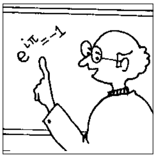
Classroom



In this section of* Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

Head-On Collision of Two Balls Revisited

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We present a dramatic demonstration which is also a simple, and an extremely low-cost experiment of head-on collision of two balls in a vertical direction. The advantages of this phenomenon in the vertical direction are clarified. Some simple estimates are made. A thorough analysis of this simple topic is then made, which includes various special and limiting cases, conditions of collision, change of reference frame, etc.

Introduction

Head-on collision of two bodies is a topic in class XI under mechanics. Given the two masses and their initial velocities, their final velocities are derived by using conservation of momentum and energy. Either examples are not given at all, or if they are given, authors of books as well as teachers mention the case of collision between two vehicles, of equal masses or one light and one heavy. But not everybody can present to watch this 'experiment', nor can one 'perform' it at will.

There is of course the linear air track where one can study collision of bodies and several other phenomena, but it is a fairly costly and bulky apparatus. Can one devise a simple low-cost

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Classical mechanics, collisions of two bodies.

experiment where we can make measurements and compare with the formulae, or at least a demonstration, which will vividly bring out the intricacies of the formulae? If we try two balls on a horizontal table, it will be impossible to avoid rotation of balls, and also the collision will be far from head-on because of different radii of the balls. Friction will also cause problems. Finally, it will also be difficult to measure the initial and final speeds of the balls just before and after the collision.

If we try two balls on a horizontal table, it will be impossible to avoid rotation of balls, and also the collision will be far from head-on because of different radii of the balls. Friction will also cause problems.

What about performing the experiment in the vertical direction? Gravity would be much easier to take account of, and friction due to air is far more negligible as compared to that of the table.

Experiment/Demonstration

Take a big ball (basket ball/foodball/volleyball) and another ball smaller than that (tennis ball). If we drop either of them separately to the ground, it bounces and rises to about 40%-60% of the original height. But if we hold them one over the other, the smaller one above the bigger one and touching it, and drop them simultaneously, lo and behold! the small ball shoots up to the ceiling height.

This itself serves as an excellent and dramatic demonstration. Having enjoyed this phenomenon, our plea now to teachers is that when they teach this topic in the classroom and derive formulae for final velocities, they should show this demo. It hardly takes a minute. The teacher can also ask questions like: ‘From where does the small ball get the energy to rise so high?’, etc.

Can we make some observations without using any gadget for measurement of velocities? Yes, provided we are willing to sacrifice precision and accuracy. We simply mark a vertical scale on the wall from the ground level to the ceiling. A least count of 2 cm or even 5 cm on this scale will be good enough.

We drop the bigger ball from a height of about 1 m and watch, very roughly, the height to which it bounces. This gives us its coefficient of restitution, and allows us to calculate its speed just before the bounce (downward) and just after the bounce (upward). Then we hold the small ball above the bigger one, with their centres along the same vertical line, and with the bigger ball at about 1 m height. We drop them together. One person can concentrate on the big ball and another one on the small ball, and try to estimate, again very roughly, the heights to which they rise. This allows us to estimate the final velocities just after the collision.

One can try this with different balls, for example, by replacing the tennis ball with a rubber ball or a ping pong (table tennis) ball. It is seen that the ping-pong ball suffers from a large air drag and does not rise to the same height as a tennis ball. Also for better measurements one must try to hold the two balls with their centres close to the same vertical so that it is a head-on collision, and the small ball rises to the maximum height. It is not difficult to achieve this in a few trials.

Theory

Let two balls A and B having masses m_1 and m_2 and initial velocities u_1 and u_2 collide head-on, which means with their centres on the same line of motion. Let v_1 and v_2 be their final velocities after collision, neglecting any loss of energy. Since the phenomenon is taking place in one dimension, we may drop the vector symbol, though still keeping in mind that velocity is an algebraic quantity (not mere magnitude) which can take negative as well as positive values. Conservation of momentum and of energy gives us the two equations.

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2 , \tag{1}$$

$$m_1 u^2_1 + m_2 u^2_2 = m_1 v^2_1 + m_2 v^2_2 . \tag{2}$$

These can respectively be put in the form

$$m_1 (u_1 - v_1) = m_2 (v_2 - u_2), \tag{3}$$

$$m_1 (u^2_1 - v^2_1) = m_2 (v^2_2 - u^2_2). \tag{4}$$

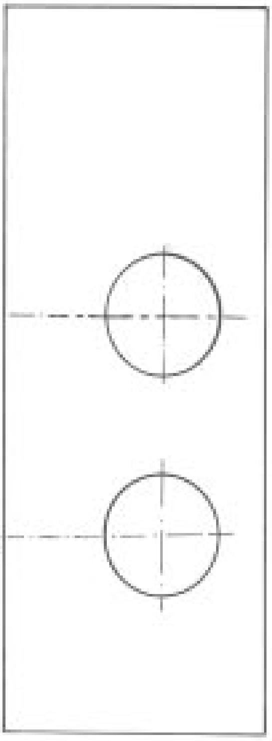


Figure 1. The bigger ball dropped from a height h_1 rises to a height h_2.

