Mathematics in Engineering – Part II

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In this part of the article I begin with somewhat more advanced mathematics as compared to the first part. I then move on to discuss insights, hierarchies of models, levels of approximation, models that are plain bad, and finally the role of truth.

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Beyond Linear Algebra

There is more to mathematics in engineering than the matrix calculations described in the first part of this article. For flavour, I will discuss two topics: one computational or numerical, and the other analytical or symbolic.

Nonlinearity

Nonlinearity is everywhere. Yet much of undergraduate engineering education in India concentrates exclusively on linear problems. The problems discussed in the first part were all linear as well.

Reproduced from *Current Science*, Vol.88, No.3, 10th Feb., pp.405-414, 2005.

Part I. *Resonance*, Vol.10, No.4, pp.51-64.

We now consider a nonlinear problem, and carry out some calculations. The trivial example discussed below hardly matches the many difficult nonlinear problems that engineers, scientists and mathematicians solve, but highlights some key points: local approximation by a simpler problem, small steps, and iteration or sequential refinement.

Consider

$$Ax + (x^T x)Bx = b,$$

Keywords

where A and B are $n \times n$ matrices, b is $n \times 1$, and the unknown x is $n \times 1$ as well. The above problem is clearly a departure from $Ax = b$, and is chosen for ease of presentation: it represents no particular physical application.

One way to solve this problem is iterative, using Newton's method. Say at some stage our estimate of the solution is x_k; our next and improved estimate $x_{k+1} = x_k + \Delta x$ (say), and should ideally satisfy the original equation

$$A(x_k+\Delta x)+(x_k^T x_k+2x_k^T\Delta x+\Delta x^T\Delta x)B(x_k+\Delta x) = b.$$

Simplifying matters by ignoring terms proportional to $\|\Delta x\|^2$ and smaller, we obtain

$$x_{k+1} = x_k - \left(A + 2Bx_k x_k^T + x_k^T x_k B\right)^{-1}$$

$$\left\{\left(A + x_k^T x_k B\right) x_k - b\right\}. \quad (1)$$

The above iterative procedure can be begun with an initial guess x_0; and if x_0 is sufficiently close to the correct solution, then the iteration will converge (under some technical conditions which usually hold)[1].

[1] More generally and formally, to solve $g(x) = 0$, we let

$$x_{k+1} = x_k - [Dg(x_k)]^{-1}\; g(x_k),$$

where $[Dg(x_k)]$ is the Jacobian matrix of partial derivatives of the vector $g(x)$ with respect to the elements of x, evaluated at x_k.

As a particular example, taking

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix},\ B = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \text{ and } b = \left\{ \begin{matrix} 1 \\ 0 \end{matrix} \right\},$$

and starting with

$$x_0 = \{1, 0\}^T,$$

we find the solution converges rapidly to

$$x = \{0.1502, 0.3623\}^T.$$

In general and large nonlinear problems, there may be multiple or no solutions; many guesses for x_0 may fail to produce convergence; and there may be severe difficulties with estimating or even repeatedly using the Jacobian $[Dg(x_k)]$. However, methods based on linearization (exact or approximate) and the above iterative scheme (possibly with simplifying approximations) are powerful and widely used.

Sometimes a good guess for x_0 might be so hard to find that we are reduced to using *continuation methods*. These involve solving a simpler problem; using its solution as the initial guess for a slightly changed problem; and using *its* solution for a further changed problem; and so on, until the simpler problem is changed completely to the original problem. These and other tricks, many of them tailored to suit specific applications, are an important part of the overall attack on nonlinear problems.

Asymptotic Approximations

Asymptotic approximations or perturbation approximations (see, e.g., [1,2]) can be useful for certain engineering problems. I will give an example of such approximations, both to display them in their own right as well as enable comparison with some non-asymptotic approximations that follow.

