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Axial symmetric rotation of a partially immersed body in a liquid with a surfactant layer

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Abstract. This paper gives a simple integral formula to evaluate the torque on a slowly rotating symmetric body partially immersed in a viscous liquid covered by an adsorbed surface film. Besides the results known earlier, new results have also been derived for small values of the surface shear viscosity parameter κ. It is seen that the effect of κ in all cases is to increase the torque.

Keywords. Slow rotation; axially symmetric bodies; surfactant layer.

1. Introduction

A boundary condition at the free surface of a liquid is the vanishing of the tangential stress. But when an adsorbed mono-molecular film covers the surface, this boundary condition needs modification. The changed condition may be derived from the work of Scriven (1960) by incorporating a surface shear viscosity parameter κ, and has been utilized by several workers for studying problems important in the field of surface chemistry. Since relevant problems involve slow motion, just the Stokes equation instead of the full Navier–Stokes equation suffices. In an extensive paper, Davis (1980) has evaluated the torque on an axially symmetric body rotating in a liquid with surfactant layer. But his method depends unnecessarily on an analogy with the potential problem of calculating the virtual mass of a heaving body. Here, we show that for small κ the torque can be directly evaluated by expressing it in the form of an integral dependent only on the zeroth [$\kappa = 0$] order solution of the rotatory Stokes equation. Using this formula, the results for the hemispherical immersed parts given by Davis (1980) have been derived, and also new results for spheroidal, dumbbell-shaped bodies and other cases of axial symmetric bodies have been generated by using the expressions derived in §3 below.

2. Formulation of the problem

Consider a substrate consisting of fluid of viscosity μ bounded by a surfactant mono-molecular layer F of surface shear viscosity κ. Let us set up a cylindrical polar coordinate

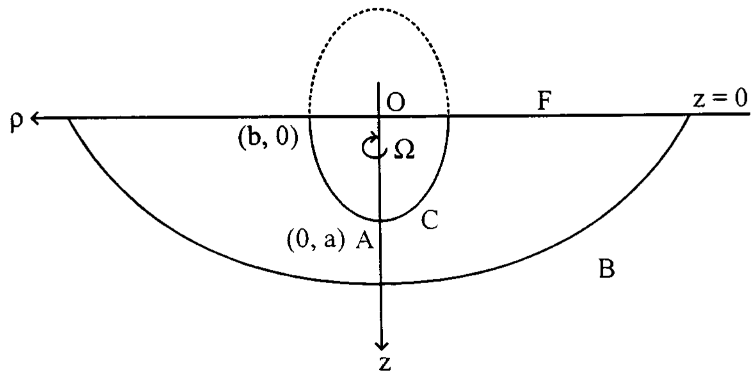

Figure 1. Rotating axial symmetric body C in a semi-infinite fluid with a surfactant layer F.

system which for the axially symmetric problem to be taken up here is designated by (ρ, z) in the meridional plane, where the z axis is vertically downwards in the fluid and $z = 0$ is occupied by the surfactant layer. The motion is caused by the axial rotation of a partially immersed axisymmetric body C about its axis of symmetry which is taken as the z-axis with angular velocity Ω. The substrate fluid may extend to infinity or be bounded by an axially symmetric solid boundary surface B preserving the symmetry of the motion. As shown in figure 1, C, F and B represent the boundary curves of the fluid mass in the meridional plane.

On account of axial symmetry, streamlines are circles round the axis of symmetry, and hence only the azimuthal component $w\ (\rho, z)$ of fluid velocity is non-vanishing. The equation of continuity is identically satisfied and w is governed by (Davis 1980),

$$\Lambda w = \frac{\partial^2 w}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial w}{\partial \rho} - \frac{w}{\rho^2} + \frac{\partial^2 w}{\partial z^2} = 0. \tag{1}$$

Equation (1) is to be solved under the following boundary conditions. No slip condition:

$$w = \Omega\rho, \quad \text{on C}, \tag{2}$$

$$w = 0, \quad \text{on B}. \tag{3}$$

Scriven's condition on surfactant layer F (Scriven 1960):

$$\frac{\partial w}{\partial z} = a\lambda \frac{\partial^2 w}{\partial z^2}, \qquad z = 0, \ \ b \le \rho \le \mathrm{R}, \tag{4}$$

the parameter λ here differs from that of Davis (1980) by a factor $1/a$.