We may assume that $v_1 \neq u_1$ and $v_2 \neq u_2$, because otherwise it would mean that there is no collision. Then dividing the respective sides of (4) by (3), we get

$$u_1 + v_1 = v_2 + u_2 \,. \tag{5}$$

Equations (1) and (5) are two linear equations, and solving these, we get

$$v_1 = \frac{m_1 - m_2}{m_1 + m_2} u_1 + \frac{2m_2}{m_1 + m_2} u_{2,}$$

$$v_2 = \frac{2m_1}{m_1 + m_2} u_1 - \frac{2m_1 - m_2}{m_1 + m_2} u_2 \,. \tag{6}$$

Consider the bounce of the big ball off the floor. Let it be dropped from a height h_1 and let it rise to a height h_2 after bouncing from the floor; see *Figure* 1. If e is the coefficient of restitution between the ball and the floor, it is the ratio of the speeds of the ball close to the floor just after and just before the bounce. Thus we shall have

$$e = (h_2/h_1)^{1/2}. \tag{7}$$

In our experiment, both the balls are dropped simultaneously, with the small ball above the big one. Both of them fall through the same distance, say h_1, before the bigger ball touches the ground. Let the subscript 1 stand for the big ball and subscript 2 for the small ball. Thus the speed of both the balls just before the bounce will be

$$|u_2| = (2gh_1)^{1/2}, \tag{8}$$

where g is the acceleration due to gravity. The big ball touches the ground with this speed and rebounds with the speed

$$u_1 = e|u_2| = e\,(2gh_1)^{1/2}. \tag{9}$$

Note that the phenomenon consists of the bounce of the bigger ball off the floor and the collision between the two balls within a fraction of a second.

Thus we require the following observations to determine the parameters. We measure the masses of the two balls. We drop the bigger ball alone from a height h_1 and observe the height h_2 to which it rises after the bounce, and this gives e. Then we drop the two balls together, as described, from the height h_1 and determine the heights h_3 and h_4 to which the big and the small ball, respectively, rise after collision; see *Figure* 2. This gives us an estimate of the final velocities v_1, v_2 after collision. They would be given by

$$v_1 = (2gh_3)^{1/2}, \qquad v_2 = (2gh_4)^{1/2}. \tag{10}$$

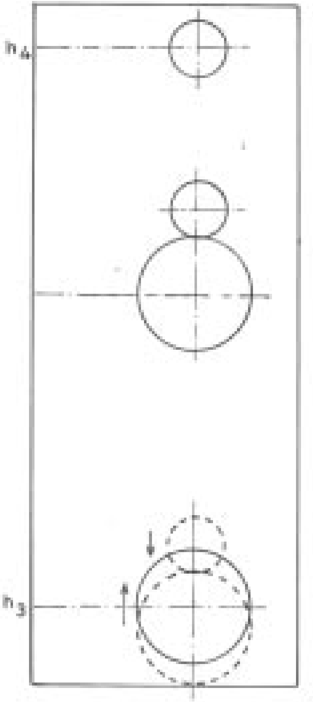


Figure 2. The two balls are dropped together from a height h_1. The bigger ball rises to height h_3 and the smaller one to h_4. The intermediate positions are shown by dotted lines.

Observations and Calculations

The masses of the basketball and a tennis ball was found to be $m_1 = 610$ g and $m_2 = 85$ g. When we dropped the basketball from the height of 1 m, it was found to rise to 40 cm, thus giving

$$e = \sqrt{0.4} = 0.633. \tag{11}$$

Then both the balls are dropped as described, the basketball being at 1 m. This gives the velocity of the tennis ball just before the collision to be

$$u_2 = -(2 \times 9.8 \times 1)^{1/2} \text{ m/s} = -4.43 \text{ m/s}. \tag{12}$$

Velocities would be taken as positive in upward direction. The speed of the basketball just before the bounce will also be $|u_2| = 4.43$ m/s. After the bounce, it becomes

$$u_1 = e|u_2| = 0.633 \times 4.43 \text{ m/s} = 2.804 \text{ m/s}. \tag{13}$$

Using these in (6), we get

$$v_1 = 0.755 \times 2.8 - 0.245 \times 4.43 \text{ m/s} = 1.03 \text{ m/s}, \tag{14}$$
$$v_2 = 1.755 \times 2.8 + 0.755 \times 4.43 \text{ m/s} = 8.26 \text{ m/s}.$$

Both the balls move upward (positive velocity) after collision. With these speeds, the balls would rise to

$$h_3 = 5 \text{ cm}, \; h_4 = 3.45 \text{ m}. \tag{15}$$

Note that in this calculation, we have neglected the loss of energy during collision, that is, we have not taken into account the coefficient of restitution between the two balls. This loss may somewhat reduce the heights to which the balls rise.

