Bernoulli Brothers

Jacob I and Johann I: A Pair of Giant Mathematicians

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Bernoulli family is a family of stalwart mathematicians. In this article, major contributions and some of the problems posed by Jacob I and Johann I, two of the outstanding mathematicians of this family, have been discussed briefly.

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Introduction

In the long, glorious history of mathematics, we come across various mathematical geniuses who have enriched the subject by their significant contributions. But if we consider the contribution of a single family in this field, then the Bernoulli family outshines all others, both in terms of versatility and the number of mathematicians it produced. A father and son combination is not very rare in the field of mathematics. Farcas Bolyai (1775-1856) and Janos Bolyai (1802-1860), George David Birkhoff (1884-1944) and Garrett Birkhoff, Emil Artin and Michael Artin, Elie Cartan and Henri Cartan, etc. belong to this class. As father-daughter pair of mathematicians we find Theon and Hypatia in Greece, Bhaskaracharya II (1114-1185) and Lilavati in India during ancient times and Max Noether and Emmy Noether in more modern times. Even if we consider the contribution of a particular family, we find the Riccati family in Italy which produced at least three mathematicians. But, as mathematicians, the members of the Bernoulli family were much superior to those of the Riccati family.

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Bernoulli Family

The ancestors of the Bernoulli family originally lived in Holland. In the year 1583, they migrated to Switzerland and settled at Basel, situated on the bank of Rhine. Initially, members of the family flourished in business, but afterwards, its descendants

Box 1. Bernoulli Family Tree

(* Refers to mathematician members)

Nicholas (1623-1708)

* Jacob I (1654-1705)	* Nicholas I (1662-1716)	* Johann I (1667-1748)	
	* Nicholas II (1687-1759)		
	Nicholas III (1695-1726)	* Daniel I (1700-1782)	* Johann II (1710-1790)
	* Johann III (1746-1807)	Daniel II (1751-1834)	* Jacob II (1759-1789)
		Kristoff (1782-1863)	
		Johann Gustav (1811-1863)	

entered into the scientific arena. Three successive generations of this family produced as many as eight (See *Box* 1) mathematicians who dominated the mathematical world during the last part of the seventeenth century and the entire eighteenth century. It is not possible to discuss the contributions of all of them. We shall confine our discussions to two Bernoulli brothers only, viz. Jacob I and Johann I, and some famous mathematical problems posed by them. As we are dealing with only two members of the family, we shall refer to them as Jacob and Johann throughout this article.

Catenary Problem

Catenary problem was one of those famous problems which generated a lot of enthusiasm among mathematicians after the invention of calculus. The word 'Catenary' comes from the Latin word 'Catena', which means chain. Jacob Bernoulli (see *Box* 2) was the originator of the catenary problem. In the May 1690 issue of '*Acta eruditorum*'[1] Jacob wrote – "And now let this problem be proposed: To find the curve assumed by a loose string hung freely from two fixed points". Jacob further assumed that the string is flexible and of uniform cross-section. Earlier, Galileo (1564-1642) had dealt with this problem and wrongly

[1] '*Acta eruditorum*', the oldest scientific journal began appearing since 1682 under the supervision of Gottfried Leibniz (1646-1716).

Box 2. Jacob Bernoulli

Jacob Bernoulli, also known as Jacques and James Bernoulli, was born in Basel on December 27, 1654. In 1671, he obtained a post-graduate degree in philosophy from Basel University. He went against his father's suggestion to study mathematics, physics and astronomy. In this connection he said – "Against my father's will I study the stars". During 1681-82 he travelled to Holland and England and met famous scientific minds like Robert Hooke (1635-1703) and Robert Boyle (1627-1691). In 1687, he became a Professor of Mathematics at the University of Basel and remained in that post till his death, on August 16, 1705.

concluded that the curve would be a parabola. In the year 1646, Christian Huygens (1629-1695), then at the age of sixteen, proved that the curve cannot be a parabola. One year after the appearance of the problem, in the June 1691 issue of '*Acta eruditorum*', three correct solutions of the problem were published. The three solutions came from Huygens, Leibniz and Johann Bernoulli, brother of Jacob. Though they attacked the problem from three different points of view, all of them concluded that the curve would be a catenary (*Figure* 1). Jacob Bernoulli himself failed to solve the problem. The Cartesian equation of a catenary is $y = (e^{ax} + e^{-ax}/2a)$, where a is a constant whose value depends on the mass per unit length of the chain and the tension of suspension. In real life, a nice example of a catenary is the Gateway Arch (See *Box* 3).

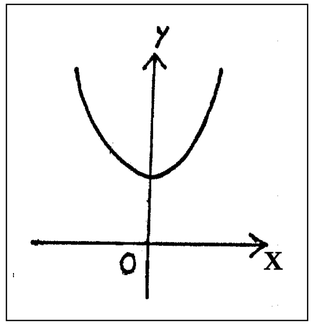


Figure 1. Catenary.