The physical quantity of interest is the couple T required to maintain the rotation. This is composed of two parts, viz., the torque M due to fluid resistance and part N due to surface film resistance.

Now, the fluid torque is given by

$$M = -2\pi\mu \int\limits_c \rho^3 \frac{\partial}{\partial n}\left(\frac{w}{\rho}\right) \mathrm{d}s, \tag{5}$$

where n is the outward normal into the fluid and ds is the element of arc along C.

The film torque is given by

$$N = -2\pi\kappa \left[\rho^3 \frac{\partial}{\partial\rho}\left(\frac{w}{\rho}\right)\right]_{(\rho,z)=(b,o)}. \tag{6}$$

For small λ, we write

$$w = w_0 + \lambda w_1, \tag{7}$$

and $M = M_0 + \lambda M_1$. (8)

The boundary conditions on neglecting terms of $O(\lambda^2)$ satisfied by w_0 and w_1 may now be obtained from (2)–(4) as follows

$$\begin{aligned} &w_0 = \Omega\rho, && \text{on C},\\ &w_0 = 0, && \text{on B},\\ &\partial w_0/\partial z = 0, && \text{on F}, \end{aligned} \tag{9}$$

and

$$\begin{aligned} &w_1 = 0, && \text{on C},\\ &w_1 = 0, && \text{on B},\\ &\partial w_1/\partial z = a\left(\partial^2 w_0/\partial z^2\right), && \text{on F}. \end{aligned} \tag{10}$$

3. Formula for couple

Using (7), (8) in (5), we see that

$$M_0 = -2\pi\mu \int_c \rho^3 \frac{\partial}{\partial n}\left(\frac{w_0}{\rho}\right) \mathrm{d}s, \tag{11}$$

and

$$M_1 = -2\pi\mu \int_c \rho^3 \frac{\partial}{\partial n}\left(\frac{w_1}{\rho}\right) \mathrm{d}s. \tag{12}$$

The first result above is the classical result for the case of stress-free surface for which the value of w_0 is explicitly known in diverse cases. We shall show that this value of w_0 suffices to evaluate M_1 too. Now using the first of the boundary conditions (9) and (10), we can write (12) as

$$\begin{aligned} M_1 &= -\frac{2\pi\mu a}{\Omega} \int_c w_0 \frac{\partial w_1}{\partial n} \rho \, \mathrm{d}s \qquad (13)\\ &= -\frac{2\pi\mu a}{\Omega} \left[\left\{-\int_F - \int_B + \int_{C+B+F}\right\} w_0 \frac{\partial w_1}{\partial n} \rho \, \mathrm{d}s\right]\\ &= \frac{2\pi\mu a}{\Omega} \int_b^R \left[w_0 \frac{\partial^2 w_0}{\partial z^2}\right]_{z=0} \rho \, \mathrm{d}\rho, \end{aligned}$$

where the integral over B vanishes on using the second of the boundary conditions (9). The integral over C + B + F may be shown to vanish on using Green's theorem for the operator Λ and making use of the relevant boundary conditions (9) and (10).

Next, for small λ, N as given by (6) may be approximated by

$$N = \lambda N_1 = -2\pi\mu\lambda a\left[\rho^3\frac{\partial}{\partial\rho}\left(\frac{w_0}{\rho}\right)\right]_{(b,0)}. \tag{14}$$

Thus, the total torque T may be approximated as

$$T = M_0 + \lambda M_1 + \lambda N_1. \tag{15}$$

4. Evaluation of couple

In this section, we shall make use of the known values of w_0 to get the approximate total couple T for various axisymmetric bodies by evaluating M_1 and N_1 through (13) and (14), and making use of known values of M_0.