Consider the equation

$$\epsilon x^5 + x^2 - 1 = 0,$$

where $0 < \epsilon \ll 1$ is small and known, and we seek x. The smallness of ϵ allows approximations. Noting that for $\epsilon = 0$ there is the solution $x = 1$, we can pose a series of the form

$$x = 1 + \sum_{k=1}^{\infty} a_k \epsilon^k.$$

By a routine procedure involving collecting and equating terms, we obtain

$$x = 1 - \frac{1}{2}\epsilon + \frac{9}{8}\epsilon^2 - \frac{7}{2}\epsilon^3 + \frac{1615}{128}\epsilon^4 + \cdots.$$

If we fix the number of terms upto those shown and take ϵ smaller and smaller, the error will eventually be proportional to ϵ^5. For the particular value $\epsilon = 0.05$, the actual root is 0.977441 while the asymptotic approximation agrees well, at 0.977454. This sort of perturbation expansion is called regular, as opposed to singular (more interesting for specialists).

Asymptotic techniques have an uncomfortable place in modern, computer-empowered engineering. Yet, many good engineers, even ones not mathematically inclined in the usual sense, think in a way that keeps track of the relative orders of magnitude of physical features and effects, using these in an intuitive way that loosely resembles asymptotic approximation.

Clever Insights

Clever insights abound in any mature subject where complex phenomena are viewed in simplified forms for better understanding (i.e., where truth is traded for understanding). These insights help us to both construct approximate solutions to seemingly difficult problems, as well as to suitably interpret numerical results from, say, a finite element code, and sometimes even to simply find a useful approximate solution to a mathematical problem which might otherwise be tedious. I will present here an example of each. These examples are drawn from within mechanical engineering due to my own limitations. They are, unavoidably, somewhat technical; however, at least the third example is free from equations.

Impact between Elastic Spheres

The first example involves the low velocity impact of two elastic spheres, for which Hertz's theory of contact was developed (see, e.g., [3] and references therein). The Hertzian contact analysis shows how a seemingly formidable problem can, by clever insights, be brought into a tractable yet meaningful form. In contrast, rigorous mathematical treatment of this problem raises many difficult issues.

Hertz's approach proceeds by noting that the size of the contact region is small compared to the size of the sphere; and that the impact occurs over a time that is long compared to the time period of vibrations in the sphere. Thus, impact proceeds as if contact between two point masses is mediated by a nonlinear spring. The behaviour of this nonlinear spring is given by

$$F = C\delta^{3/2}, \qquad (2)$$

where F is the force, δ is the compression, and C is a known constant depending on sphere material and size.

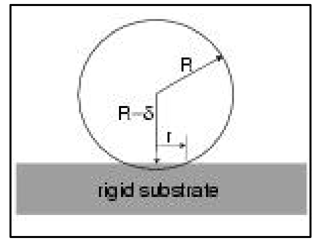


Figure 1. Contact between an elastic sphere and a rigid surface.

For those not interested in the finer details of Hertzian contact, here is an alternative and simple way to determine (or at least understand) equation (2). In *Figure* 1, an elastic sphere of radius R is pressed into a rigid surface by an amount δ (the undeformed sphere is drawn, to emphasize the amount of compression). The resulting contact patch has a radius of approximately r which satisfies, for small δ,

$$r = \sqrt{2R\delta}.$$

Now, assume that deformations are localized in a cylinder of diameter and height both equal to $2r$. The area is πr^2; and the strain is $\delta/(2r)$. F is E times strain times area,

$$F = E \times \frac{\delta}{2r} \times \pi r^2 = \frac{\pi}{\sqrt{2}} E\sqrt{R}\delta^{3/2},$$

which is off from the correct C of (2) by less than a factor of 2.

Heated Strip on a Rigid Substrate

Here is another example of nice insight into an engineering problem. Consider the system in *Figure* 2. The figure shows the cross section of a rectangular strip perfectly bonded to a rigid substrate. The strip is heated, so that it tends to expand but is restrained by the non-expanding substrate. We are interested in the interface

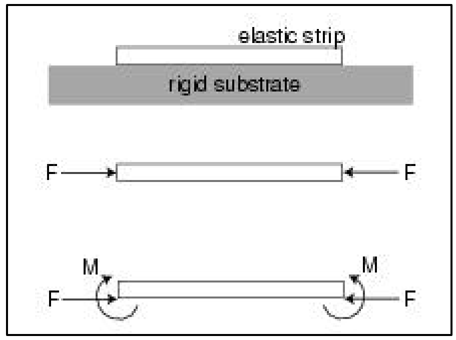


Figure 2. Heated strip on a rigid substrate.

forces and stresses generated. This idealized problem is also relevant to piezoelectric actuators used to control vibrations in structures.