Logarithmic Spiral and Jacob Bernoulli

The logarithmic spiral (*Figure* 2) was one of the curves which drew the attention of mathematicians after analytical geometry was introduced by Rene Déscartes (1596-1650) in the year 1637. This curve was Jacob's favourite. At that time, the polar equation of the curve was written as $\ln r = a\theta$,where a is a constant and 'ln' is natural logarithm (at that time known as 'hyperbolic logarithm'). Nowadays, the equation is written as $r = e^{a\theta}$, where θ is measured in radians (1 radian is approximately equal to 57 degrees) and 'a' denotes the rate of increase of the spiral. When $a > 0$, r increases with anti-clockwise rotation and we get a left-handed spiral (*Figure* 3), whereas if $a < 0$, then r decreases in anti-clockwise sense and we get a right-handed spiral (*Figure* 4).

Box 3. Gateway Arch

The Gateway Arch was built in the year 1965 under the supervision of architect Eero Saarinon. It is situated at St. Louis in Missouri. Built in the shape of an inverted catenary, its highest point is at a height of 630 ft. on the banks of the river Missouri.

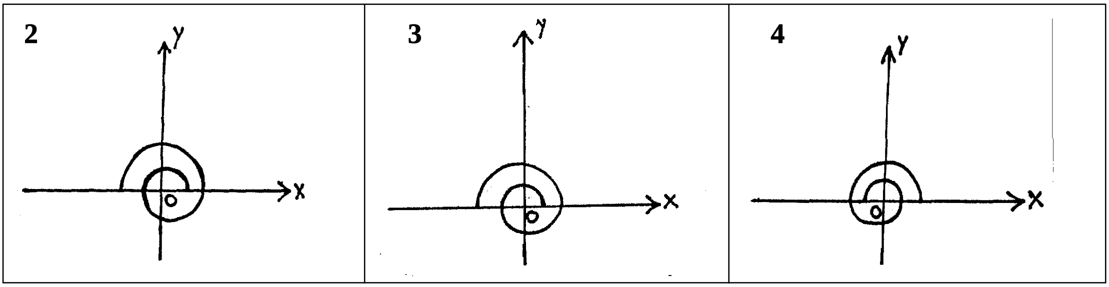


Figure 2 (left). Logarithmic spiral.
Figure 3(center). Left-handed spiral.
Figure 4 (right). Right-handed spiral.

Properties of a Logarithmic Spiral

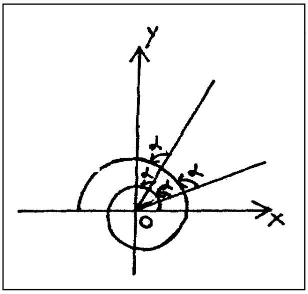
1. Any straight line drawn through the pole cuts a logarithmic spiral at the same angle (*Figure* 5). For this reason, this curve is also known as an *equiangular spiral*.

2. The property of a logarithmic spiral which surprised Jacob Bernoulli most is that it remains unaltered under many geometrical transformations. For example, typically a curve suffers a drastic change under inversion (see *Appendix* A where this and some other notions are recalled). But a logarithmic spiral generates another logarithmic spiral under inversion which is a mirror image of the original spiral, i.e. a left-handed spiral becomes a right-handed one and vice-versa.

3. The evolute of a logarithmic spiral is the very same spiral.

4. The pedal of a logarithmic spiral is the same spiral. The caustic of a logarithmic spiral is yet again the same spiral.

Figure 5. Equiangular spiral.



Jacob was so astonished by these findings about the logarithmic spiral that the christened the curve 'Spira mirabilis' (The marvellous spiral) and expressed his desire that a logarithmic spiral be engraved on his grave with the words 'Eadem mutata resurgo' (Though changed, I shall arise the same) written under it. This desire of Jacob was similar to that of Archimedes (see *Box* 4). Jacob's ambition was almost fulfilled. But, possibly from ignorance, the artisan engraved an Archimedean spiral (see Appendix A) on it instead of a logarithmic spiral. Even today, spectators at the Münster Cathedral in Basel can see the Archimedean spiral and the writing beneath it on the grave.

Box 4. Last Desire of Archimedes

If a rectangular area like *Figure* 6 is rotated about *AD* (where *AD=BD=DC), then* we get a right circular cone inscribed inside a hemisphere which is again inscribed within a right circular cylinder. Archimedes proved that the ratios of the volumes of the cone, the hemisphere and the cylinder will be a 1:2:3. This discovery pleased Archimedes so much that he desired that a figure of a cone inscribed in a hemisphere further inscribed in a cylinder be engraved in his grave. His desire was fulfilled. Since then the above theorem has gained popularity as the 'Tombstone theorem'.