In any case, it is not surprising that the tennis ball rises to the ceiling height!

Limiting Cases

One can perform this experiment with different mass ratios, although the best and dramatic effect is observed when m_1 is fairly larger than m_2. It is interesting to consider various limiting cases. For this, let us define the mass ratio

$$x = m_1/m_2. \tag{16}$$

We consider the following special/limiting cases

(a) $m_1 = m_2, x = 1$: This gives

$$v_1 = u_2, v_2 = u_1. \tag{17}$$

This is the well-known result in which the velocities are exchanged after collision.

(b) $m_1 >> m_2, x >> 1$: In this case, let $t = 1/x$, so that $t << 1$. Equation (6) can be written as

$$v_1 = \frac{1-t}{1+t}u_1 + \frac{2t}{1+t}u_2, v_2 = \frac{2}{1+t}u_1 - \frac{1-t}{1+t}u_2. \tag{18}$$

Expanding in Taylor series and retaining terms up to first order in t, we get

$$v_1 \sim (1-2t)u_1 + 2tu_2, v_2 \approx 2(1-t)u_1 - (1-2t)u_2. \tag{19}$$

The limiting values of the velocities after collision, for $t \to 0$, are seen to be

$$v_1 = u_1, v_2 = 2u_1 - u_2. \tag{20}$$

From this, we note that if the second ball is an extremely light ball, with almost negligible mass, it will shoot up with a limiting velocity $2u_1 - u_2$, which in our case comes out to be

$$v_2 = 2 \times 2.8 + 4.43 \text{ m/s} \approx 10.0 \text{ m/s.} \tag{21}$$

This will make it rise to a height of about 5 m, when dropped from 1 m.

$m_1 << m_2$, $x << 1$: This gives

$$v_1 \approx -(1-2x)u_1 + 2(1-x)u_2 = -u_1 + u_2 + 2x(u_1 - u_2),$$
$$v_2 \approx 2xu_1 + (1-2x)u_2 = u_2 + 2x(u_1 - u_2). \tag{22}$$

If the first ball is extremely light, we can take $x \to 0$. Then the final velocities, in the limit $x = 0$, are

$$v_1 = -u_1 + 2u_2, v_2 = u_2 . \tag{23}$$

Thus the velocity of the lighter ball is reversed and reduced, while that of the heavier ball remains almost unaffected in the zeroth order.

Change of Frame of Reference

Equation (6) for final velocities can also be derived by another method. We go to the frame of reference of B, the second ball, which means we move with it with a constant velocity u_2. Then for us, $u'_2 = 0$, while ball A of mass m_1 is approaching this ball with a velocity $u'_1 = u_1 - u_2$. We have used primes to denote velocities in our frame, which is moving with a velocity, which is the initial velocity of the second ball B. Then the equations for conservation of momentum and energy become

$$\begin{aligned} m_1 u'_1 &= m_1 v'_1 + m_2 v'_2 \\ m_1 u'^2_1 &= m_1 v'^2_1 + m_2 v'^2_2 , \end{aligned} \tag{24}$$

where v'_1 v'_2 are the final velocities in the new frame.

On solving these, we get

$$v'_1 = \frac{m_1 \pm m_2}{m_1 + m_2} u'_1,$$

which gives either $v'_1 = u'_1$ and therefore $v'_2 = 0$, or

$$v'_1 = \frac{m_1 - m_2}{m_1 + m_2} u'_1. \tag{25}$$

The first solution corresponds to 'no collision' (ball A may be moving away from ball B). So we discard it. Thus (25) is the only physical solution for the case of collision. This leads to

$$v'_2 = \frac{2m_1}{m_1 + m_2} u'_1. \tag{26}$$

The velocities in the laboratory frame are related to those in the frame B by translation with a velocity u_2. There v_1, v_2 would be obtained from the respective primed variables by replacing u'_1 by $u_1 - u_2$ and also adding u_2 to each final velocity. Thus

$$v_1 = \frac{m_1 - m_2}{m_1 + m_2}(u_1 - u_2) + u_2,$$

$$v_2 = \frac{2m_1}{m_1 + m_2}(u_1 - u_2) + u_2. \tag{27}$$

On simplifying, we see that these are the same as (6).

Conditions for Collision

Consider again head-on collision along a horizontal straight line. We consider that a positive velocity represents a ball moving to the right, and ball A is to the left of ball B. We can have three cases, (a) both balls moving to the right, (b) the two balls approaching each other, and (c) both balls moving to the left. For collision to take place, their velocities before collision must satisfy.

$$u_1 > u_2. \tag{28}$$

It may be verified that this inequality holds good in all the three cases, taking proper account of algebraic signs for velocities.