Those acquainted with the theory of elasticity will anticipate stress singularities near the edges of the bond. An approximate analysis based on simplifying insights can proceed as follows.

Since the substrate is rigid, the width of the strip is basically unchanged. For the strip to retain its lateral dimensions while staying flat, the net force from the substrate should be as sketched in the figure (middle), for some compressive force (per unit length in the direction into the page) that exactly counteracts the tendency to expand due to heating. Since the problem is actually three dimensional, there are similar forces at all the edges of the strip.

Taking the material's Young's modulus to be E, Poisson's ratio to be ν, coefficient of thermal expansion to be α, the change in temperature to be ΔT, and the thickness of the strip to be h, we are led to conclude that

$$F = \frac{\alpha E h}{1-\nu}\,\Delta T.$$

We note, next, that the force F at the midplane of the strip (see figure (middle)), actually comes from the interface. The correct interface forces must therefore be the equivalent system shown in the figure (bottom), with

$$M = \frac{Fh}{2}.$$

For finer details, finer analysis is needed. With the above insight, from FEM simulations, we expect strong stress concentrations near the edges of the strip.

Testing a Sprung Bicycle

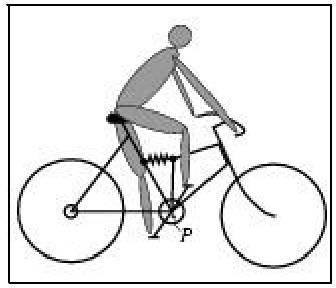


Figure 3. A sprung bicycle.

This example involves bicycles. For rider comfort, some bicycles frames are designed to flex. For example, two rigid parts of the frame might be connected at a pivot, along with a spring to resist relative rotation there. Such a sprung bicycle is sketched in *Figure* 3. The pivot is at P.

A potential buyer may wish to check two design criteria. When the bicycle goes over a bump, the frame should flex. Yet, when the rider pedals hard, the frame should *not* flex (the rider's power should go into acceleration, not deforming the spring even while on a smooth road). A quick and approximate evaluation of the second aspect might proceed as follows.

Consider an accelerating bicycle. The rider exerts forces to cause this acceleration. Assuming the rider is significantly heavier than the bicycle, and using D'Alembert's principle[2] , we need merely consider a static problem provided there is the correct horizontal backwards force acting approximately at the rider's center of mass. The rider ties a rope from his waist to a pillar and then, with rope taut, pushes on the pedal exactly as if accelerating hard. If the frame does not flex, the second criterion above is met and the design is good. Needless to say, a full analysis of the bicycle requires much additional work.

[2] Add $-m$ **a** to each point mass m, where **a** is its acceleration; and then treat the system as static, not dynamic.

Modeling

I once read on some engineer's door the following jocular lines:

> Engineers think theory approximates reality.
> Physicists think reality approximates theory.
> Mathematicians never make the connection.

Mathematicians can, and often do, work in a world that needs no contact with any physical reality. Physicists speak ardently about the fundamental and universal truths of nature. But in engineering, there is no special place for absolute truth. The flat-earth theory works if you want to build a bridge, and the point-mass-earth theory is good if you want to calculate the trajectory of the moon.

I am a mechanical engineer, and enjoy the subject of vibrations. So I will use vibrations to discuss a key idea. To that end, here is a crash course on vibration theory.

