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Fractals

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Abstract. An elementary introduction to the concept of fractals is given. Some examples of fractals drawn from nature are briefly discussed. It is suggested that fractal geometry may be useful in characterizing the grain size and shape distributions in polycrystalline solids.

Keywords. Fractals; non-integral dimensions; grains.

1. Introduction

This talk is intended as an elementary introduction to the concept of fractals for non-experts. I shall try to concentrate on some of the qualitative geometrical features, and will not discuss more technical topics such as phase transitions on fractals.

The organization of the talk is as follows: In § 2 I shall briefly discuss the motivation for studying these unconventional geometrical objects. The intuitive notion of dimension is generalized in § 3 to include non-integral values of the dimension. Some illustrative examples are given in § 4. Section 5 contains some concluding remarks.

2. Motivation

Fractals are geometrical objects having an effective dimensionality which is non-integral. They are useful in modelling several quite different physical situations. It may not be unreasonable to hope that they would also be useful in describing certain aspects of the mechanical behaviour of matter.

This situation is not entirely like that of a biochemist who tries to sell a 'wonder drug' without quite knowing which ailment it cures. In our case, the drug is acknowledged to be 'neurosensitive'. Whether the effects are desirable or undesirable in a specific proposed application is not known beforehand. In some cases, for example in the understanding of the influence of dimensionality on critical phenomena near phase transitions, fractals have been found to be very useful.

The mechanical specification of a system involves the listing of the variables in terms of which the state of the system is described (kinematics), and of the laws of mutual interaction and time-evolution of these variables (dynamics). The choice of variables in terms of which we characterize the state is important: unnatural choices lead to convoluted descriptions. An example of a bad choice of variables is the Ptolemaic insistence on describing planetary motion in terms of a superposition of uniform circular motions. A comparatively more recent example is the phrase 'temperature-dependence of the Debye temperature' used by some physicists around 1910 to describe the non-agreement of the measured temperature-dependence of the specific heat of materials with Debye's theoretical predictions.

The introduction of fractals to describe the mechanical behaviour of matter has, first, a kinematical aspect. It has not yet been established that our difficulties in understanding the mechanical properties of solids are sufficiently acute to necessitate a substantial change from the conventional picture involving an ideal solid plus defects. However, the general trend of the discussions at this Meeting would seem to indicate that, near the plastic limit, the formalism is under stress. It may be useful to explore the possibility of alternative modes of description.

Most text books on the mechanics of solids start out by classifying the defects in solids as point-, line-, surface-, or volume-defects. It is not surprising that the classification scheme is geometrical, and is almost universally preferred over other possible schemes (alphabetical, historical, mobile/immobile defects, annealed/quenched defects, etc.). The geometrical characterization of defects captures some of their most important features. It cannot be the whole story, but it is a useful place from which to begin.

3. A definition of non-integer dimensionality

It is well-known that the strength of solids is a strong function of the grain size and grain shape. Earlier it has been emphasized by Ranganathan that grains in metals are quite irregular in shape (Ranganathan 1984). A schematic representation of possible grain shapes on a two-dimensional lattice is shown in figure 1. Clearly, a material with compact hexagonal grains (as in figure 1a) would be different, mechanically, from a material with very ramified grains (figure 1d). It would be useful to have a numerical characterization of these grain shapes to be able to specify whether the grains are compact objects with a few holes, or are ramified and stringy. Roughly speaking, a compact volume, a disc-like shape, and a string-like shape respectively would behave like three-, two- and one-dimensional objects. A numerical characterization of attributes such as 'stringiness' often gives rise to nonintegral answers, when applied to objects of complex geometrical shapes. Such objects may be said to have a non-integral dimension and are called fractals (Mandelbrot 1977, 1982).

What is meant by saying that an object has dimension 1·5? For that matter, what is meant by saying that an object is three-dimensional? Before giving a definition of dimension, which is generalizable to non-integral values, it is useful to remember that the dimension we assign to an object depends very much on our choice of its representation. A ball of wool may be represented as a point, or a sphere, or as a tangled line (having dimensions 0, 3, and 1 respectively) depending on our scale of description (Mandelbrot 1977). The dimension we assign actually pertains to the mathematical representation of the object, and not to the object itself.

