The most prestigious award for mathematics is the Fields Medal which is awarded once in four years to three or four young mathematicians for their outstanding contributions. They receive the medal during the International Congress of Mathematicians held once in four years.

In the most recent International Congress of Mathematicians held in Berlin, Germany during August 18–August 27, 1998, the following four mathematicians were awarded the Fields Medals: Richard E Borcherds, W Timothy Gowers, Maxim Kontsevich and Curtis T McMullen.

The Work of the Fields Medallists: 1998 [1]

1. Richard E Borcherds

C S Rajan

The work of Borcherds draws upon diverse areas from mathematics and physics, and shows a surprising convergence of ideas from finite group theory, modular forms, Lie algebras, and conformal quantum field theory. The proof of the so-called monstrous moonshine conjecture is a major highlight of the work; in the following discussion we concentrate mainly on this topic. The moonshine conjecture predicts the existence of an intimate relationship between the monster group, the largest of the sporadic finite simple groups, and the theory of modular functions. In order to clarify the structures arising in this conjecture, Borcherds introduced the concept of vertex algebras, also known as chiral algebras, which provides a mathematically precise algebraic formulation of conformal quantum field theory, and has furthered the connection between automorphic forms and Lie algebras.

Finite groups are familiar objects not only in mathematics but also in various sciences, especially physics. Simple groups, namely those with no nontrivial normal subgroups, are the building blocks for finite groups. The classification of all finite simple groups, completed in the seventies, ranks among the major achievements of mathematics in this century. Apart from certain series of simple groups, such as the alternating groups consisting of even permutations on 5 or more symbols and the so-called Chevalley groups, there are 26 'sporadic' finite simple groups, making up the list of finite simple groups. The first sporadic groups were constructed by Mathieu in the last century, but it took more than 100 years before other sporadic groups were discovered. An interesting example was discovered by Conway, as the automorphism group of the Leech lattice, modulo {±1}. The Leech lattice, which plays an important role in Borcherds' work as well, is the unique lattice in the 24-

[1] Reproduced with permission from *Current Science*, Vol.75, pp.1290–1292, 1998.

Note to the Student Reader

In this four-part article[1], you will find accounts of the work of the four Fields medallists of 1998. A characteristic feature of mathematics is that research level ideas are often quite difficult to describe in simple terms, owing to the many-runged nature of the concepts involved – abstractions built upon abstractions. Indeed, the words themselves are frequently a source of difficulty, as there are so many technical terms involved. Nevertheless it is necessary that students make at least an attempt to read expository articles which describe current research. This remark is of particular relevance to these articles, as the work described has been judged by the mathematical community to be of far-reaching importance. We urge you to make the effort, and to read the articles in their entirety. In each case, the first one-third or so of the article gives an overview of the topic and highlights the main problems of interest, while the remainder of the article is much more technical and specific. Therefore, if you find yourself in difficulties over the pieces, do not get discouraged!

Editors

[1] The remaining three parts will appear in subsequent issues.

dimensional Euclidean space, with a fundamental domain of unit volume, such that the squared length of any element in the lattice is an even integer and there are no lattice elements of squared length 2.

The 'monster' is the largest sporadic simple group of order

$$2^{46}\times 3^{20}\times 5^{9}\times 7^{6}\times 11^{2}\times 13^{3}\times 17\times 19\times 23\times 29\times 31\times 41\times 47\times 59\times 71$$

or having approximately 10^{54} elements. The existence of such a group F was predicted independently by Fischer and Griess in 1973. Even before the monster group was proved to exist, hints of its intricate connections with the theory of modular functions began to appear. It was observed by Ogg that in a certain naturally occurring sequence S_n of modular curves, S_p has genus 0 (namely it is the Riemann sphere) for a prime p if and only if p divides the order of the monster group. McKay and Thompson found interesting connections between dimensions of vector spaces with irreducible representations of the monster, with the coefficients of the Fourier series expansion of the elliptic modular function j. In the light of their observations McKay and Thompson conjectured that there ought to be a natural infinite-dimensional representation of the monster, on $V = \oplus_{n\in Z} V_n$, where the dimension of V_n is the nth Fourier coefficient in the q-expansion of $j-744$.