Mechanical Vibrations

See *Figure* 4(a). A block slides on a frictionless surface. It is restrained by a spring. If disturbed from its equilibrium, the block oscillates or vibrates, to and fro. It is

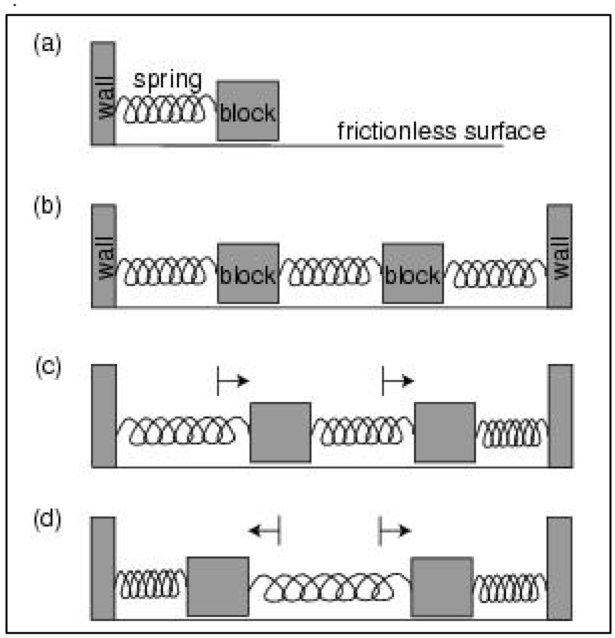
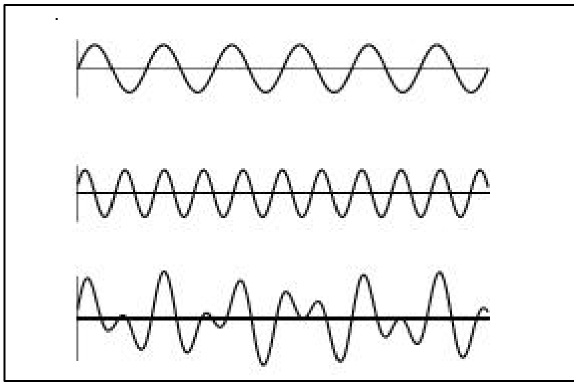


Figure 4. Vibration theory: normal modes.

intuitively clear that stiffening the spring will make the block oscillate faster (with a higher frequency, or smaller time period), while increasing the mass of the block will make it oscillate slower (lower frequency, larger time period). Now consider *Figure* 4(b), where two such blocks are held in place by three springs. A multitude of apparently complex motions are possible for this system, but there are two special motions, corresponding to *normal modes* of vibration. In one of these, the two blocks vibrate in synchrony (*in phase*), and the spring in the middle plays no role: the system in this mode is merely two copies of the first system. In the second mode, the blocks vibrate exactly in opposition (*out of phase*), which really is another face of synchrony. These two normal modes are indicated schematically in *Figures* 4(c,d). The frequency of vibration in the first mode is lower than that in the second mode, where the middle spring gets called into play as well.

These ideas are broadly applicable. If the two masses or the springs are not identical then there are still two normal modes, only they are not so easy to sketch from intuition and must be found through equations (some of that comes in the next subsection). If there are more blocks (i.e., more masses), then there are more normal modes. One mode per block. If we consider longitudinal vibrations of a rod, where mass and stiffness are merged into a continuum of material, then there are infinitely many normal modes associated with as many, and steadily increasing, frequencies; of these, in engineering, only the first several (2, 10, 50, or more, depending on the application) may be considered important.

An important idea remains: general motions are *linear combinations* of motions in the normal modes. See *Figure* 5. Let the first graph represent the displacement of the first mass in *Figure* 4(c), plotted against time. And let the second graph, showing more direction reversals within the same time (hence, higher frequency),



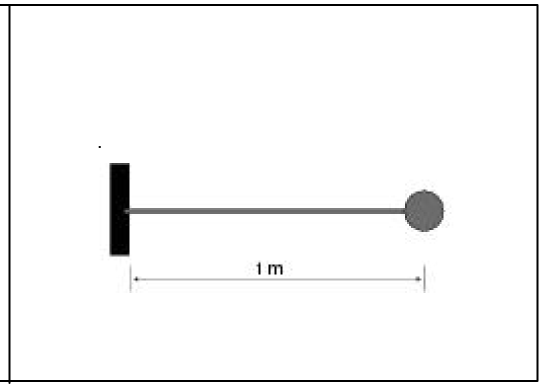


Figure 5. Vibration theory: general motions.