What are the conditions on the final velocities? We compare the final velocities with each, as well as with the respective initial velocities. After collision, these balls do not jump or overtake one another, and must recede away from each other. For this, we must have

$$v_2 > v_1. \tag{29}$$

Also the ball A on the left, after collision, must move with a velocity smaller than its initial velocity. Similarly, ball B on the right must move after collision with a velocity greater than its initial velocity. Thus we must have

$$v_1 < u_1, v_2 > u_2. \tag{30}$$

These inequalities must be understood in an algebraic sense, and one can see that they are valid in all the three cases. As said earlier, (29) and (30) are expressions of the fact that the balls do not overtake each other (a truly 1-D process).

When does Ball a Come to Rest?

We want to find the condition when the ball A comes to rest after collision, that is $v_1 = 0$. This would happen when the collision transfers just enough momentum to ball A to cancel its initial momentum. Putting $v_1 = 0$ in (1) and (6), we get

$$(m_2 - m_1)\, u_1 = 2m_2\, u_2. \tag{31}$$

(a) If $m_1 = m_2$, this gives $u_2 = 0$. This is the familiar case of two spheres with equal masses, in which case the velocities are exchanged after collision. Here the second ball is at rest initially ($u_2 = 0$), so the first ball will come to rest after collision, that is $v_1 = 0$, as desired.

But suppose neither ball is initially at rest. We can achieve $v_1 = 0$ in this case only if $m_1 \neq m_2$. It is clear even qualitatively that this can happen when (b) both balls are travelling to the right and when ball A is lighter than ball B, or (c) when the balls are approaching each other and ball A is heavier than ball B. Let us analyse these situations.

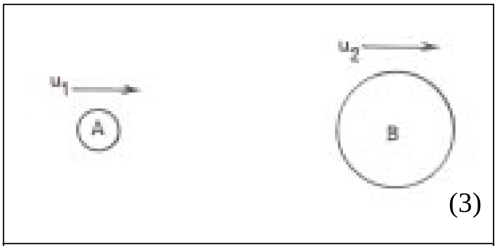
When $m_1 \neq m_2$ (31) becomes

$$u_1 = \frac{2m_2}{m_2 - m_1} u_2 .$$

(b)Thus if both the balls are travelling to the right (u_1, u_2 positive), then we must have $m_2 > m_1$, and they must be related by (32) to give $v_1 = 0$; see *Figure* 3.

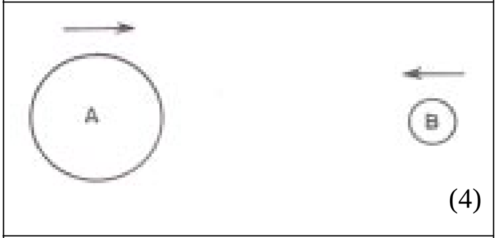
(c)If the balls are approaching each other ($u_1 > 0$ and $u_2 < 0$), then we must have $m_1 > m_2$, and again (32) must hold to yield the result $v_1 = 0$. See *Figure* 4.

(d) One might say that (32) is also satisfied for $m_2 > m_1$ when both balls are travelling to the left, so that both initial velocities are negative. This situation means that a heavier ball B is



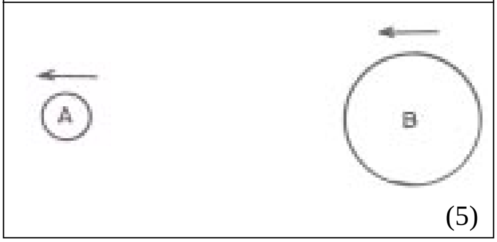
(3)

Figure 3. Smaller ball A colliding with a bigger ball B, both travelling to the right, can result in A coming to rest after collision.



(4)

Figure 4. Bigger ball A travelling to the right collides with a smaller ball B approaching it, can result in A coming to rest.



(5)

Figure 5. Smaller ball A (on the left) and bigger ball B (on the right), both travelling to the left, cannot result in A coming to rest after collision.

colliding with a lighter ball A, both travelling to the left; see *Figure* 5. Even qualitatively, we can see that this cannot result in the lighter ball coming to rest after collision. So what is wrong?

Remember that here we want $u_1 < 0$ and $v_1 = 0$. But this violates the inequality (30), where we require $v_1 < u_1$ (which means v_1 will be more negative). Thus it is not possible to satisfy all these three inequalities together, and hence this solution is discarded.

What about symmetry between the two balls? It is a collision between two balls, and we might replace A by B and B by A. But we have broken the symmetry when we required $v_1 = 0$, that is we wanted the first ball A (the one on left) to come to rest after collision. If we want the second ball B (the one on right) to come to rest, that is $v_2 = 0$, with $m_1 \neq m_2$, then it will be possible only if they are travelling to the left, and *not* if they are travelling to the right. So the symmetry is restored again in that sense.

Acknowledgements

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Suggested Reading

[1] Velocity amplification in collision experiments involving superballs, Class of William G Harter, 1971, *American Journal of Physics*, Vol.39, pp.656.

How Far Apart are Primes? Bertrand's Postulate

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It is well-known that there are arbitrarily large gaps between primes. Indeed, given any natural number n, the numbers $(n+1)!+2;(n+1)!+3;¢¢¢;(n+1)!+(n+1)$ being large multiples of $2;3;¢¢¢;n+1$ respectively, are all composite numbers.

