a + (\tilde{m}_1 - \tilde{m}_0), & X \geq a, \\ \tilde{m}_0 X + \frac{1}{2}(\tilde{m}_1 - \tilde{m}_0)(|X + 1| - |X - 1|), & |X| \leq a, \\ \tilde{m}_2 X + (\tilde{m}_2 - \tilde{m}_0)a + (\tilde{m}_0 - \tilde{m}_1), & X \leq -a. \end{cases} \quad (2)$$

The experimental setup which these equations model is described in detail in Appendix A. The data obtained from the experimental setup consists of V_1 and V_2 , or X and Y respectively after non-dimensionalization, measured at two different values of time gap for a given value of R . The non-dimensional parameter values corresponding to this setup are $a \approx 6.96$, $\tilde{m}_0 \approx R \times (-4.09) \times 10^{-4}$, $\tilde{m}_1 \approx R \times (-7.50) \times 10^{-4}$, $\tilde{m}_2 \approx R \times 4.59 \times 10^{-3}$. The system of equations is integrated using odeint from the module numpy in Python. Depending on R , the only variable parameter which we will consider after fixing all other parameters, the circuit shows chaotic behaviour or otherwise.

In order to choose parameter values both for experimental and numerical studies, we performed stability analysis of eq. (1). The bifurcation diagram shown in the left panel of figure 2 was obtained from performing long simulations at each fixed value of the resistance R and looking for the minima of the time series. It clearly shows the period doubling route to chaos with decreasing value of the resistance R . Based on these bifurcations, we chose two parameter values corresponding to two qualitatively different types of dynamics: $R = 1867$, $L = 16.6$ mH is used to study the periodic behavior, while $R = 1728$, $L = 20$ mH is used to study the chaotic behavior. The reason for varying the inductance is mainly because the peak frequency of the voltage across the inductor is a function of the resistance, and correspondingly the inductance used in eq. (1) for simulations is varied.

The right panels of figure 2 show X and Y components of both numerically generated data (without added noise), and experimentally obtained data for the same discrete time series for a typical trajectory that is used later in assimilation or synchronization.

3. Synchronization and ensemble Kalman filter

We would like to compare the methods of synchronization and ensemble Kalman filter for incorporating observations of a system into a dynamical model. In this section, we will describe very briefly these two methods mainly in order to establish some notations, and also describe the setup we use for the numerical and experimental results presented in the following section. But we begin this section with a discussion of our methodology for comparing the two techniques.

Figure 2. (Left) Period doubling bifurcations in Chua's circuit. Vertical lines mark the resistance values at which our real-life circuit shows concordant behaviour; (Right) X and Y components of the trajectory at $R = 1728\Omega$ and time gap 0.58 in dimensionless units. The numerically generated data is in green and the experimental data is in red.

In our experimental setup, we measure the voltages V_1, V_2 across the two capacitors C_1, C_2 in the Chua circuit at discrete time instances $\tau_1, \tau_2, \dots, \tau_N$, where N is the number of observations. We denote the value $\frac{\tau_N}{N}$ or the time gap between successive observation times by $\Delta\tau_{\text{comp}}$ for the simulated data. In case of experimentally obtained data, we denote the same time gap by $\Delta\tau_{\text{obs}}$. We convert the voltages and the observation times to non-dimensional quantities using the same non-dimensionalization method described in appendix B. This gives rise to two time series that we denote by $x^+(t_k), y^+(t_k)$ for $k = 1, \dots, N$. We use the first time series $x^+(t_k)$ as the observations to be assimilated, either by using synchronization or by using the ensemble Kalman filter. If we denote the time series of the assimilated trajectory by $x^a(t_k), y^a(t_k)$, the relative l_2 error of the assimilated trajectory with respect to the true trajectory is given by:

$$e_x = \left[\sum_{k=1}^N (x^+(t_k) - x^a(t_k))^2 \right]^{1/2} \quad \text{and similarly } e_y,$$

$$e_{\text{total}} = [e_x^2 + e_y^2]^{1/2}$$

$$\text{and } e_{\text{total}}^{\text{rel}} = \frac{e_{\text{total}}}{\left[\sum_{\alpha=x,y} \sum_{k=1}^N (\alpha^+(t_k))^2 \right]^{1/2}}. \quad (3)$$

