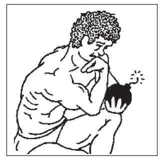
Think It Over



This section of *Resonance* presents thought-provoking questions, and discusses answers a few months later. Readers are invited to send new questions, solutions to old ones and comments, to 'Think It Over', *Resonance*, Indian Academy of Sciences, Bangalore 560 080. Items illustrating ideas and concepts will generally be chosen.

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Solution to
How many balls are there in the urn at 12 o'clock?

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The answer in the random selection case is that, with probability one, the urn will be empty at 12 o'clock. (Notice the difference between this result and that in the second situation described in the problem.)

To prove this, let B_n = event that ball no. 1 remains in the urn after the first n withdrawals. Then, prove that the probability of B_n is given by

$$P(B_n) = \frac{9}{10} \times \frac{18}{19} \times \frac{27}{28} \times \ldots \times \frac{9n}{9n+1}. \qquad (1)$$

Let A_1 = event that ball no.1 remains in the urn at 12 o'clock. Evidently $A_1 = \bigcap_{n=1}^{\infty} B_n$. As the events $B_n, n \geq 1$ are decreasing (that is, $B_1 \supseteq B_2 \supseteq \ldots$) we have

$$P(A_1) = P\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} P(B_n) \lim_{n\to\infty} \prod_{k=1}^{n} = \frac{9k+1}{9k+1} 9k,$$

Keywords

by using (1).

Now it is an interesting exercise to show that

$$\prod_{n=1}^{\infty} \frac{9n}{9n+1} = 0.$$

(**Hint:** $\prod_{k=1}^{n} \frac{9k}{9k+1} = \left[\prod_{k=1}^{n} \frac{9k+1}{9k}\right]^{-1}$, and show that $\lim_{n\to\infty} \prod_{k=1}^{n} \frac{9k+1}{9k} = \infty$.) Let A_i = event that ball no.i remains in the urn at 12 o'clock. Similar reasoning gives us $P(A_i) = 0, i = 2, 3, \ldots$.

(Note that for $i = 11, 12, \ldots, 20, P(A_i) = \prod_{n=2}^{\infty} \frac{9n}{9n+1}$.)

It is now easily seen that the probability that the urn is not empty at 12 o'clock $= P(\bigcup_{i=1}^{\infty} A_i) = 0$. This is the required result.