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2. A Route to Square Roots and Cube Roots

A Case Study in Serendipity

Introduction

Earlier this year, a student of mine reported to me an empirical discovery that he had made: if x is a rational approximation to $\sqrt{2}$, then so is

$$\frac{x+2}{x+1} \quad (= y,\ \text{say}), \tag{1}$$

and in general it is a better one; for instance, if $x = 4/3$ then $y = 10/7$; and if $x = 3/2$ then $y = 7/5$. Observe that $|y - \sqrt{2}| < |x - \sqrt{2}|$ in each case. The discovery came in response to a question posed earlier to the class, concerning methods for finding rational approximations to square roots. The topic being covered was 'Simplification and Rationalization of Irrational Surds', and I had asked for the best rational approximation to $\sqrt{2}$ with denominator less than 100, with techniques such as the following in mind: raise $(\sqrt{2} - 1)$ to a high positive integral power and express the answer in the form $A\sqrt{2} - B$, where A and B are integers; then $\sqrt{2} \approx B/A$. (For instance, raising it to power 10 yields the approximation

$$\sqrt{2} \approx \frac{3363}{2378}, \tag{2}$$

giving 7-place accuracy.) This was duly found (as I had hoped), but observation (1) reported above came as a pleasant surprise. It seems likely that the discovery came about as a result of the following train of ideas. Let $(\sqrt{2}-1)^k$ be expanded via the binomial theorem and written as $A_k\sqrt{2}-B_k$, where A_k and B_k are integers. Then

$$\begin{aligned}(\sqrt{2} - 1)^{k+1} &= (\sqrt{2} - 1)\left(A_k\sqrt{2} - B_k\right) \\ &= -\left(A_k + B_k\right)\sqrt{2} + \left(2A_k + B_k\right), \end{aligned} \tag{3}$$

showing that $A_{k+1} = -(A_k + B_k)$ and $B_{k+1} = -(2A_k + B_k)$. Writing $x = B_k/A_k$ and $y = B_{k+1}/A_{k+1}$, we find that

$$y = \frac{x+2}{x+1}, \tag{4}$$

and since y is clearly better than x as an approximation to $\sqrt{2}$, this suggests observation (1) made above.

The discovery naturally sparked off an investigation into the nature of this and other related approximation schemes, and as the study progressed an unexpected route to the Newton-Raphson scheme for computing square roots lay revealed. What follows is an account of the investigation.

Brief Resume of Iterative Function Theory

As the proposed scheme is iterative in nature, it is in place to briefly review the basic facts governing such schemes. Let $I \subset \mathbf{R}$ be an open interval (here $\mathbf{R}$ denotes the set of real numbers) and let $f : I \to I$ be a differentiable function. Suppose that f has a fixed point $\alpha \in I$; that is, $f(\alpha) = \alpha$. Suppose moreover that there exists a constant k with $0 < k < 1$ and an open interval $J \subset I$ containing α but no other fixed point of f, such that for all $x \in J$, $x \neq \alpha$, we have the inequality

$$\left| \frac{f(x) - f(\alpha)}{x - \alpha} \right| < k. \tag{5}$$

Then f is termed a *contraction mapping* with respect to α (it 'pulls' x towards α), and the fixed point α is termed an *attracting fixed point*. When this happens, the sequence

$$x,\ f(x),\ f(f(x)),\ f(f(f(x))),\ \ldots \tag{6}$$

converges to α for all $x \in J$. It can be shown that (5) is equivalent to the requirement that $|f'(\alpha)| < 1$. The value of $|f'(\alpha)|$ in this case gives an approximate indication of the rate of convergence; the smaller the value, the faster the rate of convergence. If the inequality is reversed ($|f'(\alpha)| > 1$), then α is a *repelling fixed point*, and in this case convergence will not take place for any open interval $J \subset I$, however small (unless of course $x = \alpha$). See Smital (Suggested Reading) for further details.