In the numerical simulations that we perform, the “true” trajectory $X^{\text{true}}(t)$ is obtained by solving eq. (1) with a specific initial condition $X^{\text{true}}(0)$, from which the time series $x^+(t_k), y^+(t_k)$ are obtained by adding Gaussian noise to the true trajectory at the same time instances as used in the experimental studies. We discuss the same l_2

errors even in the case of assimilation of the numerical data, even though in principle, we could consider the l_2 error in the full three-dimensional phase space, in order to compare the assimilation of experimental data and the assimilation of numerical data.

3.1 Synchronization

We represent the model in eq. (1) by $\dot{x} = F(x)$, where $x \in \mathbb{R}^3$ and $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the Lipschitz continuous right-hand side of eq. (1). We will be considering the case of observing the potential across C_1 at discrete times. Representing these observations as $y_1, \dots, y_N$, we see that $y_k = Hx^+(t_k)$ for a set of times $t_1 < \dots < t_N$, where $x^+(t)$ is the true trajectory of the system being observed, and $H = (1 \ 0 \ 0)^T$ is the projection onto the first component of x^+ .

These observations are “assimilated” into the numerical model by trying to synchronize the model with the original system. To this end we use the observations in the model in the following fashion.

$$\dot{z} = F(z) - A[H z(t) - y(t)]. \quad (4)$$

Here $y(t)$ is the linearly interpolated observation at time t (and hence is a piece-wise linear continuous function). The solution of the above equation gives $z(t)$ which is the model trajectory that we wish to synchronize with the true trajectory $x^+(t)$. Here the column matrix $A = (a_1 \ a_2 \ a_3)^T$ represents the strength of coupling between the true trajectory $x^+(t)$ (as represented in the observations) and the “slave” trajectory $z(t)$. Thus, in the notation introduced above, $x^a(t_k) = z(t_k)$ is the assimilated time series at discrete times t_k for $k = 1, \dots, N$.

We define the continuous time synchronization error $E(t)$ to be simply $E(t) = z(t) - x^+(t)$. Thus, we see that the error satisfies

$$\dot{E} = F(E + x^+) - F(x^+) - A \cdot H \cdot E, \quad (5)$$

which is a non-autonomous system with 0 as a fixed point. Note that in case we assume the observations to be noisy, the above equation will be a stochastic differential equation. One possible view of synchronization is that we need to tune the coupling constants, which are the elements of A , so that 0 is a stable fixed point. Note that experimentally, all the three components of $x^+(t)$ are not available continuously in time, since we only have access to observation of the first two components at discrete times, and hence the stability analysis of the above equation cannot be performed when assimilating experimental data, even using techniques developed for the analysis of impulsive synchronization [26].

3.2 Ensemble Kalman filter

The ensemble Kalman filter is a Monte-Carlo approximation of the Kalman Filter. We provide only a very brief introduction here, since it has been well-documented elsewhere, e.g., see [9, 10]. The main idea is to use an ensemble of trajectories of the model to represent the uncertainty in the state and use the Kalman filter method for updating the ensemble in order to use the observations in an approximate Bayesian framework. In particular, for the so-called “forecast step”, r ensemble members are generated and propagated according to the dynamical model which in our context are eq. (1). Using the same notation as above, the forecast from one observational time t_{n-1} to the next observational time t_n can be represented as

$$\dot{x} = F(x), \quad x(t_{n-1}) = x_n^{a,i} \text{ for } i = 1, \dots, r. \quad (6)$$

