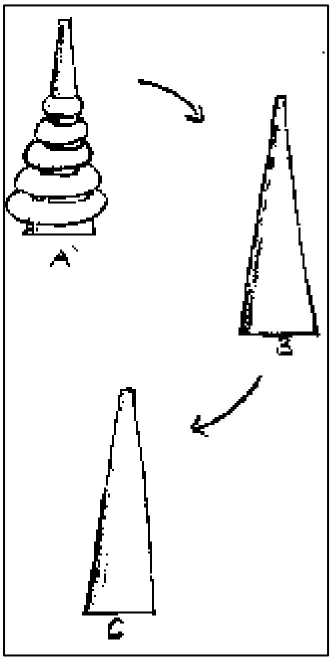
Classroom

In this section of* Resonance, *we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

Shailesh A Shirali
Rishi Valley School
Rishi Valley 517 352
Andhra Pradesh, India

The 'Oriented' Tower of Hanoi

The `Tower of Hanoi' or `Tower of Brahma' is well known. We are given n disks (numbered $1, 2, 3, \ldots, n$); in size we have $\#1 < \#2 < \#3 < \ldots < \#n$. Also given are three pegs A, B and C. The disks are initially all on A. They are required to be moved to C, using B as a halting station. The rules are simple: a larger disk is not permitted to sit on top of a smaller one. With these restrictions we are required to move the disks from A to C using the least possible number of moves. It is well known, and easy to prove via induction, that the least number of moves required to do the needful is $2^n - 1$. Note that in this version, if the terminus is B rather than C, nothing essential changes; the formula is the same.



We now impose an additional condition: the direction of movement must be clockwise. That is, we may move disks directly from A to B, from B to C, or from C to A; but not directly from B to A, from C to B, or from A to C. How does this condition affect the formula for the least number of moves? Clearly now the choice of terminus is no longer irrelevant.

Let u_n be the least number of moves required if the disks are to be moved to B, and let v_n be the corresponding number if the disks have to be moved to C. Note that u and v are not the same. For instance, $u_1 = 1$ whereas $v_1 = 2$; and it may be verified that $u_2 = 5$ whereas $v_2 = 7$. Elementary counting arguments yield the following equations:

$$\begin{aligned} u_n &= 2v_{n-1} + 1; &(1)\\ v_n &= 2v_{n-1} + u_{n-1} + 2. &(2) \end{aligned}$$

To see why (1) is true, note that before disk # n can be moved from A, disks # 1, # 2, …, # $(n-1)$ must first be moved; clearly they must all be moved to C, for only then can disk # n be moved to B. This requires v_{n-1} moves. Then disk # n is moved from A to B (1 move required), and finally disks # 1, # 2, …, # $(n-1)$ are moved from C to B via A (from the 'back'), taking another v_{n-1} moves and bringing the total number of moves to $2v_{n-1} + 1$; therefore $u_n = 2v_{n-1} + 1$. Equation (2) may be justified along similar lines.

From (1) and (2) we obtain:

$$\begin{aligned} u_n &= 2(u_{n-1} + u_{n-2}) + 3;\\ v_n &= 2(v_{n-1} + v_{n-2}) + 3. \end{aligned}$$

Observe that both sequences obey the same recursion! The initial conditions are:

$$u_1 = 1;\quad u_2 = 5;\quad v_1 = 2;\quad v_2 = 7.$$

A Mathematica program to compute v_n and u_n for various values of n is given below.

```
ClearAll[u,v];SetAttributes[{u,v},Listable];
u[1] = 1; u[2] = 5;
v[1] = 2; v[2] = 7;
u[n_] := 2 v[n - 1] + 1
v[n_] := 2 v[n - 1] + u[n - 1] + 2;
```

Here are some sample output values:

```
u[Range[1,10]] = {1,5,15,43,119,327,895,2447,6687,18271}
v[Range[1,10]] = {2,7,21,59,163,447,1223,3343,9135,24959}
```

Observe that for both sequences, each term is roughly three times the preceding term (a bit less, in fact). We shall presently be able to explain why this is so.