Let us now ask ourselves the following question. If we start with a natural number n and start going through the numbers $n+1;n+2$ etc., how far do we have to go

Keywords
Bertrand's postulate, Prime number theorem.

before hitting a prime? Trying out the first few numbers, we see that

$$1 \to 2;\ 2 \to 3;\ 3 \to 5;\ 4 \to 5;\ 5 \to 7;\ 6 \to 7;\ 7 \to 11 \text{ etc.}$$

Thus, it seems that we need to go 'at most twice the distance' i.e., we seem to be able to find a prime between n and $2n$ for the first few values of n. But there is absolutely no pattern here. In fact, although we have seen above that there are arbitrarily large gaps between primes, it is nevertheless true that 'there is regularity in the distribution of primes'. It is this fascinating clash of tendency which seems to make primes at once interesting and intriguing. It turns out indeed to be true that: there is always a prime between n and $2n$. This statement, known as Bertrand's postulate, was stated by Bertrand in 1843 and proved later by Chebychev in 1852. Actually, Chebychev proves a much stronger statement which was further generalised to yield a fundamental fact about the prime numbers known as the prime number theorem.

Although there are arbitrarily large gaps between primes, it is nevertheless true that 'there is regularity in the distribution of primes'.

Proving the prime number theorem is beyond the scope of this article but stating it certainly lies within it. For a positive real number x, let us denote by $\pi(x)$, the number of primes which do not exceed x. The prime number theorem states that the ratio $\frac{\pi(x)\log x}{x}$ approaches the limit 1 as x grows indefinitely large i.e.,

$$\lim_{x \to \infty} \frac{\pi(x)\log x}{x} = 1.$$

One usually writes $\pi(x) \sim \frac{x}{\log x}$ to describe such an asymptotic result. What Chebychev proved was that there are some explicit positive constants a, b so that

$$a\frac{\log x}{x} < \pi(x) < b\frac{\log x}{x}.$$

If p_n denotes the n-th prime, Bertrand's postulate is equivalent to the assertion that $p_{n+1} < 2p_n$ while the prime number theorem itself is equivalent to the statement $\lim_{n\to\infty} \frac{p_n}{n\log n} = 1$.

If p_n denotes the n-th prime, Bertrand's postulate is equivalent to the assertion that $p_{n+1} < 2p_n$ while the prime number theorem itself is equivalent to the statement $\lim_{n\to\infty} (p_n)/(n \log n = 1$.

It is rather startling to note that the great mathematician Gauss (1777-1855) had, at the age of 15, already conjectured the truth of the prime number theorem. Four years later, in 1796, Legendre also came independently to conjecture something similar.

Legendre conjectured, based on empirical evidence, that $\pi(x) \approx \frac{x}{A\log x + B}$ and also conjectured values of A, B which turned out to be incorrect. Gauss, on the other hand, conjectured that $\pi(x) \approx \int_2^x \frac{dt}{\log t}$; the right side is denoted by $\mathrm{li}(x)$ to stand for the 'logarithmic integral'. This seems to have exactly the content of the prime number theorem since clearly $\mathrm{li}(x) \approx \frac{x}{\log x}$. However, later research (following Riemann) has confirmed that Gauss's assertion is even more astute than what it appears to be on the face of it. In fact, the function $\mathrm{li}(x)$ has the asymptotic expansion (for any fixed n)

$$\mathrm{li}(x) = \frac{x}{\log x} + 1!\frac{x}{(\log x)^2} + 2!\frac{x}{(\log x)^3} + \cdots + (n-1)!\frac{x}{(\log x)^n} + O\left(\frac{x}{(\log x)^{n+1}}\right).$$

A refined version of the prime number theorem indeed implies that $\pi(x)$ has the same asymptotic expansion!

In particular, this implies that the best possible values for A and B in Legendre's conjecture are $A = 1, B = -1$.

The prime number theorem was proved independently by Hadamard and de la Vallee Poussin. A well-known mathematician quipped once that the proof almost immortalised these two mathematicians – they lived to be 96 and 98, respectively!

Returning to Bertrand's postulate, after Chebychev's first proof, other simpler proofs appeared. In this article, we shall discuss two of the simplest proofs due to two great minds – Ramanujan and Erdos. It is perhaps for the first time that, in Resonance, a proof due to Ramanujan is given [1]. Most of us are told stories about him and his discoveries and it is rarely that one can find a proof of his which is elementary enough to be actually discussed at this level. Erdos's proof is even more elementary and we start with it.

Erdos's Proof

We start with any natural number n and look at the product $\prod_{p\le n} p$ over all primes $p \le n$.

We shall have occasion to use the well-known and easily proved fact asserting that the highest power to which a prime divides $n!$ is given by the expression

$$[n/p]+[n/p^2]+[n/p^3]+\cdots$$

Erdos's proof starts with the following very beautiful observation:

Lemma. $\prod_{p\le n} p \le 4^n$.