The solution at time t_n is usually denoted by $x_n^{f,i}$ where the superscript indicates “forecast”. The first two moments of the forecast are given by

$$x_n^f = \frac{1}{r} \sum_{i=1}^r x_n^{f,i} \quad (7)$$

$$C_n^f = \frac{1}{r-1} \sum_{i=1}^r (x_n^{f,i} - x_n^f)(x_n^{f,i} - x_n^f)^T. \quad (8)$$

The Kalman gain matrix is then calculated as

$$K_n = C_n^f H^T (H C_n^f H^T + \Sigma^{\text{KF}})^{-1}. \quad (9)$$

Let $\varepsilon_n \sim \mathcal{N}(0, \Sigma)$ be a normal distribution of length r . We use Σ to be a multiple of identity $\Sigma = \sigma I$, where σ is a tunable parameter. The forecast members are updated using the observations and the above Kalman gain matrix as

$$x_n^{a,i} = x_n^{f,i} + K_n(y_n - H x_n^{f,i} + \varepsilon_n^i), \quad (10)$$

where the superscript “ a ” denotes the “analysis” obtained in the assimilation step. This process is then repeated for all the observations. The detailed justification for this so-called “perturbed observation” ensemble Kalman filter is given, e.g., in Refs [9, 27]. The mean of the analysis ensemble

$$x_n^a = \frac{1}{r} \sum_{i=1}^r x_n^{a,i} \quad (11)$$

can be used as an estimate of the state at time t_n .

Since we are observing only the x -component, Σ^o is a 1×1 matrix, thus $\Sigma^o = \sigma_{\text{obs}}^2$ where σ_{obs} is a real number. Similarly, Σ is a 1×1 matrix which is equal to σ^2 . Σ^{KF} is $\sigma_{\text{KF}}^2 I_{3 \times 3}$. The optimization of tunable parameters is covered in section 4.1.

4. Comparisons and consolidated results

We wish to compare the methods of synchronization and ensemble Kalman filter when they are at their best behaviour, by which we mean that the parameters (coupling constants for synchronization and the error covariance used in EnKF) are optimized such that the l_2 error, as defined in eq. (3), is minimal. In order to do this, we assimilate data and calculate l_2 error at different values of these tunable parameters, as presented in section 4.1. The parameters corresponding to which the l_2 error is minimal are the optimized parameters and are then used in section 4.2. In this section, the l_2 error with respect to the x and y coordinates is calculated for the last quarter of the assimilation time interval, in order to remove any effects of transients.

Throughout this section, we study the following qualitatively distinct cases. As mentioned earlier, we choose $R = 1867\Omega$ as a representative of periodic behaviour and choose $R = 1728\Omega$ for chaotic behaviour. For each of these two cases, we choose either a small time gap between the observations (in dimensionless units) $\Delta\tau_{\text{comp}} = 0.058$ (high frequency data) or a large time gap of $\Delta\tau_{\text{comp}} = 0.58$ (low frequency data). In turn, for each of these, we either generate simulated data with high noise level $\sigma_{\text{obs}} = 2.0$ or low noise level $\sigma_{\text{obs}} = 0.002$ or generate data from the experimental realization of the circuit as described in appendix B. This gives rise to twelve distinct cases (e.g. chaotic, high frequency, low noise). In each of these cases, we work with 900 observations. The above choices (and in particular, the number of observations) are constrained by the measurement devices used in the experimental setup, and the choices for numerically simulated data were matched with the experimentally realizable ones.

4.1 Optimization of parameters

4.1.1 Synchronization: The tunable parameters for synchronization are the coupling constants a_1 , a_2 , and a_3 that are the elements of the matrix A in eq. (6).