Now write z_n for $u_n + 1$ or $v_n + 1$; then we find that z obeys a 'homogeneous' recursion relation:

$$z_n = 2(z_{n-1} + z_{n-2}).$$

The initial conditions are: (a) for the u-sequence, $z_1 = 2$ and $z_2 = 6$; (b) for the v-sequence, $z_1 = 3$ and $z_2 = 8$.

Assuming that z is an exponential function, i.e., $z_n = t^n$ for some suitable number t, we find by substitution into the above relation that $t^2 = 2t + 2$. This equation has the roots $1 + \sqrt{3}$ and $1 - \sqrt{3}$. It follows that the sequence of powers of either of these two numbers obeys the same recursive law as z, and so therefore does any linear combination of the two power sequences. So we write

$$z_n = a(1 + \sqrt{3})^n + b(1 - \sqrt{3})^n$$

for unknown constants a and b, and then find a and b via the initial conditions $z_1 = 2, z_2 = 6$ (or $z_1 = 3, z_2 = 8$ for the v-sequence). This is easy to do, for we have only to solve a pair of simultaneous equations.

Whereas in the unoriented tower of Hanoi problem a very simple formula holds for the least number of moves required, in the present problem the formula is substantially more complicated. We find, after going through the steps suggested above, that

$$u_n = \left(\frac{3 + \sqrt{3}}{6}\right)(1 + \sqrt{3})^n + \left(\frac{3 - \sqrt{3}}{6}\right)(1 - \sqrt{3})^n - 1,$$

$$v_n = \left(\frac{3 + 2\sqrt{3}}{6}\right)(1 + \sqrt{3})^n + \left(\frac{3 - 2\sqrt{3}}{6}\right)(1 - \sqrt{3})^n - 1.$$

That may be a bit more than we had bargained for! As a matter of fact, one has also the curious formulae

$$u_n = \sum_{i \ge 0} 2^{n-i} \binom{n-i}{i} - 1$$

$$v_n = \sum_{i \ge 0} 2^{n-i} \binom{n+1-i}{i} - 1.$$

Now we can explain why in both sequences each term is a bit less than three times the preceding term. Observe that $1 - \sqrt{3} \approx -0.73205$, which in absolute value is strictly less than 1. So as n gets larger and larger, the contribution of the term $b(1-\sqrt{3})^n$ gets smaller and smaller. Since the quantity alternates in sign, and since u_n and v_n are integers, we arrive at the following:

$$u_n = \text{the integer closest to } \left(\frac{3+\sqrt{3}}{6}\right)(1-\sqrt{3})^n - 1;$$

$$v_n = \text{the integer closest to } \left(\frac{3+2\sqrt{3}}{6}\right)(1+\sqrt{3})^n - 1.$$

Sample verification.

Let $n = 4$; we have

$$\left(\frac{3+\sqrt{3}}{6}\right)(1+\sqrt{3})^4 - 1 \approx 42.9393.$$

$$\left(\frac{3+2\sqrt{3}}{6}\right)(1+\sqrt{3})^4 - 1 \approx 59.0222.$$

Indeed we find that $u_4 = 43$ and $v_4 = 59$. It follows from the above that for large enough n,

$$\frac{u_n}{u_{n-1}} \approx 1+\sqrt{3}; \quad \frac{v_n}{v_{n-1}} \approx 1+\sqrt{3}.$$

Note that $1+\sqrt{3} \approx 2.73205$. This accounts for the observation made earlier. Now we are able to make a more precise observation: For sufficiently large n we have:

$$\frac{u_n}{u_{n-1}} \approx 2.73205; \quad \frac{v_n}{v_{n-1}} \approx 2.73205.$$

Sushan Konar
Raman Research Institute
Bangalore 560 080, India.