4.1 *Case 1 – Hemisphere in fluid of infinite depth*

In this case $b = a, R \to \infty$ and then we have the well-known velocity field (Davis 1980) for a steadily rotating sphere,

$$w_0 = \left(\Omega a^3\rho\right)/\left(\rho^2 + z^2\right)^{3/2}. \tag{16}$$

Also, M_0 is given by (Davis 1980)

$$M_0 = 4\pi\mu\Omega a^3. \tag{17}$$

Now, making use of (16) in (13), we get

$$M_1 = -(3/2)\pi\mu\Omega a^3. \tag{18}$$

Next from (14) and (16), we get

$$N_1 = 6\pi\mu\Omega a^3. \tag{19}$$

Hence, the total couple

$$T = 4\pi\mu\Omega a^3\,(1 + (9/8)\lambda). \tag{20}$$

4.2 *Case 2 – Substrate bounded by a solid hemisphere of radius $R(> a)$ concentric with the rotating hemisphere*

In this case $b = a, R > a$, and the velocity is given by (Davis 1980)

$$w_0 = \frac{\Omega a^3\rho}{(R^3 - a^3)}\left[\frac{R^3}{(\rho^2 + z^2)^{3/2}} - 1\right], \tag{21}$$

and the value of M_0 is

$$M_0 = 4\pi\mu\Omega\frac{R^3 a^3}{(R^3 - a^3)}. \tag{22}$$

As in case 1, we can evaluate

$$M_1 = -\frac{3}{2}\pi\mu\Omega\frac{a^3R^2}{(R^3-a^3)}\left[\frac{R^3+R^2a+Ra^2-3a^3}{(R^2+Ra+a^2)}\right], \tag{23}$$

$$N_1 = 6\pi\mu\Omega\left[R^3a^3\big/(R^3-a^3)\right], \tag{24}$$

and hence the total couple

$$T = 4\pi\mu\Omega a^3\left[\frac{R^3}{R^3-a^3}+\frac{9}{8}\lambda\left\{1+\frac{2R^2a^3+Ra^4+a^5}{(R^3-a^3)(R^2+Ra+a^2)}\right\}\right]. \tag{25}$$

It may be seen that T as given in (19) and (20) are exactly the same as obtained by Davis (1980).

4.3 *Case 3 – Substrate bounded below by a rigid plane $z = H$*

In this case $b = a$, $R \to \infty$, and the velocity is given by (Davis 1980)

$$w_0 = \Omega\Gamma a^3\left[\frac{\rho}{(\rho^2+z^2)^{3/2}}-\frac{\rho\eta(3)}{4H^3}\right], \tag{26}$$

for large values of H. Here

$$\Gamma = 1 + a^3\frac{\eta(3)}{4H^3}, \eta(3) = \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m^3},$$

and M_0 is given by

$$M_0 = 4\pi\mu\Omega a^3\Gamma. \tag{27}$$

As in case 1, we can evaluate

$$M_1 = -6\pi\mu\Omega a^3\left[\frac{1}{4}-\frac{a^3\eta(3)}{4H^3}\right](\Gamma)^2, \tag{28}$$

$$N_1 = 6\pi\mu\Omega a^3\Gamma, \tag{29}$$

and hence the total couple

$$T = 4\pi\mu\Omega a^3\Gamma\left[1+\frac{3}{2}\lambda\left\{1-\frac{\Gamma}{4}\left(1-\frac{a^3\eta(3)}{H^3}\right)\right\}\right]. \tag{30}$$

We give in table 1 the values of M_0, M_1, N_1 and T, as normalized by $4\pi\mu\Omega a^3$. The values obtained by Davis (1980) are given in parentheses.

On comparison, it is seen that except for the small value $H = 2$, the values compare well. The discrepancy for $H = 2$ may be attributed to the fact that (10) that we have used is tenable only for large H.