Proof.

The prime number theorem was proved independently by Hadamard and de la Vallee Poussin. A well-known mathematician quipped once that the proof almost immortalised these two mathematicians – they lived to be 96 and 98, respectively!

We prove this by induction on n. It evidently holds good for small n. Look at some $n>1$ such that the result is assumed for all $m \le n$. Then,

$$\prod_{p\le n} p = \prod_{2p\le n+1} p \prod_{n+1<2p\le 2n} p \le 4^{\frac{n+1}{2}} \prod_{n+1<2p\le 2n} p$$

by the induction hypothesis.

Now, the surprisingly simple observation that each prime in the last product (i.e., each prime between $(n+1)/2$ and n) divides the binomial coefficient $\binom{n}{[\frac{n+1}{2}]}$, shows that $\prod_{p\le n} p \le 4^{(n+1)/2}2^{n-1} = 4^n$.

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[1] Ramanujan's solution of an elementary problem has been discussed in [1].

The bound $\binom{n}{[\frac{n+1}{2}]} \leq 2^{n-1}$ used above is trivially seen to be true by induction. Thus, we have proved this lemma.

Since we are interested in the possible primes between n and 2n, it is natural to consider the binomial coefficient $\binom{2n}{n}$ because it 'captures' all these primes as its divisors. Now, obviously, the binomial coefficient $\binom{2n}{n}$ is the largest term in the expansion $(1+1)^{2n}$ which has $2n+1$ terms. Therefore, we have

$$(2n+1)\binom{2n}{n} \geq 2^{2n}. \qquad (1)$$

This gives a lower bound for this middle binomial coefficient and, the idea of the rest of the proof is that lot of the contribution comes from primes between n and 2n. More precisely, if we write

$$\binom{2n}{n} = \prod_{p \leq 2n} p^{e_p} = \prod_{p \leq n} p^{e_p} \prod_{n+1 < p \leq 2n} p;$$

then we shall use the lemma to give an upper bound for the first product $\prod_{p \leq n} p^{e_p}$. Here, e_p denotes the power of p dividing the middle binomial coefficient $\binom{2n}{n}$ and, the second product stands for 1 if there are no terms.

We want to see which primes $p \leq n$ actually contribute to $\binom{2n}{n}$.

If $p^2 > 2n$ i.e., if $n \geq p > \sqrt{2n}$, then clearly, $e_p = [\frac{2n}{p}] - 2[\frac{n}{p}] = 0$ or 1.

Thus, such primes divide $\binom{2n}{n}$ either to a single power or not at all.

If $n \geq 3$, then a prime $p \leq n$ with $2n/3 < p$ must be at least 3 and so $p^2 \geq 3p > 2n$. As $1 \leq n/p < 3/2$ for $2n/3 < p \leq n$, we have $[\frac{n}{p}] = 1$ and $[\frac{2n}{p}] = 2$ i.e.,

$e_p = [\frac{2n}{p}] - 2[\frac{n}{p}] = 2 - 2 = 0$. Thus, these primes do not divide $\binom{2n}{n}$ when $n \geq 3$.

Gauss conjectured that $\pi(x) \sim \int_2^x (dt)/(\log t)$.

In other words, we have:

$e_p \le 1$ if $\sqrt{2n} < p \le 2n/3$, and $e_p = 0$ if $2n/3 < p \le n$.

Finally, for the primes with $p \le \sqrt{2n}$, we simply take the trivial bound $p^{e_p} \le 2n$. Then, we have

$$\binom{2n}{n} = \prod_{p\le n} p^{e_p} \prod_{n+1<p\le 2n} p$$

$$\le \prod_{p\le\sqrt{2n}} (2n) \prod_{\sqrt{2n}<p\le 2n/3} p \prod_{n+1<p\le 2n} p$$

$$\le 4^{2n/3} \prod_{p\le\sqrt{2n}} (2n) \cdot \prod_{n+1<p\le 2n} p.$$

using the lemma. We take $n \ge 8$ as we have verified Bertrand's postulate explicitly for $n \le 7$; so $\sqrt{2n} \ge 4$ and thus, the number of terms in the first product is at most $\sqrt{2n} - 2$ (as 1 and 4 are not primes). Therefore, we have on using (1) that

$$\frac{2^{2n}}{2n+1} \le \binom{2n}{n} \le (2n)^{\sqrt{2n}-2} 4^{2n/3} \prod_{n+1<p\le 2n} p.$$

Replacing the first term by $\frac{2^{2n}}{(2n)^2}$, we get

$$\prod_{n+1<p\le 2n} p \ge \frac{4^{n/3}}{(2n)^{\sqrt{2n}}}.$$

Thus, to show that the left side has terms (i.e., that it is not 1 according to our convention), it suffices to see whether the right hand side is bigger than 1 for all n. As usual, this will turn out to be true for large enough values of n and will fail for small values (this only means that the inequality is good enough for large values of n and we need to verify the original assertion directly for the smaller values left out).

