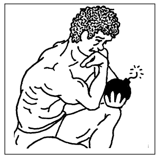
Think It Over



This section of* Resonance *is meant to raise thought-provoking, interesting, or just plain brain-teasing questions every month, and discuss answers a few months later. Readers are welcome to send in suggestions for such questions, solutions to questions already posed, comments on the solutions discussed in the journal, etc. to* Resonance *Indian Academy of Sciences, Bangalore 560 080, with "Think It Over" written on the cover or card to help us sort the correspondence. Due to limitations of space, it may not be possible to use all the material received. However, the coordinators of this section (currently A Sitaram and R Nityananda) will try and select items which best illustrate various ideas and concepts, for inclusion in this section.

Discussion of question raised in *Resonance*, Vol.1. No.9

St. Petersburg Paradox

Consider the following gambling game. First you pay a fee, say Rs 10, to play this game. Then you go on flipping a *fair* coin until a tail first appears. Your reward will be Rs 2^n if you make the coin come up heads $n-1$ times before a tail appears. For example, if the outcome is *HHHT*, you win $2^4 = 16$ rupees. Your expected gain from playing this game is

$$\sum_{n=1}^{\infty} (2^n - 10) P(n-1 \text{ heads followed by a tail}) =$$

$$\sum_{n=1}^{\infty} (2^n - 10) 2^{-n} = \infty .$$

In fact, the expected gain is ∞ not just for the fee of Rs.10 (that you need to pay first) but for any fixed amount, however large. But most of us will not play this game for a large fee. Why?

A simple explanation for this discrepancy can be immediately given if we accept that people evaluate the value of money in proportion to its quantity only in theory, while in practice, people with common sense evaluate the worth of money in proportion to the utility they derive from it. There is no doubt that a gain of 1000 rupees is more significant to a pauper than to a rich man although both gain the same amount. Similary, if you possess ten or twenty million rupees, an additional gain of 100 rupees will not be of much use whereas if you possess , say only rupees 100, the utility will be much higher. Thus it becomes evident that no valid measurement of the expected gain can be obtained without giving due consideration to its utility.

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In the paradox, the mathematical expectation is rendered infinite by the enormous amount which you can win if the coin does not come up tails for a very long time, perhaps at the hundredth or the thousandth flip. (The reward then will be 2^{100} or 2^{1000}, as the case may be.) Now, as a matter of fact, this large sum is worth no more to you (and gives no more pleasure to you forcing you to discontinue with the game) than does a sum amounting only to 10 to 20 million rupees. Let us suppose, therefore, that as far as utility of the reward is concerned, any amount which is above, say, 2^{24} rupees (for the sake of simplicity) is deemed by you as equal to 2^{24} rupees. In this case, the expected gain is:

$$\left(\frac{1}{2}\right)2^1+\left(\frac{1}{2}\right)^2 2^2+\ldots+\left(\frac{1}{2}\right)^{24}2^{24}+\left(\frac{1}{2}\right)^{25}2^{24}+\left(\frac{1}{2}\right)^{26}2^{24}+\ldots$$

$$=1+1+\ldots+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\ldots$$

$$=24+1=25$$

Thus, your real expectation is reduced in value to 25 rupees and, therefore, the fee that you are willing to pay to play the game also cannot be very large, a result which seems more reasonable than rendering it infinite. In decision theory, the above discussion provides the basis for a bounded utility function in rational behaviour.

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