The values of T given by (20), (25) and (30) are the same as the corresponding values obtained by Davis (1980) by calculating the virtual mass coefficient of the equivalent potential flow problem describing the surface waves generated by a heaving body.

Now, we proceed to exploit (13) to generate the approximate values of the couple for small λ for a number for new cases for which w_0 and M_0 have been obtained by Chwang & Wu (1974, 1975) by the singularity method.

Table 1. Variation of torques with the depth H of the substrate.

H	∞	8	6	4	2
$M_0/4\pi\mu\Omega a^3$	1	1.0004	1.0010	1.0035	1.0282
	(1)	(1.0005)	(1.0011)	(1.0035)	(1.0290)
$M_1/4\pi\mu\Omega a^3$	.3750	.3747	.3742	.3723	.3517
	(.3750)	(.3748)	(.3746)	(.3743)	(.3777)
$N_1/4\pi\mu\Omega a^3$	1.5000	1.5007	1.5016	1.5053	1.5423
	(1.5000)	(1.5007)	(1.5016)	(1.5049)	(1.5344)
$T/4\pi\mu\Omega a^3$	$1+1.8750\lambda$	$1.0004+$	$1.0010+$	$1.0035+$	$1.0282+$
		1.8754λ	1.8758λ	1.8776λ	1.8940λ
	$(1+1.8750\ \lambda)$	$(1.0005+$	$(1.0011+$	$(1.0035+$	$(1.0290+$
		$1.8755\lambda)$	$1.8761\lambda)$	$1.8792\lambda)$	$1.9121\lambda)$

4.4 *Case 4 – Prolate spheroid*

We have for the prolate spheroid,

$$\frac{\rho^2}{b^2} + \frac{z^2}{a^2} = 1, c = (a^2 - b^2)^{1/2} = ae, a > b, R \to \infty.$$

Here

$$w_0 = \frac{\beta_0}{\rho c^2}\left[(c-z)R_1 + (c+z)R_2 + \rho^2 \log\frac{R_1 - (z+c)}{R_2 - (z-c)}\right], \tag{31}$$

where

$$\begin{aligned} R_1 &= \{(z+c)^2 + \rho^2\}^{1/2}, R_2 = \{(z-c)^2 + \rho^2\}^{1/2}, \\ \beta_0 &= \frac{\Omega c^2}{f(e)}, f(e) = \frac{2e}{1-e^2} - \log\frac{1+e}{1-e}, \end{aligned}$$

and M_0 is

$$M_0 = (16/3)\pi\mu a e \beta_0. \tag{32}$$

Now, calculating $w_0 \left.\frac{\partial^2 w_0}{\partial z^2}\right|_{z=0}$ with the help of (31) and substituting it in (13) we get

$$M_1 = \frac{8\mu {B_0}^2}{\Omega a e^3}\left[2\log(1-e^2) + e\log\frac{1+e}{1-e}\right], \tag{33}$$

$$N_1 = 8\pi\mu a e\beta_0. \tag{34}$$

Hence the total couple

$$T = \frac{16}{3}\pi\mu\Omega\frac{a^3e^3}{f(e)}\left[1 + \frac{3\lambda}{ef(e)}\left\{\frac{e^2}{1-e^2} + \log(1-e^2)\right\}\right]. \tag{35}$$