Stranger in a Strange Land

Introduction

The two essays which, we carry below were both written by research students working towards a PhD in physics. They were originally submitted to a competition (where they received honourable mention) in which the aim was to convey, to a non specialist and perhaps even non scientist reader, the motives and spirit of carrying out doctoral research. Although both have some technical details of the research area, our aim is to take particularly the student reader 'behind the scenes' of a PhD thesis. Merely by looking at the final product, or even by sitting in a seminar, the human side of scientific research may not be apparent. But research – and teaching – and learning – are all human activities, and we at *Resonance* do plan to carry the occasional piece focussing on the aspect of science education.

Rajaram Nityananda
NCRA, Pune

Strange Stars? Notwithstanding his matter-of-fact manner, I had a grave suspicion that my guide was out to pull my leg. Sure the sky is full of odd objects like neutron stars (stars made entirely of neutrons) and black holes (hungry monsters that eat up everything that ventures close, including light), but this must be stranger still. And it did nothing to my morale to find out that the investigation required something I sorely lacked – a good grinding in particle physics. In the final year of PhD, I was rapidly running out of time. Yet, here was a chance to go beyond the mundane. A serious perusal gave me a queer feeling. This would either make a nice science fiction or a scientist's dream. I was hooked.

Almost everything on Earth is made up of tiny atoms in which electrons go round tinier balls of neutrons and protons. But neutrons and protons are made up of even tinier particles called 'quark's. There are six types of quarks. And particle physicists, being the ingenious lot they are, named them – up, down, top, bottom, charm and strange. Though only the 'up' and 'down' quarks are found inside the neutrons and the protons. But these quarks are extremely shy objects – they prefer to form a small group and hide behind a shell. From outside we see only the shell of such conglomerates and call them 'hadron's'. The familiar neutron and proton are two such hadrons.

Almost everything on Earth is made up of tiny atoms in which electrons go round tinier balls of neutrons and protons.

The quarks could be pried out of their shells if the hadrons are squeezed really hard. When hadrons are packed really tight so that they crush against each other out pop the quarks from inside! But the densities needed would be enormous – a million billion times that of iron. And of course to push matter to such extraordinary densities one needs extraordinarily large forces. How on earth, or above it, could one produce that kind of force? Simple – use gravity.

Wait a minute. Isn't gravity the weakest of the four fundamental forces? That's right. But ultimately gravity wins out. Because it is proportional to the total mass. When one deals with massive objects like a star gravity is the force that dominates over all the others. Actually gravity controls the complete life history of a star – from birth to death. And the individuality of a star lies in its mass which is the factor gravity depends upon. So there are stars like our Sun which on shining without much change for billions of years. And there are heavier stars which end their lives in a spectacular supernova after a short lifespan of a mere (astronomically speaking!) million years.

It is from the ashes of this later variety of stars that the bizarre objects – neutron stars, strange stars or black holes emerge. The reason is simple. Because they were very massive to begin with, they experience the maximum squeeze in the hands of gravity giving rise to final products with unthinkable densities. In a neutron star, matter is so dense that the protons gobble up the electrons and convert themselves into neutrons. A black hole, on the other hand, is the ultimate squeezed state of matter – everything gets concentrated to a point.

A strange star is intermediate between these two. If a neutron star is squeezed a little more we reach the magic destiny at which neutrons exist no more. They break up into ‘up’ and ‘down’ quarks. And we get a star made of ‘up’ and ‘down’ quarks entirely. Twenty years ago this picture changed a little. It was argued that that the ‘ups’ and the ‘downs’ don’t feel very comfortable with themselves. So some of the ‘ups’ go and convert themselves into the ‘strange’. And then they live happily ever after in the object we now call the strange star.

I started my PhD to look into the behaviour of neutron stars when more material is dumped on them. It happens if the neutron star has a companion star that is shredding its outer layers. We realised that by increasing its mass we are basically squeezing the neutron star a bit more. That is, in the process, the neutron star has been converted to a strange star! So now it is essential to understand the physics of these strange objects.