We fixed the value of $a_3 = 1.0$ and varied a_1 and a_2 to calculate the relative l_2 error of the x and y components of the synchronized trajectory with respect to x and y components of the true trajectory, as described in eq. (3). Since we are coupling the system with the observations of the x -component, the y -component is used as a test trajectory to evaluate the performance of synchronization as a tool for data assimilation. This l_2 error as a function of (a_1, a_2) is plotted in figure 3, for a few representative cases out of the twelve cases described above: low frequency, high noise, simulated data for chaotic trajectory (top left); high frequency, low noise, simulated data for periodic trajectory (top right); low frequency, experimental data for chaotic trajectory (bottom left); and finally, high frequency, experimental data for periodic trajectory (bottom right, shown as a function of a_1 for various values of a_2 , since the surface plot does not clearly show the local minimum as a function of a_1).

For all the cases except the one corresponding to the periodic trajectory at $R = 1867\Omega$ with experimental observations measured at high frequency of $\Delta\tau_{\text{obs}} = 0.054$ (as shown in bottom right panel of figure 3), the error increases with increasing a_2 but decreases with increasing $a_1 = 500$, whereas for the frequently observed periodic trajectory, there is a minimum in the l_2 error as a function of a_1 . Thus, we fixed the value $a_1 = 500$ and chose a_2 such that the l_2 error at $a_1 = 500$ attains the minimum value. For the exceptional case, a_1 was again chosen to be 500 while a higher value of $a_2 = 300$ was chosen as compared to the other cases.

4.1.2 Ensemble Kalman filter: The optimization of the tunable parameters was done as in the case of synchronization. These parameters are the standard deviation of the initial ensemble, Σ^{KF} , the number of ensemble members r , and the observations error covariance σ that is used during the update step of EnKF.

We calculated the l_2 error by fixing all other parameters and varying the initial ensemble covariance Σ^{KF} . It was observed that Σ^{KF} does not have a significant effect on the error. Similarly our numerical

Figure 3. The l_2 error in the synchronized trajectory, as a function of the coupling constants a_1, a_2 , for low frequency, high noise, simulated data for chaotic trajectory (**top left**); high frequency, low noise, simulated data for periodic trajectory (**top right**); low frequency, experimental data for chaotic trajectory (**bottom left**); and high frequency, experimental data for periodic data (**bottom right**).

Figure 4. The l_2 error in the mean of the EnKF, as a function of the observational error covariance σ used in the EnKF, for chaotic trajectory (**top row**) and periodic one (**bottom row**), for high frequency (**left column**) and low frequency (**right column**) data.

experiments showed that the ensemble size of 80 members or more was enough to obtain stable results and hence we have chosen $r = 100$ member ensemble for the EnKF. After these two values were fixed, we investigated the variation of the l_2 error between the EnKF mean and the actual trajectory, as shown in figure 4. For the simulated data, we see that for very low observational noise σ_{obs} in simulated data, the estimation error is sensitive to changes in σ , in the sense that the error increases sharply with σ , but for high observational noise levels σ_{obs} , the error remains more or less constant with respect to changes in σ . In the case of high frequency experimental data, the error remains more or less constant with respect to σ whereas for low frequency experimental data, the error decreases up to a certain level with decreasing σ . In summary, we see that the smaller the value of σ , the smaller the error, which leads to the choice of $\sigma = 10^{-3}$ as the optimal choice for almost all the cases, including the experimental data.

4.2 Comparisons

With the choices of the “optimal” coupling constants for synchronization and error covariance σ

for EnKF as described in the previous section, we compare the performance of these two methods. In particular, we focus on varying the following parameters:

- Noise in observations, i.e., σ_{obs} .
- Frequency at which observations were measured, i.e., $\Delta\tau_{\text{obs}}$.