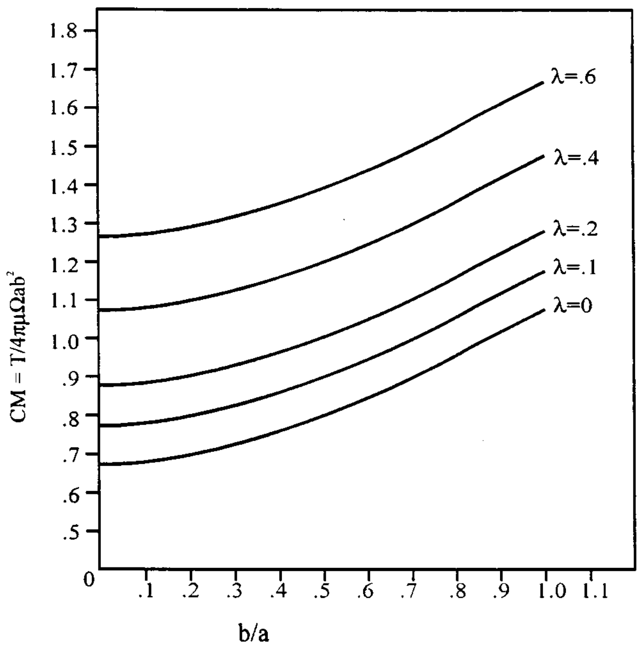

Figure 2. Variation of the moment coefficient $C_M = T/4\pi\mu\Omega ab^2$ of rotating prolate spheroids with the ratio $b/a = (1 - e^2)^{1/2}$.

We define the torque coefficient C_M with reference to $4\pi\mu\Omega ab^2$. Thus, for the prolate spheroid

$$C_M = \frac{T}{4\pi\mu\Omega ab^2} = \frac{4}{3}\frac{e^2}{f(e)(1-e^2)}\left[1 + \frac{3\lambda}{ef(e)(1-e^2)}\{e^2 + (1-e^2)\log(1+e^2)\}\right]. \quad (36)$$

Figure 2 shows the variation of moment coefficient C_M with b/a for various values of the surface shear viscosity parameter λ. C_M is seen to monotonically increase with b/a as well as with λ.

4.5 *Case 5 – Substrate bounded by a solid prolate spheroid with eccentricity e_2 concentric with the rotating prolate spheroid with eccentricity $e_1 (> e_2)$.*

In this case $b = b_1$, $a = a_1$ and $R = b_2$ and the equation of two confocal prolate spheroids is given by

$$(z^2/a_i^2) + (\rho^2/b_i^2) = 1 \qquad (i = 1, 2, a_i \geq b_i, a_2 > a_1),$$

with a common focal length $2c$ and eccentricities e_1 and e_2,

$$c^2 = a_1{}^2 - b_1{}^2 = a_2{}^2 - b_2{}^2, \qquad c = e_1 a_1 = e_2 a_2 \qquad (e_1 > e_2),$$

and the velocity given by Chwang & Wu (1975) is

$$w_0 = \left[\alpha_0\rho + \frac{\beta_0}{\rho}\left\{(c-z)R_1 + (c+z)R_2 + \rho^2\log\frac{R_1 - (z+c)}{R_2 - (z-c)}\right\}\right], \qquad (37)$$

where R_1 and R_2 are given in the preceding case (4), and

$$\alpha_0 = \frac{-\Omega f_2}{f_1 - f_2}, \beta_0 = \frac{\Omega}{f_1 - f_2}, f_i = f(e_i) = \frac{2e_i}{1 - e_i{}^2} - \log\frac{1+e_i}{1-e_i},$$

and M_0 is

$$M_0 = \frac{16}{3}\pi\mu c^3\beta_0. \tag{38}$$

As before, we can evaluate

$$M_1 = -8\pi\mu\beta_0 c^3\left[\frac{1}{\Omega}\left\{\alpha_0 + \beta_0 \log\frac{1-e_1}{1+e_1}\right\} + \frac{e_2}{\Omega e_1}\left\{\alpha_0 + \beta_0 \log\frac{1-e_2}{1+e_2}\right\} + \frac{2\beta_0}{\Omega e_1}\log\frac{1-{e_2}^2}{1-{e_1}^2}\right], \tag{39}$$

$$N_1 = 8\pi\mu\lambda\beta_0 c^3. \tag{40}$$

Hence the total couple

$$T = \frac{16}{3}\pi\mu\frac{{a_1}^3{e_1}^3\Omega}{f_1 - f_2}\left[1 + \frac{3}{e_1}\frac{\lambda}{(f_1 - f_2)}\left\{\frac{{e_1}^2}{1-{e_1}^2} + \frac{{e_2}^2}{1-{e_2}^2} - \log\frac{1-{e_2}^2}{1-{e_1}^2}\right\}\right], \tag{41}$$