In order to be useful, a definition of dimension must be simple, robust (insensitive to small changes in geometry), and should agree with our intuitive notion of dimension

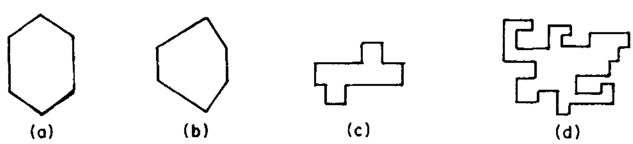

Figure 1. A sequence of increasingly complex grain shapes (schematic).

when applied to conventional spaces (like manifolds). Several such definitions may be found in the literature (Mandelbrot 1977, 1982; Dhar 1977; Engelking 1978). In the following, we consider only the simplest of these.

Consider a space in which the distance between two points, and a volume measure, are defined. Let $V(R)$ be the volume of the set of points whose distance from an arbitrarily chosen origin is less than R. If $V(R)$ varies as R^D for large R, for all choices of the origin, we define D to be the fractal dimensionality of the space. Formally,

$$D = \lim_{R \to \infty} [\log V(R)/\log R].$$

Clearly, using this definition, a straight line has dimension 1 and a plane has dimension 2. Now consider the triangular gasket shown in figure 2. It consists of an equilateral triangle, the mid-points of the sides of which are joined together and divide the triangle into four smaller, equal triangles (three upright, and the central one inverted). Each of the three upright second-generation triangles is divided similarly into four third-order triangles. The 3^2 upright triangles of the third generation are further divided to give 3^3 fourth order triangles, and so on, *ad infinitum*. The gasket consists of all the points in the plane which belong to at least one of the line segments used in this division procedure.

It is easy to see that the points belonging to the gasket have zero area, but that the total length of the line segments which constitute the gasket is infinite. It would thus seem reasonable to assign this set of points a dimensionality between 1 and 2. To determine the fractal dimensionality of the gasket we have to define the metric and a volume-measure for this case. The distance between two points is most simply defined as the Euclidean distance between them (alternatively, as the length of the shortest line joining them and lying completely on the gasket). The volume of an rth generation triangle with all inside points included is defined to be 3. Using these definitions, it is easy to see that

$$V(2R) = 3V(R).$$

Using the definition of D given earlier, this gives $D = \log_2 3 \approx 1{\cdot}54$. The triangular gasket is thus an example of a fractal.

4. Other examples

In the following we give some additional illustrative examples of fractals. For additional examples, and some very beautiful pictures, see Mandelbrot (1977, 1982).

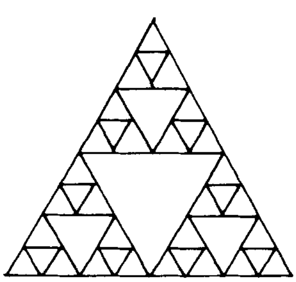

Figure 2. The triangular gasket. Only the edges belonging to the first, second, and third order triangles are shown here for clarity.

4.1 *Lattice animals*

Lattice animals are defined as connected clusters of sites embedded on a lattice (figure 3). The problem of the determination of the mean values of the properties of animals containing a given number of sites, given that all distinct shapes of such animals have equal statistical weights, is called the random animal problem. This assumption is not true for crystal grains, as compact grains have lower surface energy, and are favoured thermodynamically.

If a typical animal on a d-dimensional hypercubical lattice (d is an integer here) were compact, its maximum lateral extent R would vary as $n^{1/d}$, where n is the number of sites in the animal. It is found that random animals are typically stringy, and R varies as a *larger* power of n. In 3-dimensions, the dependence is $r \sim n^{1/2}$. Thus a random animal in 3-dimensions has a fractal dimension equal to 2 because $n \sim R^2$.

4.2 *Percolation clusters*

Consider a d-dimensional hypercubical lattice (d integral) in which each nearest-neighbour bond is, independent of others, unbroken (*i.e.*, connects the sites at its ends) with a probability p (figure 4). For small p values (p less than a critical value p_c), a site chosen at random is connected to only a finite number of other sites through occupied bonds. In this case, the effective dimension of the clusters may be defined to be zero. For $p > p_c$, the mutually connected sites constitute a finite fraction of the total number of sites of the lattice, and form a d-dimensional network (with some holes). When p is equal to the critical probability p_c, the incipient infinite cluster has zero fractional volume, and is very tenuous. However, if we construct a sphere of radius R about a point, the expected number of sites connected to the origin varies as R^D where $D \approx 1{\cdot}6$ for $d = 2$. The infinite cluster at the percolation threshold is an easily-realizable example of a fractal.