Figure 6. A cantilever beam with an end mass.

represent the displacement of the first mass in *Figure* 4(d). A general motion is the sum of two such motions, each of arbitrary amplitude or size, as sketched in the third graph of *Figure* 5. From this third graph, it is not immediately obvious that there are motions at two frequencies simply added up; that insight comes from mathematical abstraction, though it soon becomes part of the physical world of every vibration engineer.

Much remains: energy dissipation through friction, effects of forcing, resonance, nonlinearity, design issues, and practical troubleshooting. But the heart of vibration theory lies in its first step: linear combinations of normal modes.

Example

Now, some actual calculations. Consider the system sketched in *Figure* 6. A steel rod of diameter $d = 0.01$ m, length $L = 1$ m, density $\rho = 7800$ kg/m^3, and Young's modulus (material stiffness) $E = 210$ GPa is built into a strong wall at one end and has a steel ball of diameter $D = 0.1$ m at the other. What are the time periods for the normal modes of *lateral* (sideways) vibrations in this system?

The *first* normal mode (and hence its time period) can be approximated by ignoring the mass of the rod, and treating the sphere as a point mass. In this engineering approximation, the rod simply acts as a spring. By a routine calculation, we obtain the time period as

$$T = \frac{2\pi}{\sqrt{\frac{3EI}{mL^3}}} = 0.7221 \text{ sec},$$

where m is the mass at the end, and I is something called the second moment of area of the cross section (for details, see [4]).

We might approximately incorporate the beam's mass by adding an effective inertia to the point mass at the end. The mass m_r of the rod is (ignoring the overlap between the rod and the sphere)

$$m_r = \pi \frac{d^2}{4} L\rho = 0.6126 \text{ kg}.$$

Not all of this mass moves equally: it is only near its right end that the rod moves as much as the sphere; so (skipping a more complex calculation) we might add on about $m_r/3$, giving

$$T = 0.7221\sqrt{1 + \frac{m_r}{3m}} \text{ sec } = 0.7399 \text{ sec}.$$

A more careful, and presumably accurate, answer can be found by treating the rod as a cantilevered Euler-Bernoulli beam [4] with a point mass at the end. Then the governing PDE is

$$EIu_{xxxx} + \frac{m_r}{L} u_{tt} = 0,$$

with boundary conditions

$$u(0,t) = 0, \quad u_x(0,t) = 0 \quad \text{(cantilever end)}$$

along with

$$u_{xx}(L,t) = 0 \quad \text{(zero end moment);} \quad \text{and}$$

$$EIu_{xxx}(L,t) = mu_{tt}(L,t) \quad \text{(force from end mass)}.$$

Now we are no longer restricted to the first normal mode, and can seek "all" of them (within the approximations of *this* model, that is). The solution for the above, straightforward though tedious, is omitted here. The net result for the first time period is

$$T = 0.7347 \text{ sec},$$

a reasonable match with foregoing estimates.

More accurate estimates might be obtained using laborious calculations that incorporate the rotary inertia of the sphere, and then shear deformations and rotary inertia in the the beam itself; and, eventually running out of simple theories, using the finite element method.

Approximation

The above sequence of increasingly accurate models notwithstanding, errors in understanding of boundary conditions, specification of geometry, measurement of material constants and incorporation of effects like damping will always limit the numerical accuracy to which the time period of the vibrating rod can be found. And models of more complex systems with, e.g., plasticity, fracture, frictional contact and impact will be less accurate still.

In engineering there is no absolute truth, no perfect model, and no grand unified theory. Ever finer FE meshes do not take us ever closer to something infinitely pure and true. But there is no grief in this. The world is infinite and our minds are limited. Truth and understanding are in conflict: a little truth lost is the price for a little understanding gained. Consequently, engineering practice is filled with the good use of bad models.