An Iterative Approximation Scheme For Square Roots

Let $a \neq 1$ be a positive rational number, and let a function $f : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ (here $\mathbf{R}^+$ denotes the set of positive real numbers) be defined by

$$f(x) = \frac{x+a}{x+1}. \tag{7}$$

Our plan is to use the iterates of f to estimate $\sqrt{a}$, which we take to be irrational. We select a starting value $x_0 > 0$ and construct the sequence $\{x_n\}_{n \geq 1}$ via the following rule :

$$x_n = f(x_{n-1}), \qquad n > 0. \tag{8}$$

Our hope is that the sequence will converge to $\sqrt{a}$, at least for some restricted set of values for x_0.

Note that in the domain of definition, the only fixed point of f is $x = \sqrt{a}$; for

$$\frac{x+a}{x+1} = x \;\Rightarrow\; x^2 = a \;\Rightarrow\; x = \sqrt{a}. \tag{9}$$

Moreover the fixed point is an attracting one, because

$$f'(x) = -\frac{a-1}{(x+1)^2}, \qquad |f'(\sqrt{a})| = \left|\frac{\sqrt{a}-1}{\sqrt{a}+1}\right| < 1. \tag{10}$$

As per the theory outlined above, this leads us to expect that for any starting number $x_0 > 0$, the sequence $\{x_n\}_{n \geq 1}$ will converge to $\sqrt{a}$.

The conclusion is true and can be proved as follows. We observe firstly that $f(x)$ is monotone throughout its domain:

as x goes from 0 to ∞, $f(x)$ goes monotonically from a to 1. Next,

$$f(x)-\sqrt{a}=\frac{(x-\sqrt{a})(1-\sqrt{a})}{x+1}, \qquad f(x)-x=\frac{a-x^2}{x+1}. \tag{11}$$

So for $a<1$, $f(x)$ lies between x and $\sqrt{a}$, whereas for $a>1$, $f(x)$ and x lie on opposite sides of $\sqrt{a}$. Now consider the second iterate of f, namely $f_2 = f \circ f$. A quick computation shows that

$$f_2(x)=\frac{x(a+1)+2a}{2x+a+1}. \tag{12}$$

It is easy to check the following:

$$f_2(x)-\sqrt{a}=\frac{(\sqrt{a}-1)^2(x-\sqrt{a})}{2x+a+1},$$

$$f_2(x)-x=\frac{2(a-x^2)}{2x+a+1}. \tag{13}$$

From this we deduce that $f_2(x)$ always lies between x and $\sqrt{a}$; that is,

$$0<x<\sqrt{a} \Rightarrow x<f_2(x)<\sqrt{a},$$

$$x>\sqrt{a} \Rightarrow x>f_2(x)>\sqrt{a}. \tag{14}$$

So f_2 is 'better behaved' than f, and it becomes more convenient to work with f_2 than with f. From (13), we obtain, for $x>0$, $x\neq\sqrt{a}$:

$$0<\frac{f_2(x)-\sqrt{a}}{x-\sqrt{a}}=\frac{(\sqrt{a}-1)^2}{2x+(a+1)}<\frac{(\sqrt{a}-1)^2}{a+1}=$$

$$1-\frac{2\sqrt{a}}{a+1}<1. \tag{15}$$

So f_2 is a contraction mapping with respect to $\sqrt{a}$: that is, for any $x>0$, $f_2(x)$ is better than x as an approximation to $\sqrt{a}$. Therefore the sequence

$$|x_0-\sqrt{a}|,\ |x_2-\sqrt{a}|,\ |x_4-\sqrt{a}|,\ |x_6-\sqrt{a}|,\ \ldots \tag{16}$$

decreases monotonically, and as its terms are non-negative it converges to a limit. The limit must be 0, because the only fixed point of f_2 is $\sqrt{a}$. Thus the sequence

$$x_0,\ x_2,\ x_4,\ x_6,\ \ldots \tag{17}$$

converges to $\sqrt{a}$. We find similarly that the sequence

$$x_1,\ x_3,\ x_5,\ x_7,\ \ldots \tag{18}$$

converges to $\sqrt{a}$. The assertion made above thus follows: that $x_n \to \sqrt{a}$ as $n \to \infty$, for any value of $x_0 > 0$.