4.2.1 Estimation error as a function of time: In order to understand how the estimation error changes in time, we look at the difference between the y-coordinate of the true simulated trajectory (or the measured data, in case of the experimental runs), and the y-coordinate of the assimilated trajectory, both for the case of synchronization and EnKF. This is shown in figure 5 for the case of chaotic trajectory, for assimilating experimental data (left column) and the simulated data (right column), for the EnKF (top row) and synchronization (bottom row), for various values of observational noise σ_{obs} . The dotted line indicates time at which we stop assimilating. We note that with low frequency experimental observations (left column), both EnKF (second panel) and synchronization (fourth panel) show large error almost comparable to the size of the chaotic attractor, thus

Figure 5. The error in the y -coordinate, for the case of chaotic trajectory, for assimilating experimental data (**left column**) and the simulated data (**right column**), for the EnKF (**top row**) and synchronization (**bottom row**). The dotted line indicates the time at which we stop assimilating.

indicating that these observations do not provide any useful information to the model. The failure of EnKF with infrequent observations and the reasons for this failure, essentially associated to the main assumption about Gaussianity which is not applicable for nonlinear trajectories over longer time intervals, have been documented elsewhere [28] and thus are indeed expected. The top right panel shows that the EnKF with simulated low noise data is able to estimate the trajectory well, but the synchronization method is not able to take into account the low noise data as well as the EnKF.

4.2.2 Dependence on observational noise σ_{obs} and time between observations $\Delta\tau_{\text{obs}}$: We now look at the effect of variations in the observational noise σ_{obs} . As earlier, the l_2 error with respect to the x and y coordinates is calculated for the last quarter of the assimilation time interval, in order to remove any effects of transients. Note that the observational noise in the experimental data cannot be varied (and indeed the instrument noise

is not the main source of error, since it is dominated by representativeness error [29], which is difficult or impossible to estimate, at least in this context), so this section focuses only on variations of observational noise in simulated data.

In figure 6, the l_2 error for both synchronization and EnKF is plotted against σ_{obs} for both low and high frequency data. We see that the error shows an increasing trend as σ_{obs} is increased in all cases. However, we notice that the EnKF performs better than synchronization for a given frequency for data with low noise. As the noise level increases, the error in EnKF overtakes that in synchronization, and thus synchronization performs better at higher noise values. We observe this trend for both chaotic and periodic data. We also note that synchronization shows greater sensitivity to change in $\Delta\tau_{\text{obs}}$ at low noise levels, whereas EnKF seems to be less sensitive. The right panel of figure 6 shows the variations of the l_2 error as a function of the time between the observations, for the case of chaotic trajectory for different σ_{obs} (which is shown in the subscript of the legend).

Figure 6. Error in synchronization and EnKF for varying σ_{obs} for chaotic (**left panel**) and periodic (**center panel**) trajectories. In the legend, the subscript 100 refers to the case of low frequency observations and 10 refers to the case of high frequency observations. The **right panel** shows the variations of the l_2 error as a function of the time between the observations, for the case of chaotic trajectory for different σ_{obs} (which is shown in the subscript of the legend).

5. Conclusions and future directions

In this paper, we compare the two state estimation or data assimilation methods: ensemble Kalman filter, which is commonly used in the earth sciences, and synchronization, which is not as commonly thought of as a state estimation technique. The main aim was to understand the effect of observational noise as well as of the time between observations on the state estimates using these methods. We perform this comparison of these two methods for a commonly used low-dimensional chaotic dynamical system, namely, the Chua circuit. One of the noteworthy aspects of this study is the use of both simulated data, obtained by adding noise in a numerically simulated trajectory, and experimental data, obtained by implementing the circuit as described in appendix B. A summary of the main conclusions is as follows. (i) The EnKF is less sensitive to the choice of parameters used in EnKF (the noise σ) than synchronization which depends on the choice of coupling constants (figures 3–4). (ii) If the accuracy of the observations increases, the estimation error using EnKF decreases but the estimation error using synchronization does not (figures 5–6). (iii) Both the methods are not very sensitive to the time between observations (right panel of figure 5), as long as this time is not too large, since the EnKF is known to fail when the assumptions about Gaussianity may not be valid as the time between observations increases.

