

Differential Calculus in Normed Linear Spaces

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A proper understanding of multivariable calculus, usually taught at the senior undergraduate level, is crucial for the study of differential geometry and partial differential equations, among other areas. Traditionally one adopts a coordinate-dependent approach to the subject: the Jacobian matrix is introduced as the analogue of the derivative in the one-variable case. The role of the second derivative is played by the Hessian matrix and so on.

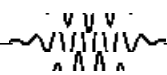
In the book under review, the author makes the point that in this approach the idea of a derivative as a linear map may fail to be emphasized. Perhaps as a consequence, the student may not develop a good grasp of fundamental theorems such as the inverse and implicit function theorems.

Also, there are clear advantages of a coordinate-free approach: Apart from clarifying the meanings of derivatives of all orders (as multilinear maps), one can generalize calculus to infinite-dimensional

spaces with little extra effort. The latter theory has several applications in partial differential equations and calculus of variations.

This book is based on courses taught by the author in the BStat programme at the Indian Statistical Institute, Calcutta. It aims at developing a 'geometric', i.e., coordinate-free version of calculus. This is done in the context of Banach spaces. According to the preface, the prerequisites assumed are a first course in linear algebra and a thorough background in analysis, corresponding, for instance, to the material covered in Halmos's *Finite Dimensional Vector Spaces* and the first eight chapters of Rudin's *Principles of Mathematical Analysis*.

The book is organized as follows. Chapter 1 contains preliminary material on Linear Algebra (matrices, inner product spaces, eigenvalues and diagonalizability). Chapter 2 has more linear algebra: dualization, multilinear functions and tensor products. Chapter 3 deals with normed linear spaces regarded as metric spaces – the usual point-set topology notions and norms on spaces of linear maps are studied. Calculus proper (derivatives, primitives and integrals, the mean value theorem and higher derivatives) begins in Chapter 4. In Chapter 5 the inverse and implicit function theorems are proved. The chapter begins with a proof of the contraction mapping principle and also has a discussion of Picard's theorem about the existence of solutions to ordinary differential equations. Chapter 6, the last chapter, deals with the Lagrange method of undetermined multi-



pliers and applies it to problems in the calculus of variations.

The exposition is somewhat unusual but, on the whole, clear and user-friendly. The author adopts a personal tone throughout and motivates the introduction of each new concept and result. Another commendable feature is that the author frequently brings in concepts from related areas of mathematics, especially in the exercises.

However, the book might not be suitable for self-study by a novice to the subject – the rather cumbersome notation makes things appear far more abstract than they actually are. This is the case, for instance, in the section on multilinear maps and tensor products.

Most sections have exercises at the end. The problems chosen are very interesting and useful, in the sense that the student is likely to encounter them in further mathematical study. In fact, one of the strong points of this book is its collection of exercises.

One important area not dealt with in this book is integration of functions of several variables and related topics (the formula for change of variable, Stokes theorem, etc). This topic cannot be developed naturally in the setting of general Banach spaces. However it should be an essential part of any course on multivariable calculus. Of course, to be fair to

the author, the book is called “Differential Calculus...” !

Finally, there are two mathematically incorrect statements that the reviewer noticed. On page 209, Section 4.4.1, it is remarked that the inverse of a C^∞ bijection is C^∞ . This is, of course, not true – the usual example is $f: \mathbb{R} \rightarrow \mathbb{R}$ with $f(t) = t^3$. A more serious one is in Chapter 6. The inner-products defined on pages 261 and 271 will not give rise to a Hilbert space structure on the space of C^1 paths, i.e. the space will not be complete with respect to these norms. Hence one will not be able to apply the Lagrange multiplier method (Theorem 6.1.1) directly to these spaces. As a consequence, Section 2 (Isoperimetric Problems) in 6.2 is, as it stands, incomplete. It should be noted that it is possible to overcome this problem by taking completions.

In conclusion, this book is highly recommended for use as a textbook for senior undergraduate or beginning graduate students. Its choice of topics, method of exposition and collection of exercises is especially noteworthy. A course based on this book and supplemented by material on integration, as mentioned above, will give a thorough and solid background in multivariable calculus from the modern perspective.

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