Example: The sequence generated with $a = 4$ and $x_0 = 1$ is the following:

$$1,\ \frac{5}{2},\ \frac{13}{7},\ \frac{41}{20},\ \frac{121}{61},\ \frac{365}{182},\ \frac{1093}{547},\ \ldots, \tag{19}$$

and the sequence of absolute errors, $|x_n - \sqrt{a}|$, is:

$$1,\ \frac{1}{2},\ \frac{1}{7},\ \frac{1}{20},\ \frac{1}{61},\ \frac{1}{182},\ \frac{1}{547},\ \ldots. \tag{20}$$

Note that the sequence is approximately a geometric progression with common ratio 1/3. This is consistent with the fact that

$$|f'(\sqrt{a})| = \frac{2-1}{2+1} = \frac{1}{3}. \tag{21}$$

A function such as $(x+a)/(x+1)$ is termed a *fractional linear function*. Within the class of such functions, can we find one that produces speedier convergence to $\sqrt{a}$ than $(x+a)/(x+1)$? Note that the coefficients we use in such a scheme must be rational, else the procedure lacks meaning. We proceed to take a closer look at this question.

The formula $(x+a)/(x+1)$ can be stumbled upon in the following heuristic manner:

$$x^2 = a \Rightarrow x^2 + x = x + a \Rightarrow x(x+1) =$$

$$x + a \Rightarrow x = \frac{x+a}{x+1}. \qquad (22)$$

Following this lead, we try the following idea:

$$x^2 = a \Rightarrow x^2 + kx = kx + a \Rightarrow$$

$$x(x+k) = kx + a \Rightarrow x = \frac{kx+a}{x+k}, \qquad (23)$$

which holds for any real number k (to avoid complications, we take $k > 0$).

Let $g(x)$ denote the function $(kx + a)/(x + k)$, defined for $x \in \mathbf{R}^+$. The plan is to use g for our iterative scheme, so we first select an appropriate value for k. Now,

$$g'(x) = \frac{k^2 - a}{(x+k)^2}, \qquad (24)$$

and this yields:

$$|g'(\sqrt{a})| = \left|\frac{k^2 - a}{(k + \sqrt{a})^2}\right| = \left|\frac{k - \sqrt{a}}{k + \sqrt{a}}\right| < 1. \qquad (25)$$

So convergence to $\sqrt{a}$ works out as earlier, and by choosing k close to $\sqrt{a}$, we can generate fairly fast schemes—the closer the better. (We cannot have $k = \sqrt{a}$ as k must be rational whereas $\sqrt{a}$ is irrational.)

Example: Number crunching helps at this point. Let $a = 10$; possible rational approximations to $\sqrt{a}$ are 3 and 19/6. We obtain the following iterative schemes, corresponding to the choices $k = 1$, $k = 3$ and $k = 19/6$ respectively:

$$x \longrightarrow \frac{x+10}{x+1}, \quad x \longrightarrow \frac{3x+10}{x+3}, \quad x \longrightarrow \frac{19x+60}{6x+19}. \qquad (26)$$

Of these, the third one provides impressive results. For instance, consider the sequence $\{x_n\}_{n\geq 1}$ generated by this scheme, starting with $x_0 = 3$:

$$x_0 = 3, \ x_1 = \frac{117}{37}, \ x_2 = \frac{4443}{1405},$$

$$x_3 = \frac{168717}{53353},\ x_4 = \frac{6406803}{2026009},\ \ldots . \tag{27}$$