4.1. *Case 6 – Body generated by uniform axial distribution of rotlets*

In this case, the velocity as given by Chwang & Wu (1974) is

$$w_0 = \frac{\beta_0}{\rho}\left[\frac{z+c}{\{(z+c)^2+\rho^2\}^{1/2}} - \frac{z-c}{\{(z-c)^2+\rho^2\}^{1/2}}\right], \tag{42}$$

where rotlets are distributed in the interval $-c \le z \le c$,

$\beta_0 = \Omega[b^2(b^2+c^2)^{1/2}]/2c$ and c is related to a and b as given by,

$$(a^2 - c^2)^2 = ab^2(b^2+c^2)^{1/2}.$$

M_0 is

$$M_0 = 4\pi\mu\Omega b^2(b^2+c^2)^{1/2}. \tag{43}$$

As before, we evaluate

$$M_1 = -\frac{3}{2}\pi\mu\Omega\left[ab^4\big/\left(b^2+c^2\right)\right], \tag{44}$$

$$N_1 = 2\pi\mu\Omega ab^2\left[\left(3b^2+2c^2\right)\big/\left(b^2+c^2\right)\right]. \tag{45}$$

Hence, the total torque

$$T = 4\pi\mu\Omega b^2(b^2+c^2)^{1/2}\left[1 + \frac{\lambda a}{8(b^2+c^2)^{3/2}}(9b^2+8c^2)\right], \tag{46}$$

or in the coefficient form

$$C_M = \frac{T}{4\pi\mu\Omega ab^2} = \left(\frac{b^2}{a^2} + \frac{c^2}{a^2}\right)^{1/2}\left[1 + \frac{\lambda}{8}\frac{1}{[(b^2/a^2)+(c^2/a^2)]^{3/2}}\left(\frac{9b^2}{a^2} + \frac{8c^2}{a^2}\right)\right]. \tag{47}$$

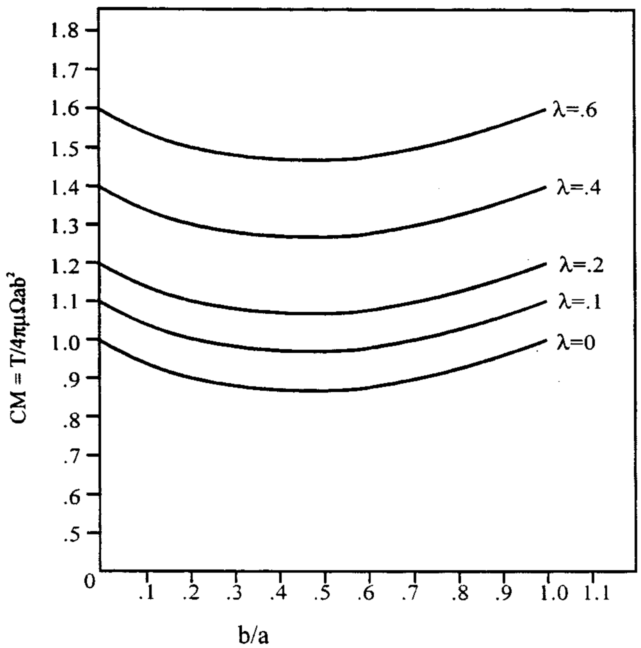

Figure 3. Variation of the moment coefficient $C_M = T/4\pi\mu\Omega ab^2$ with parameter b/a for bodies generated by uniform distribution of rotlets.

In figure 3, the variation of C_M with b/a is shown for different values of λ, as evaluated from (47). The effect of surfactant is again seen to increase C_M.