4.3 *Soap powder*

If we look at soap powder under a microscope, it is seen to consist of a large number of particles of varying sizes. As in the case of crystallites in metals, the average linear extent

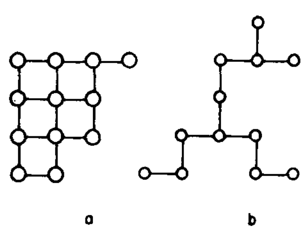

Figure 3. Animals on a square lattice. (a) is a compact cluster, while (b) is stringy.

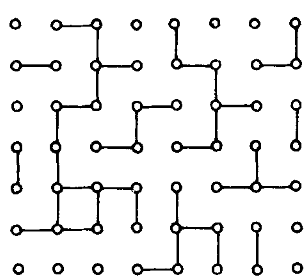

Figure 4. A random configuration of unbroken bonds on a square lattice.

of these powder particles grows faster than the cube root of their mass, and can be used to define an effective dimension of the powder. We should expect the fractal dimension of soap powder to be between 2 and 3. The precise numerical value may perhaps depend upon its chemical structure.

4.4 *The distribution of galaxies*

The precise nature of the large-scale structure of the distribution of matter in the universe is not very well-known. According to one view, the distribution of galaxies in the universe is essentially uniform. However, a different picture, in which the universe shows clustering at all scales of length, is also consistent with experimental data. According to this picture, galaxies typically reside in clusters of galaxies, which themselves are part of clusters of galaxies and so on. Each cluster contains about 3 subclusters on the average. The size of the cluster is appreciably smaller than inter-cluster distance, which implies that the number of galaxies in a sphere of radius R, centred at any particular galaxy, grows as R^D with $D < 3$. Thus the distribution of the galaxies in the universe shows a fractal structure.

4.5 *Coastlines*

The coastline of an island is typically a very zigzag line (figure 5), and its measured length depends very much on the size of the divider spacing (or calliper length) used to measure it (say, from a very detailed map). If we use a smaller divider spacing, we measure more of the small detours and turns that the line makes, and the measured length is larger. If the line is smooth at very short distances, then, as the divider spacing is decreased, the length will tend to a limiting finite value (which is defined to be the 'length of the line'). If, however, the line is rough at all length scales, we may find that $L(s)$, the length measured when the divider spacing is s, varies as s^{1-D}, where $D > 1$. This D may be called the fractal dimension of the coastline. For several actual coastlines $D = 1{\cdot}2$. For a Brownian motion path $D = 2$.

5. Concluding remarks

It should be clear from the preceding examples that fractals are not exotic, pathological objects dreamt up by mathematicians, but are very much a part of the world around us. They provide a natural description of the non-classical geometrical forms encountered in nature. It is this common geometrical character, and not the occurrence of fractional powers, which is the characteristic feature of fractals. Fractional power laws are, of course, encountered quite frequently in Nature. For two observables X and Y, if X varies as an integral power of Y (not equal to ± 1), then Y varies as a non-integral power of X. It is preferable not to use the term 'fractal' in the absence of a geometrical structure in the problem (as in, say, of stock-market prices). Fractals may also show up in the discussion of mechanical properties of matter in the context of dynamical response and the onset of chaos. A discussion of these aspects may be found elsewhere (Ramaswamy 1984). Additional references on fractals may be found in Dhar (1981).

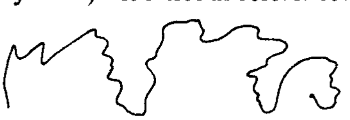

Figure 5. A schematic representation of a coastline. The line will show further wrinkles and detours, if observed on an expanded scale.

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Discussion

N Kumar: Are fractal dimensions and similar apparently geometrical concepts applicable to *dynamics*?

D Dhar: Yes, there are connections. An example is magnetisation creep.

G Venkataraman: These concepts have also been employed in describing turbulence.

R Ramaswamy: Strange attractors have been shown to have well-defined fractal dimensions.

S Ranganathan: Fractals may have some bearing on the shapes of second phase particles which influence mechanical properties.