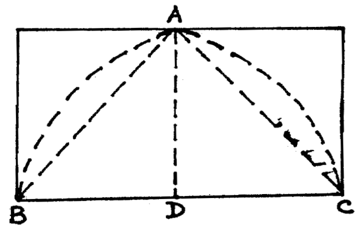


Figure 6

Brachistochrone Problem and Johann Bernoulli

The brachistochrone problem, one of the famous problems in mathematics, was posed by Johann Bernoulli (See *Box* 5), the youngest brother of Jacob Bernoulli. In the June 1696 issue of '*Acta eruditorum*' Johann posed a problem titled 'A new problem that mathematicians are invited to solve'. The problem goes like this: Given two points, to find the curve joining them along which a particle starting from rest slides under constant gravitational force in the least possible time. More generally, a brachistochrone is a curve joining two points along which a particle moves under the action of a given conservative force field in the least possible time. This problem has since then become famous in the history of mathematics as the 'brachisto-

Box 5. Johann Bernoulli

Johann Bernoulli, also known as Jean or John Bernoulli, was born at Basel on August 6, 1667. In 1683 he began his study of medicine at Basel University. Afterwards, following the footsteps of his elder brother he also studied mathematics in collaboration with Jacob, against their father's will. The two brothers, by sheer perseverance, learnt calculus which was then newly discovered by Leibniz, and started teaching calculus to contemporary renowned mathematicians including Guillaume Francoise Antoine de L'Hospital (1661-1704) who later wrote '*Analyse des infinitesimal petits*' (Analysis of the infinitely small) – the first textbook on calculus. In 1695, Johann was appointed as professor of mathematics at the University of Hale. In 1697 he joined Gröningen University. After the demise of Jacob, Johann took his elder brother's post at Basel University. He thrice won the coveted prize of the 'Paris Academy of Sciences'. Johann passed away on the first day of the year 1748.

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chrone problem'. In Greek, brachistochrone means shortest time (brachisto = shortest, chrone = time). Like the 'catenary problem', this problem was tackled earlier by Galileo, who concluded wrongly that the required curve is a circle. Johann invited "the shrewdest mathematicians in all the world" to solve this problem and allowed six months time for it. Five correct solutions were received. Of these, four solutions came from renowned mathematicians, viz. Leibniz, the Bernoulli brothers and L'Hospital whereas the fifth solution came from an anonymous mathematician. But, seeing the style of tackling the problem, experts could readily recognise that the anonymous problem solver was none other than Sir Isaac Newton (1642-1727) and Johann commented on it – "Ex ungue Leonem" (tell a lion by its claw). Afterwards Newton admitted that he solved the problem by thinking relentlessly for twelve hours over it. In all the five solutions it was concluded that brachistochrone is a cycloid. If a circle rolls on a horizontal plane without sliding, then a point on the circumference of the circle generates a cycloid (see *Box* 6). In 1673, twenty-three years after the brachistochrone problem, Huygens discovered that cycloid is also the solution of the 'tautochrone problem'. In the 'tautochrone problem', it is required to find a curve so that a particle, moving under gravity, will reach a given point on the curve in equal time irrespective of its initial position on the curve. Knowing that the cycloid is the solution of both the 'brachistochrone problem' and the 'tautochrone problem', Johann was moved to write – "But

Box 6. Cycloid

Let a circle of radius a be moving along OX in the clockwise direction. Let O be the initial position of a point on its circumference, where O is the point of intersection of two mutually perpendicular axes OX and OY (*Figure* 7), and let C be the centre of the circle. Due to rolling of the circle, the point at O reaches A at a later time where $\angle ACM = \theta$, then the parametric equation of the cycloid is

$x = a(\theta - \sin\theta)$
$y = a(1 - \cos\theta)$.

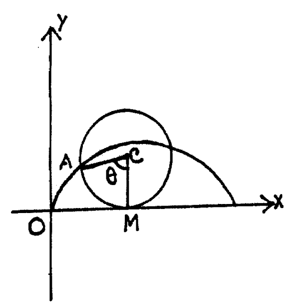

Figure 7. Cycloid.

you will be petrified with astonishment when I say that exactly this same cycloid, the tautochrone of Huygens, is the brachistochrone we are seeking". It should be mentioned here that brachistochrones and tautochrones are found to be identical for only three types of conservative force fields, viz. constant gravity field near the surface of the earth, and attractive and repulsive Hooke's type of force fields. In a constant force field the required curve is a cycloid, whereas it is a hypocycloid for an attractive type of Hooke's force and an epicycloid for a repulsive force field, the magnitude of the force being proportional to the distance from a fixed point. The solution of the brachistochrone problem gave birth to a new branch of mathematics viz. the 'calculus of variations'. In the above discussions, we have touched upon only a small fraction of the contributions of the Bernoulli brothers which is enough to show their mathematical talent (their other contributions are listed briefly in *Appendix* B). It is really surprising that though mathematics was not their original field of study (they were self-taught mathematicians), yet by the stroke of genius they opened up various new disciplines in mathematics.