4.7 *Case 7 – Dumbbell-shaped bodies*

In this case, the velocity as given by Chwang & Wu (1974), with d and b interchanged, is

$$w_0 = \frac{\Omega\rho}{2}(c^2+b^2)^{3/2}\left[\frac{1}{\{(z+c)^2+\rho^2\}^{3/2}} + \frac{1}{\{(z-c)^2+\rho^2\}^{3/2}}\right], \tag{48}$$

where c is given by

$$(a^2-c^2)^3 = a(a^2+3c^2)(c^2+b^2)^{1/2}. \tag{49}$$

The value of M_0 is

$$M_0 = 4\pi\mu\Omega(c^2+b^2)^{3/2}. \tag{50}$$

and as before, we evaluate

$$M_1 = -\frac{3}{4}\pi\mu\Omega a\frac{(2b^4-4c^2b^2-c^4)}{(c^2+b^2)}. \tag{51}$$

$$N_1 = 6\pi\mu\Omega a\frac{b^4}{(c^2+b^2)}. \tag{52}$$

Hence, the total torque is given by

$$T = 4\pi\mu\Omega(c^2+b^2)^{3/2}\left[1+\frac{3}{16}\lambda a\left\{\frac{6b^4+4c^2b^2+c^4}{(c^2+b^2)^{5/2}}\right\}\right]. \tag{53}$$

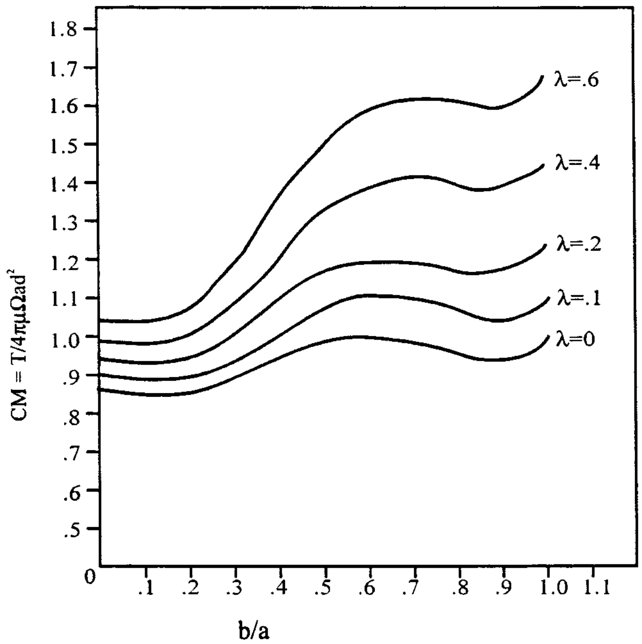

Figure 4. Variation of the moment coefficient $C_M = T/4\pi\mu\Omega ad^2$ with parameter b/a for dumbbell-shaped bodies.

or in the coefficient form

$$C_M = \frac{T}{4\pi\mu\Omega ad^2} = \frac{a^2}{d^2}\left(\frac{c^2}{a^2} + \frac{b^2}{a^2}\right)^{3/2}$$
$$\times\left[1 + \frac{3}{16}\lambda\left\{\left(6\frac{b^4}{a^4} + 4\frac{b^2}{a^2}\cdot\frac{c^2}{a^2} + \frac{c^4}{a^4}\right)\Big/\left(\frac{c^2}{a^2} + \frac{b^2}{a^2}\right)^{5/2}\right\}\right], \quad (54)$$

The variation of C_M with b/a is shown in figure 4 for various values of the parameter λ as evaluated from (54). It is seen that the effect of λ is to increase the moment coefficient C_M.

It may be verified that for $e = 0$, the values of T for cases 5–7 reduce to that of case 1 for the hemisphere, and for $e_1 = e_2 = 0$ in case 5, the values of T for the case 2 is recovered. It may also be seen that in all these cases, the role of λ is to increase the total torque.

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