The error term, $e_n = \left|x_n - \sqrt{10}\right|$, takes the following approximate values:

$$e_1 \approx 1.1 \times 10^{-4},\ e_2 \approx 8.0 \times 10^{-8},$$
$$e_3 \approx 5.6 \times 10^{-11},\ e_4 \approx 3.8 \times 10^{-14},\ \ldots . \tag{28}$$

Thus we have achieved thirteen-place accuracy within four steps—impressive enough! Observe that the error sequence is approximately a geometric progression with common ratio roughly 1/1400, and that

$$\left|\frac{19 - 6\sqrt{10}}{19 + 6\sqrt{10}}\right| = \left(19 - 6\sqrt{10}\right)^2 \approx \frac{1}{1442}. \tag{29}$$

It should be clear now that if $\sqrt{a}$ is irrational, then there is no rational number k that gives a 'best possible' fractional linear approximation scheme $x \to (kx+a)/(x+k)$, because one can select rational numbers k arbitrarily close to $\sqrt{a}$.

Stumbling upon the Newton-Raphson Scheme

It is at this point that an insight strikes us. We have seen how by choosing k close to $\sqrt{a}$ (the closer the better), we obtain increasingly better schemes of the type $x \to (kx + a)/(x+k)$ for obtaining rational approximations to $\sqrt{a}$, and that by iterating the scheme on some appropriately chosen starting value, we obtain increasingly better rational approximations to $\sqrt{a}$. *Why not then use these approximations themselves as candidates for k?* That is, *for k why not substitute x itself?* With this modification, the scheme changes to the following (it is no longer fractional linear):

$$x \to \frac{x \cdot x + a}{x + x}, \text{that is}, x \to \frac{x + a/x}{2}; \tag{30}$$

and lo and behold we have the Newton-Raphson scheme for computing square roots. On our journey to strange lands we have stumbled upon an old friend ....

The Newton-Raphson scheme is known to be tremendously fast in its rate of convergence. Considering the heuristic logic used above, it is not hard to guess that it will beat any fractional linear scheme. Consider the following data, obtained with $a = 10$ and $x_0 = 3$:

$$x_1 = 3,\ x_2 = \frac{19}{6},\ x_3 = \frac{721}{228},\ x_4 = \frac{1039681}{328776}, \tag{31}$$

$$x_5 = \frac{2161873163521}{683644320912},$$
$$x_6 = \frac{9347391150304592810234881}{2955904621546382351702304}, \ \ldots. \tag{32}$$

The error term $e_n = \left|x_n - \sqrt{10}\right|$ takes the following approximate values:

$$e_1 \approx 1.6 \times 10^{-1},\ e_2 \approx 4.4 \times 10^{-3},\ e_3 \approx 3.0 \times 10^{-6},$$
$$e_4 \approx 1.5 \times 10^{-12},\ e_5 \approx 3.4 \times 10^{-25}. \tag{33}$$

Note the rapidity with which the error term shrinks to zero!

(*Explanation:* With $h(x) = (x+a/x)/2$, we have $h'(\sqrt{a}) = 0$. With fractional linear approximation schemes, using rational coefficients, a derivative of 0 is not possible. A zero value for the derivative implies that the error term in the Newton-Raphson scheme shrinks roughly at a quadratic rate; each term is (roughly) the square of the preceding term. This is best understood in terms of the Taylor series expansion of $h(x)$ about $x = \sqrt{a}$: the linear term is absent, by virtue of the zero derivative, so the rate of convergence is controlled by the second-degree term.)

Cube Roots

Armed with this insight, we now take on the corresponding problem involving cube roots. In this case, no fractional linear scheme will do, for with rational p, q, r, s the mapping

$$x \rightarrow \frac{px+q}{rx+s} \tag{34}$$

cannot have a fixed point involving an irrational cube root. So we need to examine other types of schemes, and one possibility is to use fractional quadratic schemes.