After a few trials, we arrive at the number $n = 450$ and find that

$4^{n/3} = 4^{150} > (2n)^{\sqrt{2n}} = (900)^{30}$ since $4^5 > 900$.

There is nothing special about 450 excepting the fact that $2n$ is a perfect square and 450 is large enough for the inequality to hold good. This inequality continues to hold for $n > 450$ as the difference $4^{n/3} - (2n)^{\sqrt{2n}}$ is an increasing function. This last statement is simple to see by looking at the derivative of the difference of the corresponding logarithms. Now it is an easy exercise to verify Bertrand's postulate for $n < 450$. This was essentially Erdos's proof; ours is a slightly simplified version of his original argument which appears in [2].

Ramanujan's Proof

Let us turn to Ramanujan's proof. It is also extremely clever and completely elementary apart from the use of what is known as Stirling's formula – a proof of which has been discussed in [3].

We shall, however, give a further simplified version which avoids Stirling's formula altogether.

In the previous proof we used an estimate for $\prod_{p \leq n} p$. Here, we consider an additive version of it viz., look at the so-called Chebychev function $\theta(x) = \sum_{p \leq x} \log p$ defined for any real number $x \geq 2$. We note two things to begin with:

(i) $\theta(n)$ is simply the logarithm of $\prod_{p \leq n} p$,

(ii) Bertrand's postulate is true for a real number x if it is true for $n = [x]$; indeed, a prime between n and $2n$ is between x and $2x$ as well.

Let us also understand that since we are interested in primes between x and $2x$, we need a lower bound for $\theta(2x) - \theta(x)$. In other words, we need reasonable lower as well as upper bounds for θ values.

Now, the expression $\sum_{i \geq 1} [n/p^i]$ for the power of a prime dividing $n!$ gives us

$$\log[x]! = \sum_{i \ge 1} \psi(x/i);$$

where ψ is the function defined by

$$\psi(x) = \sum_{i \ge 1} \theta(x^{1/i}).$$

This is the reason to introduce real x.

Using an elementary trick of old vintage, we have the following:

$$\log[x]! - 2\log[x/2]! = \sum_{i \ge 1} (-1)^{i-1}\psi(x/i);$$

$$\psi(x) - 2\psi(\sqrt{x}) = \sum_{i \ge 1} (-1)^{i-1}\theta(x^{1/i}).$$

As θ, ψ are increasing functions, we get inequalities by chopping off at an odd stage and at an even stage as follows:

$$\psi(x) - \psi(\frac{x}{2}) \le \log[x]! - 2\log[\frac{x}{2}]!$$

$$\le \psi(x) - \psi(\frac{x}{2}) + \psi(\frac{x}{3}); \qquad (2)$$

$$\psi(x) - 2\psi(\sqrt{x}) \le \theta(x) \le \psi(x). \qquad (3)$$

Here, Ramanujan takes recourse to Stirling's formula which states that $n! \sim \sqrt{2\pi}\frac{n^{n+\frac{1}{2}}}{e^n}$.

We shall not use it but proceed as follows.

For any $x > 1$, we have the binomial coefficient

$$\binom{[x]}{[\frac{x}{2}]} < \sum_{r \ge 0} \binom{[x]}{r} = 2^{[x]}.$$

Taking logarithms, we obtain

$$\log[x]! - 2\log[x/2]! < x\log 2.$$

But, clearly $\log 2 < \frac{3}{4}$ since $16 < e^3$. In other words, we have

$$\log[x]! - 2\log[x/2]! < 3x/4 \quad \forall\, x > 0. \qquad (4)$$

Now, we find a lower bound. As we observed before Erdos's proof, for $x > 0$, the binomial coefficient $\binom{[x]}{[x/2]}$ (being the largest term in the expansion of $(1+1)^{[x]}$), must be bigger than $\frac{2^{[x]}}{[x]+1}$ since there are $[x]+1$ terms in the binomial expansion of $(1+1)^{[x]}$.

If x is large enough (for instance, if $x > 240$), then $\frac{2^{[x]}}{[x]+1} > e^{\frac{2[x]}{3}}$. Taking logarithms, we get

$$\log[x]! - 2\log[x/2]! > 2x/3 \quad \forall\, x > 240. \qquad (5)$$

By (2),(4) and (5), we have

$$\psi(x) - \psi(x/2) < 3x/4 \quad \forall\, x > 0; \qquad (6)$$

$$\psi(x) - \psi(x/2) + \psi(x/3) > 2x/3 \quad \forall\, x > 240. \qquad (7)$$

By replacing x by $x/2; x/4; x/8$ etc. in (6) and adding all the expressions we get

$$\psi(x) < 3x/2 \quad \forall\, x > 0. \qquad (8)$$

Note that since $\theta(x) < \psi(x)$, we have a reasonable upper bound for $\theta(x)$ by (8). For the lower bound, let us use the first inequality of (3) viz., $\psi(x) \le 2\psi(\sqrt{x}) + \theta(x)$ and the inequality $\theta(x/2) \le \psi(x/2)$ to write

$$\psi(x) - \psi(x/2) + \psi(x/3) \le 2\psi(\sqrt{x}) + \theta(x) - \theta(x/2) + \psi(x/3).$$