While this may seem remarkable enough, Thompson proposed going still further by considering, for each element $g \in F$, the series $\Sigma_{n\in Z}\mathrm{Tr}(g|V_n)q^{n-1}$. Conway and Norton then discovered that each such series coincides with the first few terms of a normalized 'Hauptmodul' for a genus 0 group Γ, a more general analogue of the elliptic function j. This led them to the celebrated moonshine conjecture, which was finally proved by Borcherds:

Theorem: There is a graded F module $V = \oplus_{n \in Z} V_n$ such that for any $g \in F$, the series $\Sigma_{n \in Z} \mathrm{Tr}(g | V_n) q^{n-1}$ is a normalized Hauptmodul for a suitable genus 0 curve.

The monster group F was finally constructed by Griess and a simpler construction was given later by Conway. The module V was constructed by Frenkel, Lepowsky and Meurman, using certain techniques from the representation theory of infinite dimensional Lie algebras and conformal field theory. Borcherds introduced the notion of vertex algebras and showed that the module V has the structure of a vertex operator algebra. Roughly speaking the vertex algebra is formed by considering the 'algebra' generated by all the vertex operators. Vertex algebras have an infinite number of locally finite bilinear pairings satisfying an analogue of the Jacobi identity. A vertex operator algebra, has additionally a distinguished element w, called a conformal vector, such that the associated vertex operator generates the action of what is known as a Virasoro algebra on V. An example of a vertex operator algebra is given by the Fock space of a string propagating on a torus. The moonshine module is obtained by combining a twisted as well as an untwisted vertex operator module associated to the Leech lattice, and amounts to a theory of a string propagating on an orbifold that is not a torus.

The coefficients of the q-expansion of these Hauptmoduls also satisfy some polynomial identities, known as replication formulas. Borcherds had also introduced a more general class of Lie algebras, called the Generalized Kac–Moody (GKM) algebras, and shown that the Weyl character formula holds for these algebras too. To prove the moonshine conjecture, Borcherds adopted the strategy of constructing a suitable Lie algebra with an action of the monster group such that the computation of the twisted Euler–Poincaré characteristic for any element of the monster yields the necessary replication formulas, in a manner similar to the derivation of the Weyl denominator formula. Starting with the module V, Borcherds twists it with the vertex algebra associated to a certain 2-dimensional lattice. The desired monster Lie algebra is a suitable subquotient of this vertex algebra, and turns out to be a GKM-algebra. The structure of this algebra is analysed using results from physics (notably the 'no ghost theorem' proved by Goddard and Thorn for the purposes of string theory) and arithmetic.

We now touch briefly on other aspects of Borcherds' work. The denominator formulas of GKM-algebras give rise to modular forms on the real orthogonal group of signature $(n+2, 2)$. Starting with a modular form on the upper half plane (which can have poles at the cusps) satisfying some extra conditions, Borcherds constructs modular forms on the orthogonal groups, whose singularities are explicitly determined, and with an explicit product expansion. In an interesting application of the method involved, he also gave an explicit product expansion for the *j*-function, and many modular forms with product expansions.

Together with Katzarkov, Pantev and Shepherd–Barron, Borcherds proved the following result which is a particular case of Shafarevich's conjecture for surfaces: any family of minimal surfaces with Kodaira dimension 0 and constant Picard number, parametrized by a complete variety, is isotrivial. The proof of this for Enriques surfaces was obtained earlier by Borcherds.

The modular forms constructed by Borcherds also seem to have interesting relationships with mirror symmetry and Donaldson polynomials. He has also certain interesting results relating Heegner cycles and modular forms, generalizing earlier results of Hirzebruch–Zagier and Gross–Zagier.

To conclude, as the automorphism group of a distinguished conformal field theory in the critical dimension 26, the monster is fundamentally related to string theory of theoretical physics. What seems amazing about the moonshine is that finite group theorists, interested in understanding finite simple groups, and theoretical physicists, interested in unifying the fundamental forces, have converged to similar mathematical structures, almost independently and around the same time.

C S Rajan, Tata Institute of Fundamental Research, Mumbai 400 005, India.

Suggested Reading

[1] *Scientific American*, pp.21–24, November 1998.
[2] Notices of the *Am. Math. Soc.*,pp.17–26, January 1999, and pp.1158–1160, November 1998.

"Does nature, as is often said, obey mathematical rules? Such an obedience suggests that the outside world is somehow constrained by mathematical principles. Or do nature and mathematics exhibit parallel but essentially unrelated behaviour? ... Perhaps the rhythms and structures of intangible mathematics simply mimic the rhythms and structures of tangible reality with neither obeying the other."

William Dunham
in '*The Mathematical Universe*',
John Wiley & Sons, Inc., 1994.