Good use of Bad Models

Informed use of bad models to good effect is an honorable part of engineering. Engineers are not allowed the luxury of saying, for example (from [5]):

> ...... if sliding friction forces are present, ... we must exclude such systems from our formulation. The restriction is not unduly hampering, since the friction is essentially a macroscopic phenomenon.

In the above, my emphasis is on the second line and not the first: engineers learn how to analyze frictionless systems too, but must bear in mind that there will be situations where friction will need to be accounted for. Physicists, unlike engineers, can disregard "macroscopic" phenomena if and when they wish. Engineering, being rooted to the macroscopic world of friction, contact, impact, plasticity, rattling, looseness, dimensional imperfections, inexactly known environments and poorly understood loading, must account for these effects as well as it can. In this light, I offer the following line (brought to my notice by S. Lahiri, and attributed to A R Dykes):

> Engineering is the art of modelling materials we do not wholly understand, into shapes we cannot precisely analyse so as to withstand forces we cannot properly assess, in such a way that the public has no reason to suspect the extent of our ignorance.

The Burden of Truth

Statements of results in technical subjects must bear what may be called their subject's burden of truth.

In mathematics, truth is nothing less than absolute proof. This can be very hard. However, the leeway allowed to mathematicians is that difficult problems can be attacked, by any and all, over a long period of time (see [6] for the story of a 350 year problem). In the interim, offshoots that lead to other truths, as well as partial results, are all acceptable contributions to the steady march of knowledge.

In physics, truth has a high place as well. But here, it is interpreted as consistency with every known relevant experiment of the past, pending possible invalidation by just one experiment, as yet unforeseen, that throws up an inconsistency. The more determined, exhaustive, and dramatic the search for that invalidating experiment is, the more satisfying and reassuring is the failure to find it. Limitations of time, space or money do not impinge on this philosophy.

In comparison, the engineer's burden of truth may seem lighter. The ceiling of that tunnel through the mountain should merely remain intact as a million cars pass under; that radioactive material should just safely reach its resting place at the bottom of the sea; that office in Hyderabad should merely remain comfortable even on crowded days in the middle of summer; and that gold should just reach its intended place on the moon. Once the design succeeds, and the system functions as needed, the burden of truth is met. The issue is not reopened (nor design fee refunded) when someone else does it better, faster or cheaper. The unique charm of mathematics in engineering lies in the many levels and forms in which it is invoked, revoked, used, abused, developed, implemented, interpreted and ultimately *put back in the box of tools*, before the final engineering decision, made within the allotted resources of time, space and money, is given to the end user.

Mathematicians are not blamed for failing to prove theorems. And physicists are not blamed for failing to disprove theories. In their subjects, truth has divine status; and its burden is borne by the gods themselves. But the engineer's burden of truth, though lighter in comparison, rests on human shoulders. Consider, if you will, these lines from the *Hymn of Breaking Strain* by Rudyard Kipling:

The careful text-books measure
(Let all who build beware!)
The load, the shock, the pressure
Material can bear.
So, when the buckled girder
Lets down the grinding span,
The blame of loss, or murder,
Is laid upon the man.

The Hertz contact approximation presented in this article was shown to me by J Papadopoulos. The heated strip analysis is close to something H D Conway showed me. The bicycle example is from J Papadopoulos *via* A Ruina, who has also much influenced the way I think about approximations in engineering. R N Goverdhan, D Chatterjee and A Ruina read the manuscript and helped me improve it. Kipling's *Hymn of Breaking Strain* is a good poem, but the rest of it is not about engineering.

Suggested Reading

[1] E J Hinch, *Perturbation Methods*, Cambridge University Press, 1991.

[2] C M Bender and S A Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill, 1978.

[3] W Goldsmith, *Impact: The Theory and Physical Behaviour of Colliding Solids*, Edward Arnold, 1960.

[4] R W Clough and J Penzien, *Dynamics of Structures*, McGraw-Hill, 1975.

[5] H Goldstein, *Classical Mechanics*, second edition, Addison-Wesley, 1980.

[6] S Singh, *Fermat's Last Theorem*, 1997, Fourth Estate, Paperback edition: 2002.

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