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Appendix A.

Inversion: Inversion is a geometrical transformation in which a point (r, q) is mapped to the point $(1/r, q)$.

Evolute: The locus of the centre of curvature of different points on a curve is called evolute of that curve.

Pedal of a curve*:* The locus of the foot of the perpendiculars of the orthogonal projections drawn on the tangents of a curve from the pole is called the pedal of that curve.

Caustic*:* Let rays of light emanating from the pole be reflected by the curve. The envelope of the reflected rays is called the caustic of that curve.

Archimedean Spiral*:* The equation of an Archimedean spiral is $r = aq$. Perhaps it was Archimedes who discovered various properties of this curve and hence it bears his name.

Appendix B. Mathematical Works of Jacob Bernoulli

1. In 1682, Jacob did research work on comets, theory of gravitation, etc.

2. Since 1683, he was a regular contributor to '*Journal des scavans*' and '*Acta eruditorum*'. He published many new theorems on algebra in '*Acta eruditorum*'.

3. He was one of the pioneers of the mathematical theory of probability. His first paper on probability theory was published in 1685. The greatest contribution of Jacob Bernoulli is '*Ars conjectandi*', a book on the theory of probability, published posthumously in 1713.

4. In the year 1689, he published many research papers on the theory of infinite series. He was the first to think about the convergence of an infinite series and proved that the series $1/1^2 + 1/2^2 + 1/3^2 + ...$ is convergent.

5. Jacob was the first mathematician to use the term 'integral calculus'. In 1690, he published a paper in '*Acta eruditorum*' in which he, for the first time used this term. Earlier Leibniz had designated it by 'Calculus summatorium'.

6. In 1692, he found out the evolutes of the logarithmic spiral and parabola.

7. In 1696, Jacob invented the method of solving the differential equation of the form $dy/dx + Py = Qy^n$ where P, Q are either functions of x or constants. This equation is well known to students of undergraduate classes as 'Bernoulli's form' of the linear differential equation. Jacob used differential equations in tackling various geometrical and mechanical problems.

8. He introduced the idea of polar coordinates in analytical geometry and used it for finding various properties of spiral shaped curves. Though the idea of polar coordinates was used earlier in some cases, Jacob was the first mathematician to use the idea extensively for finding equations and properties of various curves including spirals.

9. The relation between compound interest and lim $(1+1/n)^n$ was shown by Jacob for the first time. Using the binomial expansion of $(1+1/n)^n$, he showed that this limit lies between 2 and 3.

10. The curve 'Lemniscate of Bernoulli' (*Figure* 8) is named after Jacob Bernoulli. One can obtain the polar equation of the lemniscate by putting $n=2$ in the equation $r^n = a^n \cos n\theta$.

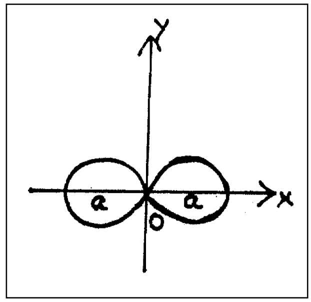

Figure 8. Lemniscate.

11. Jacob obtained the formulae for finding the radius of curvature in both Cartesian and polar coordinates.

Mathematical Contributions of Johann Bernoulli

1. Johann did extensive research on the function $y = x^x$.

2. He obtained the formula

$$\int_0^x y\,dx = xy - \frac{x^2}{2}\cdot\frac{dy}{dx} + \frac{x^3}{3}\cdot\frac{d^2y}{dx^2} - \cdots$$

using 'integration by parts'.

3. In 1695, he worked on the summation of the series

$$\sum (-1)^{k-1} k^{-1} \binom{n}{k}.$$

4. In 1699, Johann published a paper on the cycloid.

5. Johann made significant contributions to the field of elasticity and fluid dynamics and in 1738 published a book '*Hydraulica*'.

Suggested Reading

[1] Eli Maor, *The Story of a Number*, Universities Press, 1999.

[2] V M Tikhomirov, *Stories about Maxima and Minima*, Universities Press, 1998.

[3] Stuart Hollingdale, *Makers of Mathematics*, Penguin Books, 1989.

[4] Howard Eves, The Bernoulli family of Mathematicians, *Mathematics Today*, October 1987.

[5] N C Rana and P S Joag, *Classical Mechanics*, Tata-McGraw Hill Co., 1998 (fifth reprint).

[6] Pradip Kr. Majumder, *Ganitsastre Swaranio Jara*, Vol. 1 (in Bengali), Asiatic Society (Calcutta), 1995.

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