At the moment we struggle to cut through the red-tape to obtain extra time to finish the investigation (I am supposed to submit my thesis in a month) while the stars in the sky remain as oblivious of us as ever and as constant. And through the millenia they pose as daunting a problem to us as they did to the first humans.

Just Shear Luck

Rangan Lahiri
Department of Physics
Indian Institute of Science
Bangalore 560 012, India.

‘Milk? Soap solution?!’ I was aghast. Here I was, just joined one of the best theoretical physics groups in India, with dreams of winning the Nobel Prize in five years flat, and this guy, who I had just signed up with has the nerve to tell me that I was to work on colloidal solutions like milk and blood and chicken broth! Where are the Feynman diagrams, the quantum uncertainty, the principle of least action?

'Why only milk' – he goes on to reassure me – 'even clay, or muddy river water is a colloid. Just look around the room and you will find some colloidal substance.' But which of them am I supposed to work on? Well, none really. The stuff I started my research on and continued for the next five years was an artificial colloid made by dropping plastic balls in water. And when I crossed my initial mental block, I found in them a rich and diverse world.

The rigidity of a solid is basically the amount of energy you pack in a fixed volume.

The best thing about them was the fact that they were a blown up version of everyday solids and liquids. While the atoms in the keyboard I am punching are roughly a few Angstroms (that is less than a millionth of a millimeter) apart, the 'polyballs' in the colloid are spaced by a few thousand Angstroms. The balls are charged, so, like atoms they don't like to come too close to each other. This means that they too can form regular crystalline structures – thats the way the balls are farthest apart and therefore the happiest.

How do they differ from real–world solids? They are soft, *real* soft! The rigidity of a solid is basically the amount of energy you pack in a fixed volume. In the colloidal crystal the same amount of energy is packed in a much larger region of space, so they can be twisted and sheared easily. What happens to them at a given force will happen to a real solid at a billion times that much, but since you can't generate such forces in your lab, experiments on colloids tell you what real solids do under heavy stress (inside the earth's crust, for example).

What happens? People who did the experiments had found that under steady shear stress, the colloidal crystal 'melts' – loses its crystalline structure; the polyballs are no longer in a regular array but scattered at random in space – just like the molecules in a liquid. The question is: why?

But why does any solid melt? Any undergraduate will tell you its because of entropy. Entropy is the number of ways you can do things – like you can stand on two legs in only own way, but there are two ways of standing one legged, so the latter state has a larger entropy. In a crystal the atoms are fixed but in a liquid they have more room to move around, so they can 'be liquid' in many different ways. At higher temperatures, a system prefers to be in a state of higher entropy, so it is a competition between that and the energetics which prefers a regular crystalline arrangement. When these two are most evenly matched, we are at the melting point.

But that is all 'equilibrium' thermodynamics, explained by Boltzmann and Maxwell and Gibbs almost a century ago. The problem with a sheared colloid is that it is never in equilibrium – you are constantly driving it so that the well set foundations of equilibrium statistical mechanics are no use at all.

Entropy is the number of ways you can do things – like you can stand on two legs in only own way, but there are two ways of standing one legged, so the latter state has a larger entropy.

Well, almost, but not quite. As we struggled through the months to find a formulation for the problem, we learned to use equilibrium ideas in a limited way – quantities that were valid instantaneously but not all the time. The biggest problem was the huge number of variables (as many as the number of polyballs) – we had to identify the most important ones and throw away the rest. That is really what theoretical physics is about – throwing into the dustbin anything that's not essential.

After a mere two years of attempts, failures, desolation and desperation we hit upon the right model – we had managed to reduce the problem to just two variables! But still, we couldn't calculate the objects we needed to – we had to resort to numerical computation. There followed the usual period of anxiety and uncertainty, but at last we had a phase diagram, which looked just like the experimental one. Well, except for the actual numbers, you see – the shape of the phase boundary was sort of similar. But in spite of this drawback we were quite satisfied – we had taken the first step towards a theory of shear induced melting – obtained by shear luck!