Understanding the generality of the above conclusions requires extensive further studies in other chaotic systems, as well as theoretical studies of sensitivity of EnKF and synchronization to the time between observations and observational noise. Another major avenue for further studies, and indeed one of the main motivations for us to use experimental, low-dimensional, chaotic dynamical systems such as the Chua circuit is to understand the role of representativeness error and

model error [29] in data assimilation. We hope to utilize experimental data for such a system to make a clear distinction between these two types of errors and thus eventually leading to development of better methodologies for data assimilation. A concrete step in this direction is to undertake a study similar to the one presented in this paper, but using observations over a long time period, in order to understand asymptotic in time behaviour of the EnKF and the synchronization.

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Appendix A: Experimental realisation of Chua’s circuit

The circuit that we used to generate the actual observational data is as shown in figure 7. The inductor is an op-amp realization of a true inductor. The setup and build is based on the information given on a website by Valentin Siderskiy. [30] The reason behind using an electronic circuit in place of an inductor is to minimize the value of the internal resistance associated with the inductor. The internal resistance, r , measured on our circuit was 10.46Ω . The values of the circuit components used are given in table 1. The op-amps were connected to a $\pm 8V$ DC power supply. We note that all the circuit elements are of course active elements and as such the values of the inductor L , its internal resistance r , the capacitance C_1 and C_2 , all depend on the frequency at which the circuit is operated. Thus, we measured these

Figure 7. Chua's circuit used in the experiments.

values at different frequencies. Since Chua's circuit is chaotic and goes through a period doubling route to chaos, the power spectrum depends on the bifurcation parameter R and indeed is peaked at different frequencies. Hence, the value of the inductor at $R = 1867\Omega$ was chosen to be 16.6 mH and that at $R = 1728\Omega$ was chosen to be 20 mH.

Appendix B: Non-dimensionalization of Chua's circuit equation

Eq. (1) is a non-dimensionalized form of the original equation derived using Kirchoff's laws:

$$g(V_1) = \begin{cases} m_2 V_1 + (m_0 - m_2)E_{\max} + (m_1 - m_0)E_{\min} & V_1 \geq E_{\max}, \\ m_0 V_1 + \frac{1}{2}(m_1 - m_0)(|V_1 + E_{\min}| - |V_1 - E_{\min}|) & |V_1| \leq E_{\max}, \\ m_2 V_1 + (m_2 - m_0)E_{\max} + (m_0 - m_1)E_{\min} & V_1 \leq -E_{\max}. \end{cases}$$

We used the following substitutions to non-dimensionalize the above equations to derive eq. (1).

$$\begin{aligned} X &= \frac{V_1}{E_{\min}}, & Y &= \frac{V_2}{E_{\min}}, & Z &= \frac{IR}{E_{\min}}, \\ \widetilde{m}_0 &= Rm_0, & \widetilde{m}_1 &= Rm_1, & \widetilde{m}_2 &= Rm_2, \\ \tau &= \frac{t}{RC_2}, & \widetilde{g}(X) &= \frac{Rg(V_1)}{E_{\min}}, & a &= \frac{E_{\max}}{E_{\min}} \approx 6.96. \end{aligned}$$

Table 1. Table of circuit components and their values

Component	Description	Values
Op-amps	TL082	NA
R	Potentiometer	2.5k Ω
R1	Resistor	220 Ω
R2	Resistor	220 Ω
R3	Resistor	2.2 Ω
R4	Resistor	22k Ω
R5	Resistor	22k Ω
R6	Resistor	3.3k Ω
R7	Resistor	100 Ω
R8	Resistor	1k Ω
R9	Resistor	1k Ω
R10	Potentiometer	2.5k Ω
C1	Capacitor	10 nF
C2	Capacitor	100 nF
C	Capacitor	100 nF

$$\frac{dV_1}{dt} = \frac{1}{RC_1}(V_2 - V_1 - Rg(V_1)),$$

$$\frac{dV_2}{dt} = \frac{1}{RC_2}(V_1 - V_2 + I),$$

$$\frac{dI}{dt} = \frac{1}{L}(-V_2),$$

where $g(V_1)$ is a non-linear function

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