If we use the upper bound for ψ given in (8), we obtain

$$\psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) < \theta(x) - \theta\left(\frac{x}{2}\right) + \frac{x}{2} + 3\sqrt{x}. \qquad (9)$$

For $x > 240$, the left side has a lower bound given in (7) so that we finally obtain

$$\theta(x) - \theta(x/2) > x/6 - 3\sqrt{x} \quad \forall\, x > 240.$$

Evidently, the right side is positive if $x > 324$. Therefore, there is a prime between x and $2x$ if $x > 162$. For smaller values of x, we find primes explicitly as before. This finishes the beautiful proof due to Ramanujan.

More Comments on Primes

The above elementary methods and their modifications are sufficient to prove Chebychev's theorem viz., the assertion

$$\frac{1}{6}\frac{x}{\log x} < \pi(x) < 6\frac{x}{\log x}.$$

The reader is urged to try and use this to prove the following bound for the n-th prime:

$$\frac{1}{6}n\log n < p_n < 12(n\log n + n\log\frac{12}{e}).$$

This last inequality (in fact, the upper bound) shows easily that the series $\sum_p \frac{1}{p}$ over all primes diverges. How fast does it diverge?

Here is a proof showing that the divergence is at least as fast as $\log\log x$ i.e.,:

$\exists c > 0$ such that $\sum_{p \leq x}\frac{1}{p} \geq c\log\log x \ \forall x.$

To see this, given $x > 1$, let us look at the area under the curve $y = \frac{1}{x}$ between 1 and x. Evidently, this area $\int_1^x \frac{dx}{x} = \log x$ is less than $\sum_{n=1}^{[x]}\frac{1}{n}$. (draw a figure to see this).

So, if we consider the product $\prod_{p \leq x}(1 - \frac{1}{p})^{-1}$, then clearly, $\prod_{p \leq x}(1 - \frac{1}{p})^{-1} \geq \sum_{n=1}^{[x]}\frac{1}{n} \geq \log x.$

Now, $(1 - \frac{1}{p})^{-1} = (1 + \frac{1}{p})(1 + \frac{1}{p^2 - 1})$. Therefore, $\prod_{p \leq x}(1 + \frac{1}{p}) \geq a\log x$ where $a = \prod_p (1 + \frac{1}{p^2 - 1})^{-1}$. Using $e^x > 1 + x$ for all $x > 0$, we get $\prod_{p \leq x} e^{1/p} \geq \prod_{p \leq x}(1 + \frac{1}{p}) \geq a\log x$ so that $\sum_{p \leq x}\frac{1}{p} \geq c\log\log x$ for some $c > 0$.

This finishes the proof of the lower bound.

The more adventurous reader may like to use the Abel summation formula (see [4]) and prove that this lower bound is of the correct order i.e., one has the rather interesting statement:

$$\sum_{p \le x} \frac{1}{p} \sim \log\log x.$$

We end the note by giving a curious application of Bertrand's postulate to finding a recursive expression for primes which appeared in ([5], p.289).

We consider the function

$$f_n = \mathrm{Sign}\left(\frac{2((n-1)!)}{n} - \left[\frac{2((n-1)!)}{n}\right]\right.$$

defined for all $n \ge 3$. Here, the sign function is the function which takes the value 0 at 0 and the value $\frac{x}{|x|}$ for any $x \neq 0$. Clearly, $f_n = 1$ or 0 according as n is prime or composite. Now, by Bertrand's postulate, if $p_n \ge 3$, then p_{n+1} occurs as the first prime among $p_n + 2, p_n + 4, \cdots, 2p_n - 1$. Therefore, (writing p_n as p for simplicity of notation),

$$\begin{aligned} p_{n+1} = {} & (p+2)f_{p+2} + (p+4)f_{p+4}(1 - f_{p+2}) + \cdots \\ & + (p+6)f_{p+6}(1 - f_{p+2})(1 - f_{p+4}) + \cdots + \\ & (2p-1)f_{2p-1}(1 - f_{p+2}) \cdots (1 - f_{2p-3}). \end{aligned}$$

Suggested Reading

- **[1] S A Shirali, Mathematical Analysis, *Echoes from Resonance*, (Universities Press, Hyderabad), pp.30-33, 2001.**
- **[2] I Niven and Zuckermann, *Introduction to the theory of numbers*, John Wiley and Sons, New York, 1960.**
- **[3] S Ramasubramanian, Mathematical Analysis, *Echoes from Resonance*, Universities Press, Hyderabad, 2001.**
- **[4] T Apostol, *Introduction to analytic number theory*, Springer International Students Edition, Narsoa Publishers, New Delhi, 1986.**
- **[5] *American Math. Monthly*, 1975.**