Let $a > 0$; we need to estimate the value of $a^{1/3}$. Assume that a is rational while $a^{1/3}$ is irrational. Proceeding heuristically as we did earlier, we obtain the following:

for any rational number $k > 0$,

$$x^3 = a \Rightarrow x^3 + kx^2 = kx^2 + a \Rightarrow x = \frac{kx^2 + a}{x^2 + kx}. \tag{35}$$

Let f be the function defined thus on $\mathbf{R}^+$:

$$f(x) = \frac{kx^2 + a}{x^2 + kx}. \tag{36}$$

The only fixed point of f is $x = a^{1/3}$, as is easily verified. Next,

$$f'(x) = \frac{k^2x^2 - 2ax - ak}{x^2(x+k)^2}. \tag{37}$$

We must choose k so that $f'\left(a^{1/3}\right) \approx 0$. This is accomplished by ensuring that

$$k^2a^{2/3} - ak - 2a^{2/3} \approx 0. \tag{38}$$

Solving the quadratic equation for k, we find:

$$k \approx 2a^{1/3} \text{ or } -a^{1/3}. \tag{39}$$

We discard the negative root. The conclusion is, then, that the iterative scheme

$$x \rightarrow \frac{kx^2 + a}{x^2 + kx}, \tag{40}$$

with $k \approx 2a^{1/3}$ (k rational) should provide us with good rational approximations to $a^{1/3}$.

Numerical experimentation can be attempted at this point. For $a = 2$, we know that $5/4$ is close to $a^{1/3}$, so we choose $k = 5/2$. The scheme is therefore:

$$x \rightarrow \frac{5x^2 + 4}{x(2x + 5)}. \qquad (41)$$

With a starting value of $x_0 = 1$, the convergents obtained are the following:

$$x_1 = \frac{9}{7},\ x_2 = \frac{601}{477},$$
$$x_3 = \frac{2716121}{2155787},\ x_4 = \frac{55476236790681}{44031518284417},\ \ldots, \qquad (42)$$

and $\left|2^{1/3} - x_4\right| \approx 10^{-9}$. This can be considered as fairly satisfactory.

As for the case of square roots, if we limit ourselves to rational values for k, there is no best possible fractional quadratic scheme. Also, just as we did earlier, we can use the updated value of x itself to select k; that is, choose $k = 2x$. The scheme now becomes:

$$x \rightarrow \frac{2x \cdot x^2 + a}{x(x + 2x)}, \qquad \text{that is,} \qquad x \rightarrow \frac{2x^3 + a}{3x^2}, \qquad (43)$$

and this is exactly the Newton-Raphson scheme for cube roots.

Further Investigations

In terms of pedagogy, one can envisage several investigations in these general directions. Here is one possibility.

Let $x \rightarrow f(x)$, $x \rightarrow g(x)$ be two iterative schemes that converge (for suitable starting values) to the same fixed point α. Let $f'(\alpha) = \gamma$, $g'(\alpha) = \delta$. Suppose that γ and δ are non-zero and distinct. Let

$$h(x) = \frac{\gamma g(x) - \delta f(x)}{\gamma - \delta}. \qquad (44)$$

Suggested Reading

- **Jaroslav Smital. *On Functions and Functional Analysis.* Adam Hilger, 1988.**

Then $h(\alpha) = \alpha$ and $h'(\alpha) = 0$. Now let γ', δ' be distinct rational numbers close to γ, δ respectively. It seems reasonable to expect that the scheme

$$x \rightarrow \frac{\gamma' g(x) - \delta' f(x)}{\gamma' - \delta'} \qquad (45)$$

will converge rapidly. In general the question of how one can produce a good approximation scheme from two not-so-good approximation schemes seems a worthwhile one to take up for investigation by student groups.

Conclusion

As every mathematician and every mathematics teacher knows, one of the great pleasures in doing mathematics is the joy of stumbling upon hidden connections serendipitously. The above investigation would seem to bear this out in full measure.

"It's the 'jumping-gene' theory. That's why you hear through your nose and breathe through your ears."

From *Gene Antics*